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ON A COMPARISION OF FOUR ESTIMATES OF A COMMON MEAN BY MULTIPLE CRITERIA DECISION MAKING METHOD

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summary

In this paper we consider the problem of estimating the common mean μ of two independent normal populations with unknown and possibly unequal variances. The purpose of this paper is to compare four unbiased estimates of μ based on Multiple Criteria Decision Making (MCDM) procedure in order to rank the estimates from the best to the worst in terms of their variances. A simulation study is also used to compare the competing estimates in small samples.

Keywords and phrases: Common mean, multiple criteria decision making

AMS Classification: 62C25, 62G05

1 Introduction

One of the interesting problems in statistical inference is the estimation of a common mean of two independent normal populations with unknown and unequal variances. It has been extensively discussed by Graybill and Deal [2], Sinha [7], and Pal and Sinha [5]. The objective of this paper is to compare various estimates of a common mean based on Multiple Criteria Decision Making (MCDM) which has been advocated by Hwang and Yoon [3], Zeleny [10], and Yoon and Hwang [9]. A lot of research work has been concerned with MCDM problem such as Filar *et al.* [1] and Maitra *et al.* [4]. The MCDM method is briefly described in Section 2 and Section 3 contains the main results for our problem.

2 A brief description of MCDM procedure

In the context of a '*discrete*' data matrix $X = (x_{ij}) : K \times N$ where $x'_{ij}s$ represent '*risk*' of ith '*source*' for jth '*category*', and we need to compare the K rows simultaneously with respect to all the N columns, MCDM is a novel statistical procedure to integrate the multiple indicators $(x_{i1}, \ldots, x_{iN})$ for row i across all indicators into a single meaningful and overall index. This is done by defining an Ideal Row (*IDR*) with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N)$$

and a Negative-ideal Row *(NIDR)* with the largest observed value for each column as

$$NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N).$$

For any given row$i$,we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm, we compute

$$L_1(i, IDR) = \sum_{j=1}^{N} [x_{ij} - u_j] \; w_j$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} [v_j - x_{ij}] \; w_j$$

where $w'_j s$ are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, \ldots, K. \tag{2.1}$$

Similarly, under L_2-norm, we compute

$$L_2(i, IDR) = [\sum_{j=1}^{N} (x_{ij} - u_j)^2 w_j]^{1/2}$$

$$L_2(i, NIDR) = [\sum_{j=1}^{N} (x_{ij} - v_j)^2 w_j]^{1/2}$$

and compare the rows based on

$$L_2(Index_i) = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1, \ldots, K. \tag{2.2}$$

A '*continuous*' version of this setup would involve $x'_{ij}s$ where the index j would vary '*continuously*'. In the context of our problem of comparing four unbiased estimates for estimation of a common mean μ of two normal populations with unknown variances σ_1^2 and σ_2^2 (see Section 3 below), obviously$K = 4$, $x_{ij}'s$ are chosen to represent variances of the four estimates for various value of $\tau = \sigma_2^2/\sigma_1^2$. L_1-norm and L_2-norm would be redefined as

$$L_1(\,i\,, IDR) \quad = \quad \int_0^\infty [x_i(\tau) - u(\tau)]\ w\,(\tau)\,d\tau \tag{2.3}$$

$$L_1(\,i\,, NIDR) \quad = \quad \int_0^\infty [v(\tau) - x_i(\tau)]\ w\,(\tau)\,d\tau \tag{2.4}$$

$$L_2(\,i\,, IDR) \quad = \quad \sqrt{\int_0^\infty (x_i(\tau) - u(\tau))^2\,w\,(\tau)\,d\tau} \tag{2.5}$$

$$L_2(\,i\,, NIDR) \quad = \quad \sqrt{\int_0^\infty (v(\tau) - x_i(\tau))^2\,w\,(\tau)\,d\tau} \tag{2.6}$$

where $u(\tau) = \min\limits_i \{x_i(\tau)\}$ and $v(\tau) = \max\limits_i \{x_i(\tau)\}$.

3 Main results

Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be iid observations from $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, respectively. Let $\bar{x}$, $\bar{y}$,s_1^2, s_2^2 be sample means and sample variances based on sample size n. Define $D = \bar{y} - \bar{x}$. The following four unbiased estimates of μ are quite standard in the literature:

$\hat{\mu}_1 \;=\; \left(\frac{\bar{x}}{s_1^2} + \frac{\bar{y}}{s_2^2}\right)\Big/\left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right)$ (Graybill and Deal [2]),

$\hat{\mu}_2 \;=\; \bar{x} + D\left(\frac{s_1^2 + D^2}{s_1^2 + s_2^2 + D^2}\right)$ (Sinha and Mouqadem [8]),

$\hat{\mu}_3 \;=\; \bar{x} + D\min\left\{\frac{s_1^2}{s_1^2 + s_2^2}, \frac{s_2^2}{s_1^2 + s_2^2}\right\}$(Sinha [6]),

$\hat{\mu}_4 \;=\; \bar{x} + D\left(\frac{s_1}{s_1 + s_2}\right)$ (Sinha and Mouqadem [8]).

To compare the above four unbiased estimate of μ based on their variances, it should be noted that exact analytical expression for the variances are quite complicated. We do this by simulation. It should also be noted that, apart from a common factor σ_1^2, all the variances depend on $\tau = \sigma_2^2/\sigma_1^2$. We have generated 50 sets of values for $\hat{\mu}_1$, $\hat{\mu}_2, \hat{\mu}_3$ and $\hat{\mu}_4$,when $n = 5$, 10, 15 with $\tau = 0.2$, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, taking $\mu = 0$and $\sigma_1^2 = 1$,and computed their variances. We report below the simulated values of variance for each estimate in Tables 1-3 for different values of n.

Finally, we apply MCDM to these four estimates. Figures 1-3 depict the variances of these estimates.

Write $x_{1k}(\tau) = Var_k(\hat{\mu}_1), \ldots, x_{4k}(\tau) = Var_k(\hat{\mu}_4)$ for interval k, $k =$ 1,2,. . . ,9. We observe from the graphs that there are 9 intervals of τ(0.2<0.4<0.6<0.8<1.0<1.2<1.4<1.6<1.8<2.0).

Table 1: Values of variances for $n = 5$

τ	$Var(\mu_1)$	$Var(\mu_2)$	$Var(\mu_3)$	$Var(\mu_4)$
0.2	0.05725	0.05427	0.16221	0.07517
0.4	0.04082	0.04555	0.09103	0.04090
0.6	0.10196	0.08436	0.13951	0.09152
0.8	0.10867	0.10417	0.12325	0.10336
1.0	0.11361	0.11292	0.14471	0.10631
1.2	0.07718	0.10339	0.08236	0.06705
1.4	0.09673	0.13547	0.08218	0.08971
1.6	0.14825	0.15990	0.13636	0.13831
1.8	0.15172	0.15305	0.13705	0.12479
2.0	0.13989	0.16571	0.15138	0.13040

Table 2: Values of variances for $n = 10$

τ	$Var(\mu_1)$	$Var(\mu_2)$	$Var(\mu_3)$	$Var(\mu_4)$
0.2	0.01609	0.01584	0.06086	0.01879
0.4	0.02162	0.02216	0.04282	0.02138
0.6	0.03330	0.03186	0.05040	0.03348
0.8	0.03887	0.03745	0.05115	0.03794
1.0	0.03853	0.03909	0.04945	0.03456
1.2	0.05163	0.04965	0.05379	0.05091
1.4	0.06254	0.06406	0.06375	0.05846
1.6	0.05246	0.05547	0.05215	0.04650
1.8	0.07246	0.07301	0.07360	0.06951
2.0	0.07904	0.08147	0.07557	0.07346

Table 3: Values of variances for $n = 15$

τ	$Var(\mu_1)$	$Var(\mu_2)$	$Var(\mu_3)$	$Var(\mu_4)$
0.2	0.01092	0.01090	0.04406	0.01322
0.4	0.01738	0.01716	0.03362	0.01790
0.6	0.03029	0.03070	0.03512	0.02972
0.8	0.02946	0.02992	0.02808	0.02837
1.0	0.03455	0.03480	0.03549	0.03186
1.2	0.03462	0.03627	0.03281	0.03221
1.4	0.03666	0.04187	0.03421	0.03528
1.6	0.05323	0.05579	0.04896	0.04914
1.8	0.05029	0.05450	0.05016	0.04833
2.0	0.04629	0.05257	0.04677	0.04682

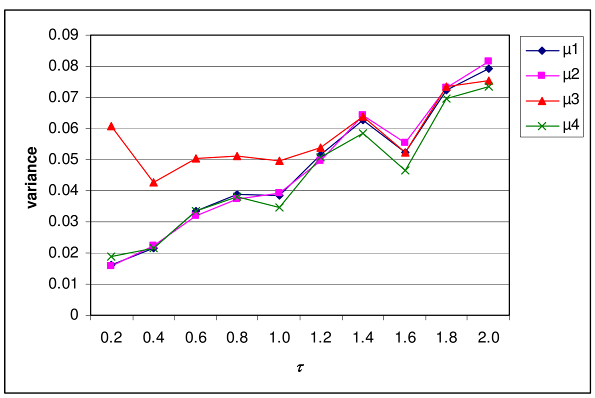

Figure 1: Graphical illustration of variances for four estimates when $n = 5$

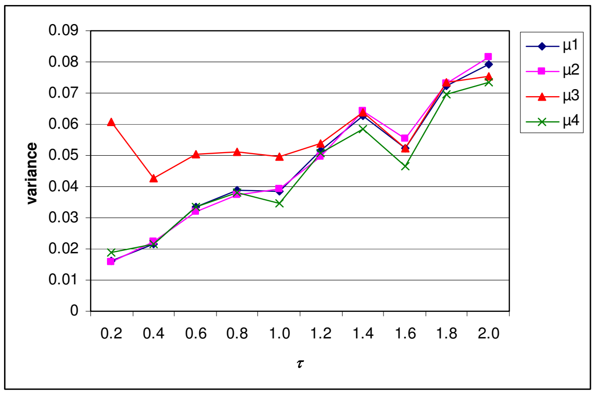

Figure 2: Graphical illustration of variances for four estimates when $n = 10$

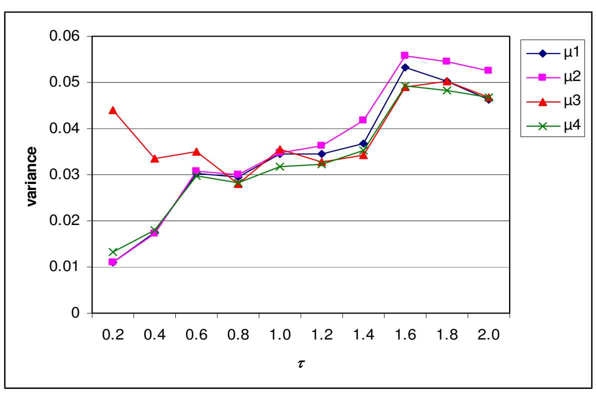

Figure 3: Graphical illustration of variances for four estimates when $n = 15$

Moreover some intervals have intersection point between graphs so we can subdivide these intervals. The main Ideal row and Negative-ideal row are as follows:

$$\begin{aligned} IDR: u(\tau) &= \{u_1(\tau): 0.2<\tau<0.4,\ u_2(\tau): 0.4<\tau<0.6,\ u_3(\tau): 0.6<\tau<0.8,\\ &\quad u_4(\tau): 0.8<\tau<1.0,\ u_5(\tau): 1.0<\tau<1.2,\ u_6(\tau): 1.2<\tau<1.4,\\ &\quad u_7(\tau): 1.4<\tau<1.6,\ u_8(\tau): 1.6<\tau<1.8,\ u_9(\tau): 1.8<\tau<2.0\}, \end{aligned}$$

$$\begin{aligned} NIDR: v(\tau) &= \{v_1(\tau): 0.2<\tau<0.4,\ v_2(\tau): 0.4<\tau<0.6,\ v_3(\tau): 0.6<\tau<0.8,\\ &\quad v_4(\tau): 0.8<\tau<1.0,\ v_5(\tau): 1.0<\tau<1.2,\ v_6(\tau): 1.2<\tau<1.4,\\ &\quad v_7(\tau): 1.4<\tau<1.6,\ v_8(\tau): 1.6<\tau<1.8,\ v_9(\tau): 1.8<\tau<2.0\}. \end{aligned}$$

Since we are dealing with a continuous parameter τ,(0.2<0.4<0.6<0.8<1.0<1.2<1.4<1.6< 1.8 < 2.0), a proper formulation of the MCDM procedure can be given as follows.

4 Analysis based on L_1-norm

For i = 1,2,3,4, we get

$$\begin{aligned} L_1(i, IDR) = & \int_{0.2}^{0.4} (x_{i1}(\tau)-u_1(\tau))\ w(\tau)\,d\tau + \int_{0.4}^{0.6} (x_{i2}(\tau)-u_2(\tau))\ w(\tau)\,d\tau \\ & + \int_{0.6}^{0.8} (x_{i3}(\tau)-u_3(\tau))\ w(\tau)\,d\tau + \int_{0.8}^{1.0} (x_{i4}(\tau)-u_4(\tau))\ w(\tau)\,d\tau \\ & + \int_{1.0}^{1.2} (x_{i5}(\tau)-u_5(\tau))\ w(\tau)\,d\tau + \int_{1.2}^{1.4} (x_{i6}(\tau)-u_6(\tau))\ w(\tau)\,d\tau \\ & + \int_{1.4}^{1.6} (x_{i7}(\tau)-u_7(\tau))\ w(\tau)\,d\tau + \int_{1.6}^{1.8} (x_{i8}(\tau)-u_8(\tau))\ w(\tau)\,d\tau \\ & + \int_{1.8}^{2.0} (x_{i9}(\tau)-u_9(\tau))\ w(\tau)\,d\tau \end{aligned}$$

$$\begin{aligned} L_1(i, NIDR) = & \int_{0.2}^{0.4} (v_1(\tau)-x_{i1}(\tau))\ w(\tau)\,d\tau + \int_{0.4}^{0.6} (v_2(\tau)-x_{i2}(\tau))\ w(\tau)\,d\tau \\ & + \int_{0.6}^{0.8} (v_3(\tau)-x_{i3}(\tau))\ w(\tau)\,d\tau + \int_{0.8}^{1.0} (v_4(\tau)-x_{i4}(\tau))\, w(\tau)\,d\tau \\ & + \int_{1.0}^{1.2} (v_5(\tau)-x_{i5}(\tau))\ w(\tau)\,d\tau + \int_{1.2}^{1.4} (v_6(\tau)-x_{i6}(\tau))\ w(\tau)\,d\tau \\ & + \int_{1.4}^{1.6} (v_7(\tau)-x_{i7}(\tau))\ w(\tau)\,d\tau + \int_{1.6}^{1.8} (v_8(\tau)-x_{i8}(\tau))\ w(\tau)\,d\tau \\ & + \int_{1.8}^{2.0} (v_9(\tau)-x_{i9}(\tau))\ w(\tau)\,d\tau \end{aligned}$$

The overall index can then be computed. It is clear that for the purpose of comparison of the four estimates, we can work with

$$L_1(Index_i) = \frac{L_1(i\,, IDR)}{L_1(i\,, IDR)+L_1(i\,, NIDR)}, \quad i = 1,2,3,4.$$

5 Analysis based on L_2-norm

For i = 1,2,3,4, we get

$$L_2(i,IDR) = \sqrt{\begin{array}{l}\int_{0.2}^{0.4}(x_{i1}(\tau)-u_1(\tau))^2\, w(\tau)\,d\tau+\int_{0.4}^{0.6}(x_{i2}(\tau)-u_2(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{0.6}^{0.8}(x_{i3}(\tau)-u_3(\tau))^2\, w(\tau)\,d\tau+\int_{0.8}^{1.0}(x_{i4}(\tau)-u_4(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.0}^{1.2}(x_{i5}(\tau)-u_5(\tau))^2\, w(\tau)\,d\tau+\int_{1.2}^{1.4}(x_{i6}(\tau)-u_6(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.4}^{1.6}(x_{i7}(\tau)-u_7(\tau))^2\, w(\tau)\,d\tau+\int_{1.6}^{1.8}(x_{i8}(\tau)-u_8(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.8}^{2.0}(x_{i9}(\tau)-u_9(\tau))^2\, w(\tau)\,d\tau\end{array}}$$

$$L_2(i,NIDR) = \sqrt{\begin{array}{l}\int_{0.2}^{0.4}(v_1(\tau)-x_{i1}(\tau))^2\, w(\tau)\,d\tau+\int_{0.4}^{0.6}(v_2(\tau)-x_{i2}(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{0.6}^{0.8}(v_3(\tau)-x_{i3}(\tau))^2\, w(\tau)\,d\tau+\int_{0.8}^{1.0}(v_4(\tau)-x_{i4}(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.0}^{1.2}(v_5(\tau)-x_{i5}(\tau))^2\, w(\tau)\,d\tau+\int_{1.2}^{1.4}(v_6(\tau)-x_{i6}(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.4}^{1.6}(v_7(\tau)-x_{i7}(\tau))^2\, w(\tau)\,d\tau+\int_{1.6}^{1.8}(v_8(\tau)-x_{i8}(\tau))^2\, w(\tau)\,d\tau\\ \quad+\int_{1.8}^{2.0}(v_9(\tau)-x_{i9}(\tau))^2\, w(\tau)\,d\tau\end{array}}$$

Under L_2-norm also, the overall index can be computed.

6 Choice of weight functions

Our first weight function $w_1(\tau)$ is defined by $w_1(\tau) = 1$ for every interval. Following Filar *et al.*[1], we also consider two additional choices of $w(\tau)$. The first one, denoted by $w_2(\tau)$, is based on the notion of entropy among $x_{1k}(\tau)$, $x_{2k}(\tau)$, $x_{3k}(\tau)$ and $x_{4k}(\tau)$ for interval k with various values of τ, denoted as $w_{2k}(\tau)$,when k = 1,2,. . . ,9, and the second one, denoted by $w_3(\tau)$, is based on the coefficient of variation of $x_{1k}(\tau)$, $x_{2k}(\tau)$, $x_{3k}(\tau)$ and $x_{4k}(\tau)$ for interval k with various values of τ, denoted as $w_{3k}(\tau)$,when k = 1,2,. . . ,9. It turns out that

$$w_{2k}(\tau) = \frac{1-\phi_k(\tau)}{\int_0^\infty [\,1-\phi_k(\tau)\,]\,d\tau}$$

where

$$\phi_k(\tau) \quad = \quad -\frac{1}{\log 4}\sum_{i=1}^{4}\left\{\frac{x_{ik}(\tau)}{\sum\limits_{i=1}^{4} x_{ik}(\tau)} \ . \log\left[\frac{x_{ik}(\tau)}{\sum\limits_{i=1}^{4} x_{ik}(\tau)}\right]\right\}, \ k = 1, 2, ..., 9$$

and

$$w_{3k}(\tau) \quad = \quad \frac{\sqrt{\frac{1}{4}\sum\limits_{i=1}^{4}\left(x_{ik}(\tau) - \bar{x}_k(\tau)\right)^2}}{\frac{1}{4}\sum\limits_{i=1}^{4} x_{ik}(\tau)}$$

therefore,

$$w_3(\tau) \ = \ \frac{\sqrt{3x_{1k}^2(\tau) + 3x_{2k}^2(\tau) + 3x_{3k}^2(\tau) + 3x_{4k}^2(\tau) - 2x_{3k}(\tau)x_{4k}(\tau) - 2x_{2k}(\tau)(x_{3k}(\tau) + x_{4k}(\tau))}}{\sum\limits_{i=1}^{4} x_{ik}(\tau)}$$

$$k = 1, 2, ..., 9.$$

7 Comparison of estimates

We report the ranks of the four estimates when compared on the basis of the weight functions $w_1(\tau), w_2(\tau)$and $w_3(\tau)$ using L_1-norm and L_2-norm in Tables 4-6.

8 Conclusion

Based on the above analysis under L_1- and L_2- norms, we conclude that our preference is uniformly for Sinha and Mouqadem [9] estimate $\hat{\mu}_4$ under three weights $w_1(\tau)$, $w_2(\tau)$ and $w_3(\tau)$. The familiar Graybill and Deal [3] estimate $\hat{\mu}_1$ which often holds rank 2 is also a good candidate for estimation of μ.

9 Acknowledgements

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References

[1] Filar, J.A., Ross, N.P. and Wu, M.L. (1999). *Environmental Assessment Based on Multiple Indicators.* Technical Report: Department of Applied Mathematics, University of South Australia.

Table 4: Analysis based on L_1-norm and L_2-norm using $w_1(\tau)$

n	μ	L_1 (IDR)	L_1 $(NIDR)$	I	Rank	L_2 (IDR)	L_2 $(NIDR)$	I	Rank
5	$\hat{\mu}_1$	0.01877	0.05409	0.25764	2	0.01634	0.04907	0.24982	2
	$\hat{\mu}_2$	0.03303	0.03983	0.45334	3	0.03336	0.04711	0.41453	3
	$\hat{\mu}_3$	0.04992	0.02294	0.68515	4	0.05004	0.02721	0.64778	4
	$\hat{\mu}_4$	0.00422	0.06864	0.05786	1	0.00606	0.05443	0.10024	1
10	$\hat{\mu}_1$	0.00464	0.01845	0.20105	2	0.00420	0.01959	0.17640	2
	$\hat{\mu}_2$	0.00513	0.01797	0.22209	3	0.00548	0.01991	0.21600	3
	$\hat{\mu}_3$	0.02191	0.00118	0.94876	4	0.02078	0.00188	0.91694	4
	$\hat{\mu}_4$	0.00062	0.02248	0.02691	1	0.00095	0.02039	0.04431	1
15	$\hat{\mu}_1$	0.00300	0.01092	0.21542	2	0.00277	0.01289	0.17680	2
	$\hat{\mu}_2$	0.00653	0.00739	0.46909	3	0.00604	0.01237	0.32833	3
	$\hat{\mu}_3$	0.00876	0.00517	0.62881	4	0.01253	0.00542	0.69790	4
	$\hat{\mu}_4$	0.00055	0.01338	0.03940	1	0.00082	0.01311	0.05881	1

Table 5: Analysis based on L_1-norm and L_2-norm using $w_2(\tau)$

n	μ	L_1 (IDR)	L_1 $(NIDR)$	I	Rank	L_2 (IDR)	L_2 $(NIDR)$	I	Rank
5	$\hat{\mu}_1$	0.20022	0.74575	0.21165	2	0.05103	0.18607	0.21521	2
	$\hat{\mu}_2$	0.42675	0.51921	0.45113	3	0.11840	0.17252	0.40698	3
	$\hat{\mu}_3$	0.67777	0.26819	0.71649	4	0.18602	0.09312	0.66639	4
	$\hat{\mu}_4$	0.04645	0.89951	0.04911	1	0.02082	0.20015	0.09421	1
10	$\hat{\mu}_1$	0.10033	0.32733	0.23460	2	0.01967	0.08112	0.19512	2
	$\hat{\mu}_2$	0.11737	0.31029	0.27444	3	0.02586	0.08182	0.24016	3
	$\hat{\mu}_3$	0.40465	0.02301	0.94619	4	0.08645	0.00816	0.91371	4
	$\hat{\mu}_4$	0.00889	0.41877	0.02078	1	0.00364	0.08553	0.04081	1
15	$\hat{\mu}_1$	0.04305	0.12945	0.24956	2	0.01071	0.03920	0.21462	2
	$\hat{\mu}_2$	0.08903	0.08346	0.51614	3	0.02167	0.03674	0.37101	3
	$\hat{\mu}_3$	0.10431	0.06819	0.60469	4	0.03765	0.01953	0.65839	4
	$\hat{\mu}_4$	0.00512	0.16737	0.02970	1	0.00223	0.04126	0.05125	1

Table 6: Analysis based on L_1-norm and L_2-norm using $w_3(\tau)$

n	μ	L_1 (IDR)	L_1 $(NIDR)$	I	Rank	L_2 (IDR)	L_2 $(NIDR)$	I	Rank
5	$\hat{\mu}_1$	0.03310	0.06225	0.34712	2	0.02244	0.04562	0.32969	2
	$\hat{\mu}_2$	0.06630	0.02905	0.69535	4	0.04785	0.03714	0.56302	4
	$\hat{\mu}_3$	0.04790	0.04745	0.50238	3	0.04250	0.03899	0.52150	3
	$\hat{\mu}_4$	0.00430	0.09105	0.04505	1	0.00542	0.05816	0.08525	1
10	$\hat{\mu}_1$	0.00271	0.00841	0.24396	2	0.00328	0.01354	0.19497	2
	$\hat{\mu}_2$	0.00318	0.00795	0.28537	3	0.00436	0.01371	0.24146	3
	$\hat{\mu}_3$	0.01034	0.00079	0.92871	4	0.01437	0.00156	0.90211	4
	$\hat{\mu}_4$	0.00029	0.01084	0.02631	1	0.00067	0.01415	0.04543	1
15	$\hat{\mu}_1$	0.00079	0.00424	0.15642	2	0.00146	0.00899	0.13957	2
	$\hat{\mu}_2$	0.00176	0.00327	0.35039	3	0.00322	0.00880	0.26770	3
	$\hat{\mu}_3$	0.00359	0.00144	0.71387	4	0.00885	0.00290	0.75333	4
	$\hat{\mu}_4$	0.00023	0.00480	0.04543	1	0.00058	0.00885	0.06110	1

Table 7: Summary of rank of four estimates using $w_1(\tau), w_2(\tau)$ and $w_3(\tau)$

n	μ	L_1-norm			L_2-norm		
		w_1	w_2	w_3	w_1	w_2	w_3
5	$\hat{\mu}_1$	2	2	2	2	2	2
	$\hat{\mu}_2$	3	3	4	3	3	4
	$\hat{\mu}_3$	4	4	3	4	4	3
	$\hat{\mu}_4$	1	1	1	1	1	1
10	$\hat{\mu}_1$	2	2	2	2	2	2
	$\hat{\mu}_2$	3	3	3	3	3	3
	$\hat{\mu}_3$	4	4	4	4	4	4
	$\hat{\mu}_4$	1	1	1	1	1	1
15	$\hat{\mu}_1$	2	2	2	2	2	2
	$\hat{\mu}_2$	3	3	3	3	3	3
	$\hat{\mu}_3$	4	4	4	4	4	4
	$\hat{\mu}_4$	1	1	1	1	1	1

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