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BAYESIAN ANALYSIS OF $P[Y < X]$ IN ITS LIMITING FORM

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SUMMARY

The persent study deals with the Bayesian estimation of the remodeled stress-strength system reliability, $P[Y < X]$. Here, X and Y represent strength and stress variables respectively.The random variables X and Y have been respectively redefined as $U = \min(X_1, X_2, \ldots, X_m)$ and $V = \max(Y_1, Y_2, \ldots, Y_n)$ to conceptualize the concept of limiting stress-strength reliability, $P[V < U]$, for meeting the requirements of the systems in defense. For such systems, the designer wishes to attach high probability to the event that the system remains operable at its minimum strength encountering with the maximum stress.

Keywords and phrases: Stress-strength reliability model, Limiting reliability model, Bayesian analysis, Squared error loss function, Linex loss function.

1 Introduction

Studies in [2, 4, 5, 6, 7, 8] dealt with the classical estimation of stress-strength reliability, $P[Y < X]$. This reliability model, known as the stress-strength model, is concerned with reliability of a component's strength X subject to a stress Y. Assuming prior variations in the parameters of the strength and stress variables, the study in [3] analyzed the problem of estimating $P[Y < X]$ in the Bayesian framework. The study considered the multi-component stress-strength system which functions if at least s out of k identical components simultaneously operate. Exact and approximate asymptotic posterior distributions for the reliability are derived when stress and strength variables are assumed to be independently exponentially distributed.

However, in practice, especially in defense, the designers and reliability engineers wish to attach high probability to the event that the system remains operable at its minimum strength encountering maximum stress at that time epoch subject to the following practical considerations:

1. The strength of the system deteriorates with time.

2. The system experiences increasing trend in stress due to advancements in science and technology.

Thus, to meet the objective of designers and reliability engineers, it makes sense to conceptualize the limiting stress-strength reliability, $P[V < U]$, where $U = \min(X_1, X_2, \ldots, X_m)$ and $V = \max(Y_1, Y_2, \ldots, Y_n)$. Here, it is assumed that a system's strength and stress are measurable during its operation at m and n random time epochs respectively. Accordingly, the limiting stress-strength reliability defines the probability that the system will function successfully till its minimum strength exceeds the maximum stress encountered during its operation.

Further, variations in material and manufacturing processes give random character to the strength variable that is assumed to be continuous in nature, whereas advancements in technology and environmental variations introduce randomness to stress variable, which is considered to be discrete. As an example, we consider bunkers or tanks that are bombed by aeroplanes with the assumption that a single direct hit after some failures is enough to damage the target. Here, bunkers or tanks have continuous strength X which is subjected to discrete stress Y.

In view of the above discussion, the present study gives a Bayesian treatment to the limiting stress-strength reliability, P[V<U]. Initially, the squared error loss function (SELF), symmetric in nature, has been used in the analysis.However, recognizing the fact that over-estimation of the reliability function is more serious than its under-estimation, therefore, the LINEX (linear-exponential) loss function, which is asymmetric in nature, is used to counter this situation. The theoretical results are highlighted with examples. For analyzing the data, relevant computer programs in C++ are developed and available with the authors.

2 Notation

X	Strength variable.
Y	Stress variable.
<x>	Least positive integer value greater than x.
$(X_1, X_2, ..., X_m)$	Measures on X.
$(Y_1, Y_2, ..., Y_n)$	Measures on Y.
U	$\min[X_1, X_2, ..., X_m]$.
<u>	Least positive integer value greater than u.
V	$\max[Y_1, Y_2, ..., Y_n]$.
$\underline{u} = (u_1, u_2, ..., u_{n_1})$	Simulated sample information on U.
$\underline{v} = (v_1, v_2, ..., v_{n_2})$	Simulated sample information on V.
p.d.f.	Probability density function.
p.m.f	Probability mass function.
SELF	Squared error loss function.
LINEX	Linear exponential.
LLF	Linex loss function.
R_1	P [Y<X].
R_2	P [V<U].
$R*_{1S}$ $(R*_{1L})$	Bayes estimates of R_1 when SELF (LLF) is used.
$R*_{2S}$ $(R*_{2L})$	Bayes estimates of R_2 when SELF (LLF) is used.

3 Statistical Background

It is assumed that-
(a) The r.v X follows exponential distribution with p.d.f

$$f_1(x, \theta) = \frac{1}{\theta} exp(-\frac{x}{\theta}) \qquad ; x>0 \quad , \theta>0. \tag{3.1}$$

(b) The r.v Y, the number of failures before the first success, has the geometric distribution with p.m.f.

$$P[Y = y] = q^y p \qquad ; 0 < p < 1 \qquad , y = 0, 1, 2, ... \tag{3.2}$$

Here, p is the probability of success. Here, success means hitting the target, and q=(1-p), probability of failure, i.e., not hitting the target.

(c) In view of (a), the p.d.f. of the r.v. U will be

$$f_2(u,\theta) = \frac{m}{\theta} exp(-\frac{mu}{\theta}) \qquad ;u>0 \quad ,\theta>0. \tag{3.3}$$

$$E(u) = \frac{\theta}{m} \qquad Var(u) = (\frac{\theta}{m})^2.$$

(d) In view of (b), the p.m.f. of the r.v. v will be

$$P[V=v] = \prod_{i=1}^{n} P[Y_i \leq v] - \prod_{i=1}^{n} P[Y_i \leq v-1]$$

$$= (1-q^{v+1})^n - (1-q^v)^n \qquad ;o<q<1 \quad ;p=1-q \quad ,v=0,1,2,.... \tag{3.4}$$

(e) θ is a r.v. with inverted gamma prior having p.d.f.

$$h_1(\theta) = \frac{a^b e^{-\frac{a}{\theta}}}{\theta^{b+1}\Gamma(b)} \tag{3.5}$$

$$E(\theta) = \frac{a}{(b-1)}; b>1 \quad ,V(\theta) = \frac{a^2}{(b-1)^2(b-2)}; b>2.$$

(f) p is a r.v. with Beta one prior having p.d.f.

$$h_2(p) = \frac{p^{c-1}(1-p)^{d-1}}{B(c,d)} \qquad ;(c,d)>0 \quad ,0<p<1. \tag{3.6}$$

$$E(p) = \frac{c}{(c+d)} \quad ,V(p) = \frac{cd}{(c+d)^2(c+d+1)}.$$

(g) $L(\theta,\hat{\theta})$ be the loss incurred in estimating θ by the statistic $\hat{\theta}$. Then, a function defined as

$$L(\theta,\hat{\theta}) = k(\theta-\hat{\theta})^2 \qquad ;(k \quad being \quad a \quad constant). \tag{3.7}$$

is called a quadratic loss function. For k = 1, (3.7) reduces to a SELF. The SELF is a symmetric function of $\hat{\theta}$ and θ and gives equal weightage to both over-estimation and under-estimation of the parameter. But symmetric loss functions are not found suitable in the estimation of reliability characteristics.

For accounting such over and under-estimation, asymmetric loss functions have been proposed in the literature. In the process, the study in [1] considered linear asymmetric loss functions. Further, the studies in [9, 10] introduced an asymmetric convex loss function called as LINEX (linear exponential) that has the following form:

$$L(\Delta) = be^{q_1\Delta} - c\Delta - b \qquad ;q_1 \neq 0, c \neq 0, b>0. \tag{3.8}$$

Here,$\Delta = (\theta - \hat{\theta})$denote the scalar estimation error in using $\hat{\theta}$ to estimate θ. It is seen that L (0) = 0. Also, for a minimum to exist at $\Delta = 0$, we must have $(q_1 b) = c$ and thus (3.8) can be re-expressed as

$$L(\Delta) = b[e^{q_1\Delta} - q_1\Delta - 1] \qquad ; q_1 \neq 0, b > 0. \tag{3.9}$$

There are the two parameters, q_1 and b, involved in (3.9) with b serving to scale the loss function and q_1 serving to determine its shape.

4 R_1 and R_2 in parametric terms

In view of the respective distributions of X and Y in (3.1) and (3.2), one gets-

$$R_1 = \int_o^\infty \left[\sum_{y=0}^{<x>-1} P[Y = y] \right] f_1(x,\theta)dx$$

$$= \sum_{j=1}^{\infty} \int_{j-1}^{j} \left[\sum_{y=0}^{j-1} P[Y = y] \right] f_1(x,\theta)dx$$

$$= \sum_{j=1}^{\infty} \sum_{y=0}^{j-1} \left[\int_{j-1}^{j} \frac{1}{\theta} exp(-\frac{x}{\theta})dx \right] q^y p$$

$$= p\left(exp(\frac{1}{\theta}) - 1 \right) \sum_{j=1}^{\infty} \sum_{y=0}^{j-1} q^y exp(-\frac{j}{\theta}).$$

$$= \left[\frac{p}{1 - q \quad exp(-\frac{1}{\theta})} \right] \tag{4.1}$$

Similarly, on using the respective distributions of U and V in (3.3) and (3.4), one gets-

$$R_2 = \int_o^\infty \left[\sum_{v=0}^{<u>-1} P[V = v] \right] f_2(u,\theta)du$$

$$= \left[exp(\frac{m}{\theta}) - 1 \right] \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{q^k}{\left[exp(\frac{m}{\theta}) - q^k \right]}. \tag{4.2}$$

For n=m=1, R_2 in (4.2) equals in (4.1).

5 Bayes estimates of R_1 and R_2 when SELF is used

In view of (3.3) and (3.5) and for the simulated sample information $\underline{u}$, the posterior p.d.f. of θ becomes

$$\Pi_1(\theta \mid \underline{u}) = \frac{L(\underline{u} \mid \theta)h_1(\theta)}{\int_0^\infty L(\underline{u} \mid \theta)h_1(\theta)d\theta}$$

$$= \frac{exp\left[-\frac{(mn_1\bar{u}+a)}{\theta}\right](mn_1\bar{u}+a)^{n_1+b}}{\theta^{n_1+b+1}\Gamma(n_1+b)} \quad ;\theta > 0, \bar{u} > 0. \tag{5.1}$$

Here, $L(\bar{u} \mid \theta)$ is the liklihood function and $\bar{u}$ is the sample mean.
Similarly, in view of (3.4), (3.6) and simulated sample information $\underline{v}$, the posterior p.d.f. of p will be

$$\Pi_2(p \mid \underline{v}) = \frac{L(\underline{v} \mid p)h_2(p)}{\int_0^1 L(\underline{v} \mid p)h_2(p)dp}$$

$$= \frac{\prod_{s=1}^{n_2}\left[(1-q^{v_s+1})^n - (1-q^{v_s})^n\right]p^{c-1}(1-p)^{d-1}}{\int_0^1 \prod_{s=1}^{n_2}\left[(1-q^{v_s+1})^n - (1-q^{v_s})^n\right]p^{c-1}(1-p)^{d-1}dp} \quad ;0 < p < 1, q = 1-p. \tag{5.2}$$

On using the respective posterior distributions of θ and p in (5.1) and (5.2), the Bayes estimates of R_2 becomes

$$R_{2S}^* = \int_0^\infty \int_0^1 R_2 \quad \pi_1(\theta \mid \underline{u})\pi_2(p \mid \underline{v})d\theta dp$$

$$= \int_0^\infty \int_0^1 \left[exp(\frac{m}{\theta}) - 1\right]\sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{q^k}{\left[exp(\frac{m}{\theta}) - q^k\right]} exp\left[-\left(\frac{mn_1\bar{u}+a}{\theta}\right)\right].$$

$$\cdot\frac{(mn_1\bar{u}+a)^{n_1+b}}{\theta^{n_1+b+1}\Gamma(n_1+b)}\frac{\prod_{s=1}^{n_2}\left[(1-q^{v_s+1})^n - (1-q^{v_s})^n\right]p^{c-1}(1-p)^{d-1}dpd\theta}{\int_0^1 \prod_{s=1}^{n_2}\left[(1-q^{v_s+1})^n - (1-q^{v_s})^n\right]p^{c-1}(1-p)^{d-1}dp}. \tag{5.3}$$

In particular, when n=m=1, then (5.3) reduces to

$$R_{1S}^* = \int_0^\infty \int_0^1 R_1 \quad \pi_1(\theta \mid \underline{x})\pi_2(p \mid \underline{y})d\theta dp$$

$$= \sum_{r=0}^\infty \frac{B(n_2+c+1, n_2\bar{y}+d+r)}{B(n_2+c, n_2\bar{y}+d)\left(1+\frac{r}{n_1\bar{x}+a}\right)^{n_1+b}}. \tag{5.4}$$

6 Bayes estimates of R_1 and R_2 when LLF is used

On using the respective posterior distributions of θ and p in (5.1) and (5.2), the Bayes estimate of R_2 becomes

$$R^*_{2L} = \frac{-1}{q_1} In \quad E\Big[exp(-q_1 R_2)\Big]$$

$$= \frac{-1}{q_1} In\Bigg[\int_0^{\infty}\int_0^1 \sum_{r=0}^{\infty} \frac{(-q_1)^r}{r!}\Bigg[\Bigg(exp(\frac{m}{\theta}) - 1\Bigg)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{q^k}{\Bigg(exp(\frac{m}{\theta}) - q^k\Bigg)}\Bigg]^r.$$

$$.exp\Bigg[-\Bigg(\frac{mn_1\bar{u} + a}{\theta}\Bigg)\Bigg]\frac{(mn_1\bar{u}+a)^{n_1+b}}{\theta^{n_1+b+1}\Gamma(n_1+b)}\frac{\prod_{s=1}^{n_2}\Bigg[(1-q^{v_s+1})^n - (1-q^{v_s})^n\Bigg]p^{c-1}(1-p)^{d-1}dpd\theta}{\int_0^1\prod_{s=1}^{n_2}\Bigg[(1-q^{v_s+1})^n - (1-q^{v_s})^n\Bigg]p^{c-1}(1-p)^{d-1}dp}. \tag{6.1}$$

For n=m=1, equation (6.1) reduces to

$$R^*_{1L} = \frac{-1}{q_1} In\Bigg[\int_0^{\infty}\int_0^1 \sum_{r=0}^{\infty}\frac{(-q_1)^r}{r!}\Bigg[\frac{p}{1-q \quad exp(-\frac{1}{\theta})}\Bigg]^r \frac{exp\Big(-\frac{n_1\bar{x}+a}{\theta}\Big)}{B(n_1+c, n_2\bar{y}+d)}.$$

$$.\frac{(n_1\bar{x}+a)^{n_1+b}p^{n_2+c+1}(1-p)^{n_2\bar{y}+d-1}dpd\theta}{\theta^{n_1+b+1}\Gamma(n_1+b)}. \tag{6.2}$$

7 Discussion

While developing systems, equipments and establishments in defense, the designer has to meet the objective of attaching high probability to the event that the system remains operable at minimum strength encountering with the maximum stress. For meeting the above-mentioned objective, the concept of a limiting stress-strength reliability model has been introduced. Initially, in section 4.0, R_1 and R_2 have been defined in parametric terms. Later, on using the past sample information, Bayes estimates of R_1 and R_2 with SELF and LLF are respectively obtained in sections 5.0 and 6.0.

8 An Example

For analyzing the respective values of R_1 and R_2 in respect of m, n and involved parameters, we assume $\theta = 24$ and q = 0.4 and 0.8. Using the expressions in (4.1) and (4.2), the values for R_1 and R_2 for varying m, n and q = (1-p) are summarized in Table-1. For developing Bayes estimates, i.e, R^*_{2S}, R^*_{1S}, R^*_{2L} and R^*_{1L} , as given in sections 5.0 and 6.0, we

Table 1: R_1 and R_2 for varying m, n and q

	q=0.4				q=0.8		
	n				n		
m	1	2	10	m	1	2	10
1	0.974	0.955	0.895	1	0.860	0.790	0.610
2	0.949	0.914	0.804	2	0.758	0.640	0.383
3	0.927	0.877	0.724	3	0.680	0.533	0.250

(i) Generated samples of sizes n_1 and n_2 from the distributions in (3.3) and (3.4).
(ii) Developed relevant computer programs in C++ and available with the authors.
The respective estimated values, i.e., R^*_{2S}, R^*_{1S}, R^*_{2L} and R^*_{1L}, for varying and fixed sets of parameters have been summarized in Table-2.

9 Analysis

Here, it should be recognized that the intensity of strength and stress mainly depends on the parameters involved in their respective distributions. For example, θ for the distribution in (3.1) stands for the mean strength. Similarly, p for the distribution in (3.2) stands for the probability of hitting the target. Obviously, the designer has to monitor the trends in the estimated values of the remoduled reliability with variations in these parameters. Similarly, for meeting the objective in the Bayesian set-up, the trends in estimated reliabilities can be monitored in respect of the variations in the means of the respective priors of θ and p, i.e. E(θ) and E(p). Studying the trends from tables, we conclude that:
(i) R_1, R_2 and their Bayes estimates under both the loss functions tend to decrease uniformly as q, the probability of not hitting the target, increases.
(ii) R_1, R_2 and their Bayes estimates under both the loss functions tend to increase uniformly as E(θ), i.e. mean strength increases.
(iii) R^*_{1S}, R^*_{2S} tend to increase uniformly as the means of the respective priors in (3.5) and (3.6) increase. The same trends are observed in R^*_{1L} and R^*_{2L}.
(iv) $R^*_{1L}(R^*_{2L})$ tends to R^*_{1S} (R^*_{2S}) as q1$\rightarrow$0
In this way, by analyzing the above trends, the designer can make a trade off between R_1, R_2,R^*_{1S}, R^*_{2S}, R^*_{1L} and R^*_{2L} and q, n, m, E(θ) , E(p), etc. to meet his system reliability goals. Note that, some of the tables showing these trends could not be included due to space restriction. However, the same are available on request.

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Table 2: Bayes estimates, R^*_{1S}, R^*_{2S}, R^*_{1L} and R^*_{2L}, for fixed n_1=5[10], n_2=4, E(θ)=24, v[1]=0, v[2]=2, v[3]=2, v[4]=1 (when n=2), v[1]=2, v[2]=5, v[3]=2, v[4]=1 (when n=10), v[1]=1, v[2]=1, v[3]=3, v[4]=0 (when n=1) for [q=0.4] and varying m, n, $E(p)$ and $\bar{u}$

		m=1, $\bar{u}$=23.467 24.605]			m=3, $\bar{u}$=8.076 [7.7924]		
E(p)	n	R^*_{1S} and R^*_{2S}	R^*_{1L} and R^*_{2L} at q_1=0.1	R^*_{1L} and R^*_{2L} at q_1=0.00001	R^*_{2S}	R^*_{2L} at q_1=0.1	R^*_{2L} at q_1=0.00001
0.2	1	0.88	0.921	0.88	0.74	0.769	0.74
		[0.895]	[0.938]	[0.895]	[0.75]	[0.78]	[0.75]
	2	0.877	0.918	0.877	0.715	0.742	0.715
		[0.89]	[0.932]	[0.89]	[0.72]	[0.747]	[0.72]
	10	0.845	0.883	0.845	0.633	0.654	0.633
		[0.86]	[0.899]	[0.86]	[0.64]	[0.661]	[0.64]
0.6	1	0.945	0.993	0.945	0.87	0.91	0.87
		[0.951]	[0.999]	[0.951]	[0.882]	[0.923]	[0.882]
	2	0.94	0.987	0.94	0.844	0.882	0.844
		[0.944]	[0.992]	[0.944]	[0.848]	[0.886]	[0.848]
	10	0.886	0.928	0.886	0.717	0.744	0.717
		[0.90]	[0.943]	[0.90]	[0.723]	[0.751]	[0.723]
0.9	1	0.956	1.00	0.956	0.89	0.932	0.89
		[0.96]	[1.00]	[0.96]	[0.894]	[0.937]	[0.894]
	2	0.948	0.996	0.948	0.87	0.91	0.87
		[0.954]	[1.00]	[0.954]	[0.873]	[0.914]	[0.873]
	10	0.89	0.932	0.89	0.733	0.761	0.733
		[0.901]	[0.944]	[0.901]	[0.764]	[0.795]	[0.764]