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EDGEWORTH SIZE CORRECTED W, LR AND LM TESTS IN THE FORMATION OF THE PRELIMINARY TEST ESTIMATOR

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summary

This paper defines the preliminary test estimator (PTE) of the univariate normal mean under the original as well as the Edgeworth size corrected Wald (W), likelihood ratio (LR) and Lagrange multiplier (LM) tests. The bias and mean squared error (MSE) functions of the estimators are derived. The conflicts among the biases and the MSEs of the PTEs under the three original and the size corrected tests have been obtained. It is found that instead of the original W, LR and LM tests, the use of the Edgeworth size corrected W, LR and LM tests in the formation of the PTEs reduces the conflict among the biases and MSEs of the estimators remarkably.

Keywords and phrases: Preliminary test estimator; Wald, likelihood ratio and Lagrange multiplier tests; bias; mean squared error; and conflict.

AMS Classification: Primary 62H12, Secondary 62G05.

1 Introduction

The normal distribution is the most widely used statistical model for many real life phenomena, and hence the estimation of its unknown mean is very important. The most commonly used estimator of the mean of a normal distribution is the maximum likelihood estimator. This estimator is exclusively based on the sample information and posses some good statistical properties. Often credible non-sample prior information about the value of the mean is available from previous experience or expert knowledge. Inclusion of such information in the formation of the estimator is likely to improve some of the statistical properties of the estimator.

The non-sample prior information can be expressed in the form of a null hypothesis. The value of the parameter specified by such a null hypothesis is known as the restricted estimator (RE). Unlike the MLE. the RE is biased. But under the null hypothesis it performs better than the MLE. As we are not sure about the accuracy of the prior information, the performance of RE is in doubt. Under the alternative hypothesis, the MLE performs better when we use the sample information only, and the RE performs better when the null hypothesis is true. Therefore, an estimator that combines the sample and non-sample prior information is likely to perform better than both MLE and RE. Preliminary test estimator, originally proposed by Bancroft [1], combines both sample and non-sample prior information in its definition. By definition the PTE involves an appropriate test statistic to remove the uncertainty in the null hypothesis.

Traditionally, the likelihood ratio (LR) test statistics is used in the definition of the PTE. But there are alternative tests available in the literature to test the same hypothesis. Tests such as Wald (W) (Wald [14]), and Lagrange multiplier (LM) (Rao [10]) tests can be used to define PTE. Billah [3], and in a series of papers Billah and Saleh ([4], [5] and [6]) introduce the W, LR and LM tests in the formation of the SPTE and PTE of the regression vector of linear multiple regression model.

Berndt and Savin [2] show that the systematic inequality relation $W \leq LR \leq LM$ exists among the three test statistics. The exact sampling distributions of these test statistics are complicated. In practice, the critical regions are determined based on their common asymptotic distribution. Under the null hypothesis, the three test statistics are asymptotically distributed as chi-square. Evan and Savin [8] show that in small sample the three tests lead to conflicting conclusions. The previous studies show that the performance of the SPTEs under these conflicting tests are also conflicting. As a result, the practitioners of the preliminary test estimator might be in a dilemma as to which of the three tests should be used to achieve the optimal statistical properties.

To reduce the conflict among the tests Evan and Savin [8] proposes some modifications for the three test statistics. They use the degrees of freedom correction (Gallant [9]) for the W and LM tests, and Edgeworth correction (see Rothenberg [11]) for the LR test. The study of Evans and Savin [8] shows that the amount of conflict among the modified tests is smaller than that among the original tests. But the amount of conflict among the modified tests is still substantial.

The conflict among the three tests with or without modification is due to the fact that the tests do not have the correct significance level. Based on the Edgeworth expansions of the exact distributions of the test statistics, and under the null hypothesis Evans and Savin [8] consider some correction factors for the chi-square critical values which make the significance levels of each test correct to certain order. For details of the corrections readers may see Rothenberg [11]. The tests with the adjusted critical values are known as the size corrected tests. The amount of conflict among the size corrected tests is negligible (cf. Evans and Savin [8]. In this study we introduce the size corrected W, LR and LM tests in the formation of the PTEs of the univariate normal mean. Our study reveals that the amount of conflict among the properties of the PTEs under the size corrected tests is very small, practically negligible. Therefore, the practitioners of the PTE can use any of the three size corrected tests, rather than the original and modified tests, in the formation of the PTE. This will guard the users against the risk of conflicting decisions that may arise while using the original or modified tests in the definition of the PTE.

The organization of the paper is as follows. The tests and the PTEs are presented in section 2. The bias functions of the PTEs are studied in section 3. Section 4 deals with the MSE of the PTEs. Finally, some concluding remarks are provided in section 5. A brief derivation of the bias and MSE functions is given in the appendix.

2 The Tests and the Estimators

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from a univariate normal population with unknown mean μ and variance σ^2. The above sample observations can be expressed in the following form

$$\boldsymbol{X} = \mu \mathbf{1} + \boldsymbol{e} \tag{2.1}$$

where $\boldsymbol{X}$ is an $n \times 1$ vector of sample observations, $\mathbf{1} = (1, ..., 1)'$ is a vector of n-tuple of 1's, μ is the unknown mean, and $\boldsymbol{e} = (e_1, ..., e_n)'$ is a vector of independent error components distributed as $N_n(\mathbf{0}, \sigma^2 I_n)$. Here I_n is an identity matrix of order n. Based on the sample information the maximum likelihood estimator of μ is defined as

$$\tilde{\mu}_n = (\mathbf{1}_n' \mathbf{1}_n)^{-1} \mathbf{1}_n' \boldsymbol{X}_n = \bar{X} \tag{2.2}$$

where $\bar{X}$ is the sample mean. The sampling distribution of $\tilde{\mu}$ is normal with mean μ and variance σ^2/n. Hence, the MLE of μ is unbiased, and it's mean squared error is the same as it's variance. The MLE of σ^2 is given by

$$\tilde{\sigma}^2 = \frac{1}{n}(\boldsymbol{X}_n - \tilde{\mu}_n \mathbf{1}_n)'(\boldsymbol{X}_n - \tilde{\mu}_n \mathbf{1}_n) \tag{2.3}$$

which is biased, and the unbiased estimator of σ^2 can be obtained by replacing the denominator of (2.3) with $(n-1)$. The distribution of the unbiased estimator of σ^2 is scaled chi-square with $(n-1)$ d.f.

Following Fishers recipe, the non-sample prior information is expressed in the form of the null hypothesis as follows:

$$H_0 : \mu = \mu_0 \tag{2.4}$$

where μ_0 is the suspected value of the unknown mean μ.

To remove the uncertainty in the prior information an appropriate statistical test is performed to test the null hypothesis in (2.4) against the alternative

$$H_1 : \mu \neq \mu_0. \tag{2.5}$$

The above hypotheses in (2.4) - (2.5) can be tested by using the three original W, LR and LM test statistics

$$W = \frac{n}{n-1} F \tag{2.6}$$

$$LR = n \ \ln \left(1 + \frac{nF}{n-1} \right) \tag{2.7}$$

$$LM = \frac{nF}{n-1+F} \tag{2.8}$$

respectively. Under the alternative hypothesis the distribution of F is non-central F with 1 and $(n-1)$ d.f., and non-centrality parameter $\Delta^2 = n(\mu-\mu_0)^2/\sigma^2$. The asymptotic distribution of the above three test statistics are the same, and is chi-square with 1 d.f. (cf. Evans and Savin [8]). In this study, we define the PTE of μ based on the asymptotic distribution of the three test statistics.

The PTE of μ under the test statistics in (2.6) - (2.8) are defined as

$$\hat{\mu}_{\text{W}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(W < \chi^2_\alpha) \tag{2.9}$$

$$\hat{\mu}_{\text{LR}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(LR < \chi^2_\alpha) \tag{2.10}$$

$$\hat{\mu}_{\text{LM}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(LM < \chi^2_\alpha) \tag{2.11}$$

where χ^2_α is the critical value of the test at significance level α, $I(\cdot)$ is the indicator function which assumes value unity when the inequality in the argument holds, and 0 otherwise.

The modified form of the W, LR and LM test statistics are

$$W_* = F \tag{2.12}$$

$$LR_* = (n-3/2)\ \ln\left(1+\frac{F}{n-1}\right) \tag{2.13}$$

$$LM_* = \frac{nF}{n-1+F} \tag{2.14}$$

respectively. The asymptotic distribution of the modified test statistics are also chi-square with 1 d.f. Using Edgeworth expansions of the exact distributions of the test statistics under the null hypothesis the adjusted critical values of the test statistics in (2.12) - (2.14) are obtained as

$$\mathcal{C}_{W_*} = \chi^2_\alpha \left\{1+\frac{\chi^2_\alpha+1}{2(n-1)}\right\} \tag{2.15}$$

$$\mathcal{C}_{LR_*} = \chi^2_\alpha \tag{2.16}$$

$$\mathcal{C}_{LM_*} = \chi^2_\alpha \left\{1+\frac{\chi^2_\alpha-3}{2(n-1)}\right\}. \tag{2.17}$$

The tests with the modified test statistics and the adjusted critical values are known as the size corrected tests. Evans and Savin [8] show that the amount of conflict among the size corrected tests is negligible. Based on the size corrected tests the PTEs of μ are defined as

$$\hat{\mu}_{\text{W*}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(W_* < \mathcal{C}_{W_*}) \tag{2.18}$$

$$\hat{\mu}_{\text{LR*}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(LR_* < \mathcal{C}_{LR_*}) \tag{2.19}$$

$$\hat{\mu}_{\text{LM*}}^{\text{PT}} = \tilde{\mu} - (\tilde{\mu} - \hat{\mu})\ I(LM_* < \mathcal{C}_{LM_*}) \tag{2.20}$$

where $\mathcal{C}_{(\cdot)}$ is the critical value of the respective size corrected test, and $I(\cdot)$ is the indicator function as defined earlier.

3 The Bias Function

In this section we state the bias functions of the PTEs of μ under the original as well as the Edgeworth size corrected W, LR and LM tests. The conflict among the biases of the PTEs are obtained and presented graphically. Some analytical comments on the biases are provided at the end of this section.

The bias functions of the estimators based on the original tests are given by

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{W}};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{W_1};\Delta\right) \tag{3.1}$$

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{LR}};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{LR_1};\Delta\right) \tag{3.2}$$

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{LM}};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{LM_1};\Delta\right) \tag{3.3}$$

respectively where $C_{W_i} = \frac{(n-1)\chi^2_\alpha}{(2i+1)n}$, $C_{LR_i} = \frac{(n-1)(e^{\chi^2_\alpha/n}-1)}{(2i+1)n}$ and $C_{LM_i} = \frac{(n-1)\chi^2_\alpha}{(2i+1)(n-\chi^2_\alpha)}$, $i = 1,2$; $G_{p,q}(a;b)$ is the distribution function of the non-central F variable with (p,q) d.f., non-centrality parameter b, and evaluated at a. See appendix for the derivation of the above bias functions.

The bias functions of the estimators of μ based on the three size corrected tests in (2.18) - (2.20) are given by

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{W}^*};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{W^*_1};\Delta\right) \tag{3.4}$$

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{LR}^*};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{LR^*_1};\Delta\right) \tag{3.5}$$

$$\mathrm{B}\big(\hat{\mu}^{\mathrm{PT}}_{\mathrm{LM}^*};\mu\big) = -\delta^{*}G_{3,n-1}\left(C_{LM^*_1};\Delta\right) \tag{3.6}$$

respectively where $C_{W^*_i} = \frac{\chi^2_\alpha(\chi^2_\alpha+2n-1)}{2(2i+1)(n-1)}$, $C_{LR^*_i} = \frac{(n-1)(e^{\chi^2_\alpha/(n-3/2)}-1)}{(2i+1)}$ and $C_{LM^*_i} = \frac{\chi^2_\alpha(\chi^2_\alpha+2n-5)(n-1)}{(2i+1)\{2n(n-1)-\chi^2_\alpha(\chi^2_\alpha+2n-5)\}}$, $i = 1,2$; $G_{p,q}(a;b)$ is the distribution function of the non-central F variable with (p,q) d.f., non-centrality parameter b and evaluated at a.

From Figure 1 it is observed that for $\Delta = 0$ the biases of the PTEs under both original and size corrected tests are zero regardless of the sample size. As Δ deviates from zero, the biases grow larger up to some moderate value of Δ, and then the biases decrease and merge to zero at some large values of Δ. Interestingly, in both original and size corrected cases, the same inequality relation exists among the biases of the PTEs as their tests counterparts. Under both original and size corrected tests, as the sample size increases, the conflict among the biases of the PTEs decreases. This is due to the fact that the conflict among the tests is more inevitable for small sample, and hence among the PTEs.

The original W, LR and LM tests do not have the correct significance level. The Edgeworth size correction ensures that the tests have significance level correct to order $1/n$. As a result, the use of the size corrected tests in the formation of the PTEs considerably reduces the conflict among the biases of the three PTEs as compare to the use of the three original tests.

4 The Mean Squared Error Function

This section states the MSE functions of the PTEs of μ under both original and Edgeworth size corrected W, LR and LM tests. The amount of conflict among the MSEs of the three PTEs are obtained and presented in both graphical and numerical forms. Some analytical comments on the MSEs are also provided in this section.

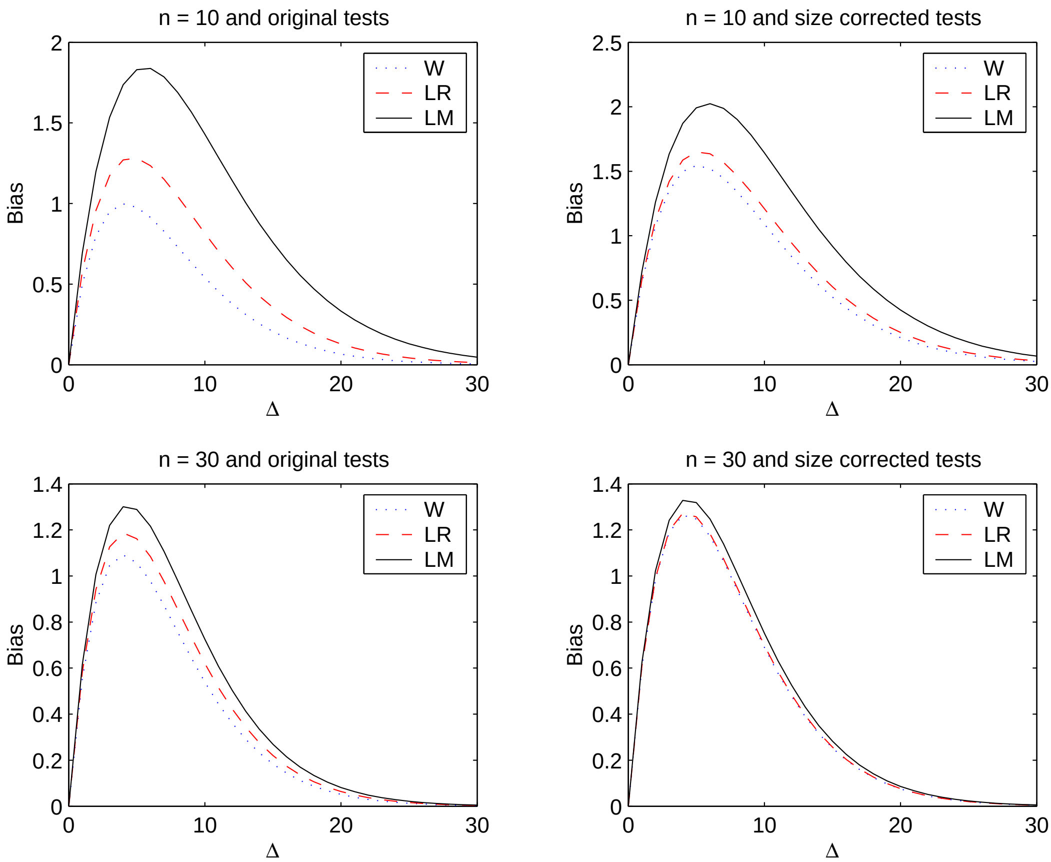

Figure 1: The bias of the PTE of μ for $\alpha = 0.05$ and varying n.

The MSE functions of the PTEs of μ under the original W, LR and LM tests are obtained as

$$\text{MSE}(\hat{\mu}_{\text{W}}^{\text{PT}}; \mu) = \frac{\sigma^2}{n}\left[1 - G_{3,n-1}\left(C_{W_1}; \Delta\right) + \Delta^2 \left\{2G_{3,n-1}\left(C_{W_1}; \Delta\right) - G_{5,n-1}\left(C_{W_2}; \Delta\right)\right\}\right] \quad (4.1)$$

$$\text{MSE}(\hat{\mu}_{\text{LR}}^{\text{PT}}; \mu) = \frac{\sigma^2}{n}\left[1 - G_{3,n-1}\left(C_{LR_1}; \Delta\right) + \Delta^2 \left\{2G_{3,n-1}\left(C_{LR_1}; \Delta\right) - G_{5,n-1}\left(C_{LR_2}; \Delta\right)\right\}\right] \quad (4.2)$$

$$\text{MSE}(\hat{\mu}_{\text{LM}}^{\text{PT}}; \mu) = \frac{\sigma^2}{n}\left[1 - G_{3,n-1}\left(C_{LM_1}; \Delta\right) + \Delta^2 \left\{2G_{3,n-1}\left(C_{LM_1}; \Delta\right) - G_{5,n-1}\left(C_{LM_2}; \Delta\right)\right\}\right] \quad (4.3)$$

where $C_{W_i} = \frac{(n-1)\chi_\alpha^2}{(2i+1)n}$, $C_{LR_i} = \frac{(n-1)(e^{\chi_\alpha^2/n}-1)}{(2i+1)n}$ and $C_{LM_i} = \frac{(n-1)\chi_\alpha^2}{(2i+1)(n-\chi_\alpha^2)}$, $i = 1, 2$; $G_{p,q}(a; b)$ is the distribution function of the non-central F variable with (p, q) d.f., non-centrality parameter b and evaluated at a. See appendix for the derivation.

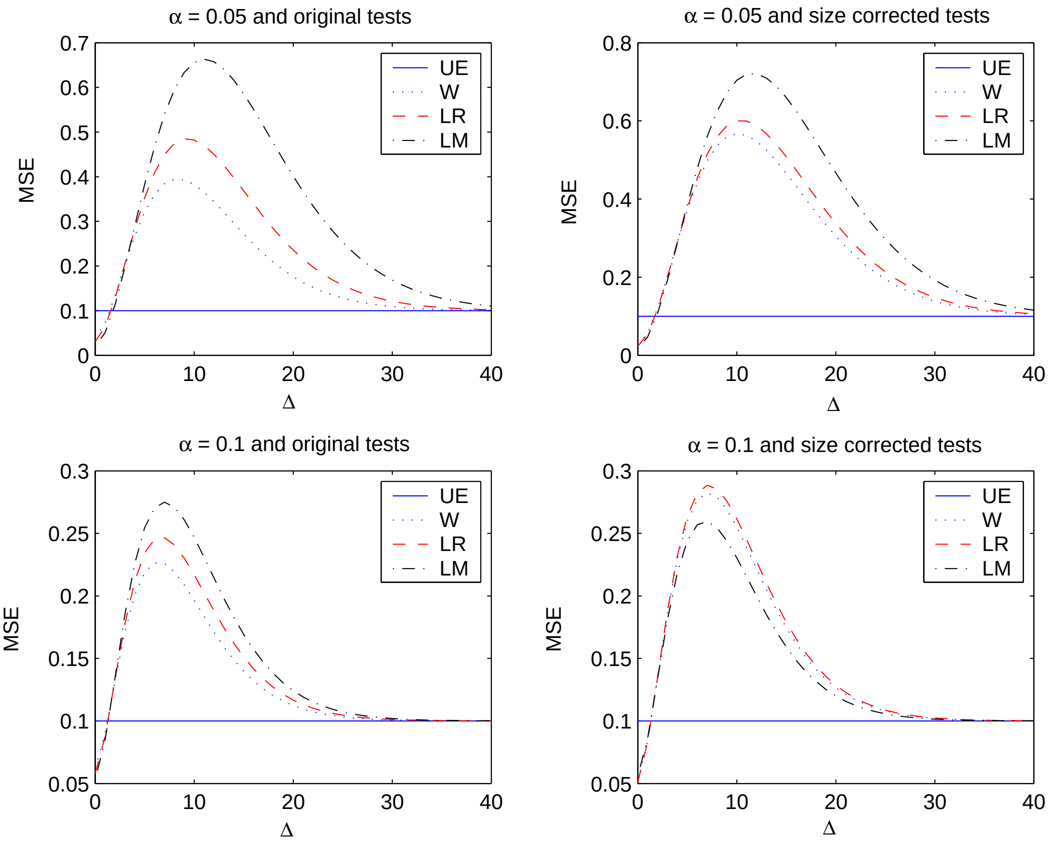

Figure 2: The MSE of the PTE of μ for $n = 10$, $\sigma = 1$ and varying α.

The MSE functions of the PTEs of μ under the size corrected W, LR and LM tests are obtained as

$$\begin{aligned}\text{MSE}(\hat{\mu}^{\text{PT}}_{\text{W}^*};\mu) = \frac{\sigma^2}{n}\Big[1 - G_{3,n-1}\left(C_{W_1^*};\Delta\right) + \Delta^2\left\{2G_{3,n-1}\left(C_{W_1^*};\Delta\right)\right. \\ \left.-G_{5,n-1}\left(C_{W_2^*};\Delta\right)\right\}\Big]\end{aligned} \tag{4.4}$$

$$\begin{aligned}\text{MSE}(\hat{\mu}^{\text{PT}}_{\text{LR}^*};\mu) = \frac{\sigma^2}{n}\Big[1 - G_{3,n-1}\left(C_{LR_1^*};\Delta\right) + \Delta^2\left\{2G_{3,n-1}\left(C_{LR_1^*};\Delta\right)\right. \\ \left.-G_{5,n-1}\left(C_{LR_2^*};\Delta\right)\right\}\Big]\end{aligned} \tag{4.5}$$

$$\begin{aligned}\text{MSE}(\hat{\mu}^{\text{PT}}_{\text{LM}^*};\mu) = \frac{\sigma^2}{n}\Big[1 - G_{3,n-1}\left(C_{LM_1^*};\Delta\right) + \Delta^2\left\{2G_{3,n-1}\left(C_{LM_1^*};\Delta\right)\right. \\ \left.-G_{5,n-1}\left(C_{LM_2^*};\Delta\right)\right\}\Big]\end{aligned} \tag{4.6}$$

respectively where $C_{W_i^*} = \frac{\chi^2_\alpha(\chi^2_\alpha+2n-1)}{2(2i+1)(n-1)}$, $C_{LR_i^*} = \frac{(n-1)(e^{\chi^2_\alpha/(n-3/2)}-1)}{(2i+1)}$ and $C_{LM_i^*} = \frac{\chi^2_\alpha(\chi^2_\alpha+2n-5)(n-1)}{(2i+1)\{2n(n-1)-\chi^2_\alpha(\chi^2_\alpha+2n-5)\}}$, $i = 1, 2$; $G_{p,q}(a;b)$ is the distribution function of the non-central F variable with (p, q) d.f., non-centrality parameter b and evaluated at a.

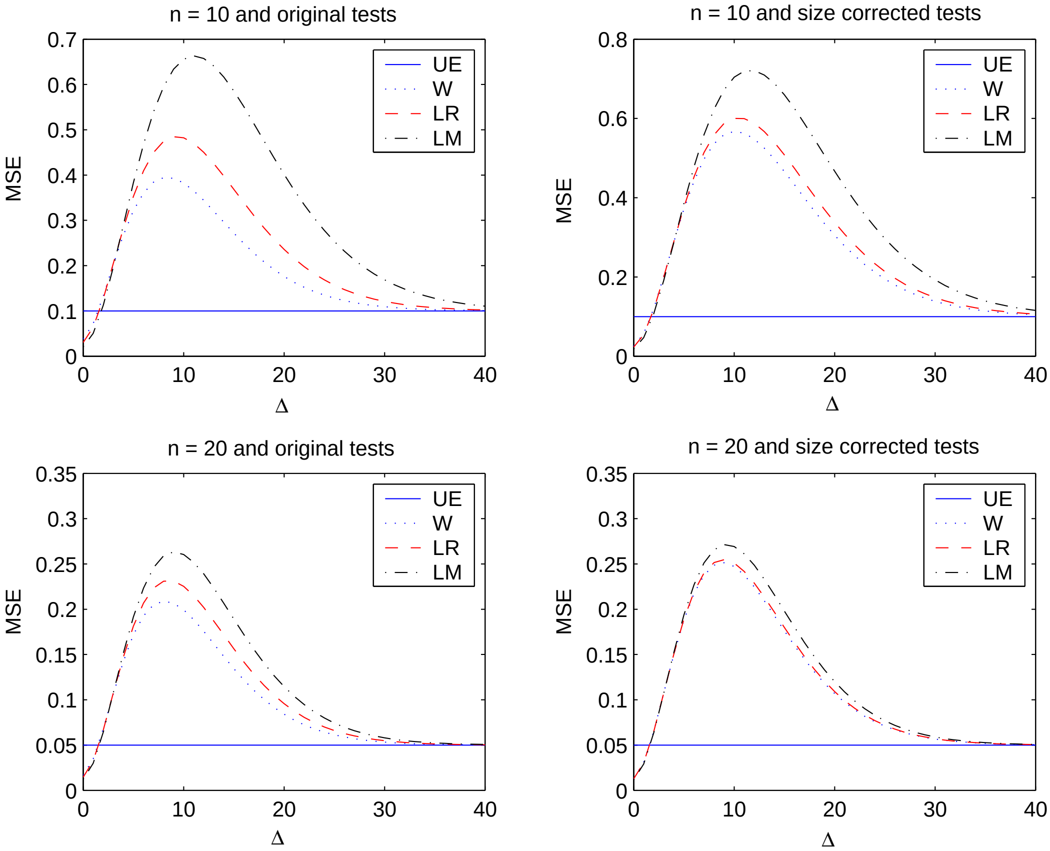

Figure 3: The MSE of the PTE of μ for $\alpha = 0.05$ $\sigma = 1$ and varying n.

From Tables 1 and 2, and Figures 2 and 3 it is observed that for any fixed n, α, and σ, from zero to some small value of Δ (say Δ_*) the MSE of the PTE under the Edgeworth size corrected tests is less than that of the PTE under the original tests. For $\Delta > \Delta_*$, the MSE of the PTE under the original tests is less than that of the PTE under the size corrected tests. However, the conflict among the MSEs of the PTEs under the size corrected tests is less than that under the original tests. Moreover, the amount of conflict among the PTEs under the size corrected tests is negligible. For example, when $n = 10$, $\alpha = 0.10$, $\sigma = 1$ and $\Delta = 3$, the amount of conflict among the MSEs under the original tests is 0.2012. On the other hand, for same n, α, σ and Δ, the amount of conflict among the PTEs under the size corrected tests is only 0.001. The performance of the PTE depends on the level of significance of the test on the null hypothesis. For an optimum choice of the level of significance readers may see Chiou and Saleh [7]. As the sample size increases, the MSE of the PTEs decreases regardless of the test used in the formation of the estimator. Similarly, the amount of conflict among the PTEs also decreases with the increase in sample size. For example, when $\alpha = 0.05$, $\sigma = 1$, $n = 10$, $\Delta = 2$, the original tests used in the formation of the PTEs, the amount of conflict among their MSEs is 0.0201. On the other hand, for the same values of α, σ,

Δ and tests used, the amount of conflict among the MSEs of the PTEs is decreased by 0.0186 for doubling the sample size.

5 Conclusion

We have defined the PTE of μ under the three original as well as the size corrected tests. The conflicting bias and mean squared error functions of the estimators are derived. The findings of this study show that the conflict among the PTE under the W, LR and LM tests (with or without modification) is because of the fact that the tests do not have the correct significance level. For the Edgeworth size corrected tests the conflict is reduced significantly. The bias and MSE based conflicts reflect the same behaviours. Our findings also illustrate the danger of sizeable conflict in using the original W, LR and LM tests in the formation of the PTE. We observe that when Edgeworth correction is applied on the three tests it not only reduces the conflict of test decisions for testing hypotheses on μ, but also reduces the conflict among the PTEs, based on such corrections, with respect to both bias and MSE criteria. Although use of such size corrected tests does not remove the conflict of the PTEs altogether, it reduces the conflict significantly. The size of the conflict is so negligible that it can be ignored for almost all practical problems. Therefore, use of any of the size corrected W, LR and LM tests in the formation of the PTE would lead to almost identical bias and MSE for the PTEs. Thus the users may use the liberty to use any of the three size corrected tests without risking of increased conflict in the performance of the PTEs.

References

[1] Bancroft, T.A. (1944). On biases in estimation due to the use of the preliminary test of significance. *Ann. Math. Statist.*, **15**, 190-204.

[2] Berndt, E. and Savin, N.E. (1977). Conflict Among Criteria for Testing Hypotheses in the Multivariate Linear Regression Model, *Econometrica*, **45**, 1263-1278.

[3] Billah, M.B. (1997). Small Sample Studies of the W, LR, and LM Based SPTEs of Regression Parameters, *Journal of Statistical Research*, **30**(2), 93-108.

[4] Billah, M.B. and Saleh, A.K.Md.E. (1998). Conflict Between Pretest Estimators Induced by Three Large Sample Tests Under a Regression Model With Students t-error, *The Statistician*, **47**(4), 593-606.

[5] Billah, M.B. and Saleh, A.K.Md.E. (2000a). Performance of Three Large Sample Tests in the Actual Formation of the Pretest Estimators for Regression Model With t-error, *Journal of Applied Statistical Science*, **9**(3), 237-251.

[6] Billah, M.B. and Saleh, A.K.Md.E. (2000b). Performance of the PTE Based on the Conflicting W, LR and LM Tests in Regression Model, *Advances on Methodological and Applied Aspects of Probability and Statistics*, N. Balakrishnan (ed.), Gordon and Breach Publishing Company, 263-276.

[7] Chiou, P. and Saleh, A.K.Md.E. (2002). Preliminary test confidence sets for the mean of a multivariate normal distribution, *Journal of Propagation in Probability and Statistics*, **2**, 177-189.

[8] Evans, G.B.A. and Savin, N.E. (1982). Conflict Among the Criteria Revisited: the W, LR and LM tests, *Econometrica*, **50**(3), 737-748.

[9] Gallant, A.R. (1975). Seemingly Unrelated Nonlinear Regression, *Journal of Econometrics*, **3**, 35-50.

[10] Rao, C.R. (1947). Large Sample Tests of Statistical Hypotheses Concerning Several Parameters with Applications to Problems of Estimation, *Proceedings of the Cambridge Philosophical Society*, **44**, 50-57.

[11] Rothenberg, T.J. (1977). Edgeworth Expansions for Multivariate Test Statistics, *Working Paper in Economic Theory and Econometrics IP-255*, Center for Research in Management Science, Institute of Business and Economic Research, University of California, Berkeley.

[12] Saleh, A.K.Md.E. (2006). *Theory of Preliminary Test and Stein-Type Estimation with Applications*, Wiley, New York.

[13] Savin, N.E. (1976). Conflict Among Testing Procedures in a Linear Regression Model With Autoregressive Disturbances, *Econometrica*, **44**, 1303-1315.

[14] Wald, A. (1943). Tests of Hypotheses Concerning Several Parameters When the Number of Observations is Large, *Transaction of the American Math Society*, **54**, 426-482.