Journal of Statistical Research *ISSN 0256 - 422 X*
2007, Vol. 41, No. 7, pp. 17–35
Bangladesh

SHRINKAGE TESTIMATION FOR THE VARIANCE OF A NORMAL DISTRIBUTION UNDER ASYMMETRIC LOSS FUNCTION

Gyan Prakash
Department of Statistics, Harishchandra P. G. College, Varanasi, India.
Email: ggyanji@yahoo.com

B. N. Pandey
Department of Statistics, Banaras Hindu University, Varanasi, India.

summary

In present paper, some shrinkage testimators for the variance of a Normal distribution have been suggested under the invariant form of the LINEX loss function assuming initiate estimate of σ^2 is available in the form of a point estimate σ_0^2. The comparisons of the proposed testimators have been made with the improved estimator under the squared error loss and the LINEX loss criteria.

Keywords and phrases: Mean square error, LINEX loss function, Shrinkage factor and Shrinkage testimator.

AMS Classification: 62F03, 62F04, 62F10.

1 Introduction

Let $x_1, x_2, \ldots, x_n$ be the random samples of size n taken from a Normal population with mean μ and variance σ^2 respectively. Some initial guessed value for σ^2 is known in the form of a point estimate σ_0^2 either from past data or from some reliable sources. The unbiased estimate of σ^2 is

$$s^2 = \frac{1}{v} \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{with} \quad Var\left(s^2\right) = \frac{2\sigma^4}{v} \ ; \ v = (n-1).$$

The shrinkage estimator (Thomson 1968) for the variance σ^2 is given by

$$Y = \sigma_0^2 + k\left(s^2 - \sigma_0^2\right) \ ; \ 0 \le k \le 1, \tag{1.1}$$

where k is the shrinkage factor specified by the experimenter according to his belief in the guess value. A value of k near to zero implies strong belief in σ_0^2 and near to one implies a strong belief in the sample values. It was found that the estimator Y performs better if σ_0^2 is in the vicinity of the true value σ^2 i.e., $\left(\frac{\sigma_0^2}{\sigma^2} \cong 1\right)$. Pandey and Singh (1977) have studied the shrinkage estimator (1.1) under the mean square error criterion.

The squared error loss is inappropriate because it gives equal importance to over estimation and under estimation. In some situations, we may come across the condition where negative error may be more serious than a given positive error. Varian (1975) and Zellner (1986) proposed a very useful asymmetric loss function known as the LINEX loss function. Basu and Ebrahimi (1991) considered its invariant form and is defined for any estimate $\hat{\theta}$ of the parameter θ as

$$L(\Delta^*) = \left\{ e^{a\Delta^*} - a\Delta^* - 1 \right\} \ ; \ a \neq 0 \ and \ \Delta^* = \left\{ \frac{\hat{\theta}}{\theta} - 1 \right\}. \tag{1.2}$$

The LINEX loss function is convex and its shape is determined by the value of a (the sign of a reflects the direction of asymmetry, $a > 0$ ($a < 0$) if over estimate is more (less) serious than the under estimation) and its magnitude reflects the degree of asymmetry. The LINEX loss criterion changes to mean square error criterion if $|a| \to 0$.

Pandey et al. (1988) considered some shrinkage testimators for the estimate of variance under mean square error criterion. Parsian and Farsipour (1999), Singh et al. (2002), Misra and Meulen (2003), Pandey et al. (2004), Ahmadi et al. (2005), Xiao et al. (2005), Prakash and Singh (2006), Singh et al. (2007) and others have considered the estimation procedures under the LINEX loss function in different contexts.

A class of estimators for the variance σ^2 is given by

$$P = c\ s^2\ , \tag{1.3}$$

where $c > 0$ be the scalar.

The risk for the estimator P under the invariant form of the LINEX loss is

$$R(P) = \left\{ e^{-a} \left(1 - \frac{2ac}{v} \right)^{-\frac{v}{2}} - ac + a - 1 \right\}.$$

The value of c, which minimizes the risk, $R(P)$ is

$$\frac{v}{2a} \left(1 - \exp\left(-\frac{2a}{v+2} \right) \right) = c_1 \ (say). \tag{1.4}$$

Therefore, the improved estimator for the variance is $P_1 = c_1\ s^2$. The risk under the invariant form of the LINEX loss and the squared error loss are obtained respectively as

$$R(P_1) = \left\{ \left(1 + \frac{v}{2} \right) \left(\exp\left(-\frac{2a}{v+2} \right) - 1 \right) + a \right\}$$

and

$$MSE(P_1) = \sigma^4 \left\{ \frac{v+2}{v}\ c_1^2 - 2c_1 + 1 \right\}.$$

In the present paper, we study the performance of the shrinkage testimators under the invariant form of the LINEX loss and the squared error loss functions with the different choices of the shrinkage factor. The recommendations regarding the choices of testimators have also been made.

2 The Shrinkage Estimators and their Properties

The LINEX loss function of the estimator Y (1.1) is given by

$$L(Y) = \frac{a^2}{2}\left\{(k\beta_1+\beta_2)^2 + \frac{a}{3}(k\beta_1+\beta_2)^3 + \cdots\right\},$$

where $\beta_1 = \frac{s^2}{\sigma^2} - \delta$, $\beta_2 = \delta - 1$, $\delta = \frac{\sigma_0^2}{\sigma^2}$.

The risk of the estimator Y may by obtained by ignoring the higher order terms (more than one) of a , is given by

$$R(Y) = \frac{a^2}{2}\left\{E(k\beta_1+\beta_2)^2 + \frac{a}{3}E(k\beta_1+\beta_2)^3\right\}.$$

The value of k , which minimizes the risk $R(Y)$ is

$$k_{min} = \frac{-\acute{B} + \sqrt{\left(\acute{B}^2 - \acute{A}\acute{C}\right)}}{\acute{A}} \;\; ; \; \acute{B}^2 > \acute{A}\acute{C},$$

where $\acute{A} = a\left\{\frac{(v+2)(v+4)}{v^2} - \delta^3 - 3\delta\left(\frac{v+2}{v}\right) + 3\delta^2\right\}$, $\acute{B} = \left\{\frac{v+2}{v} + \delta^2 - 2\delta\right\}\{1 + a(\delta-1)\}$ and $\acute{C} = (\delta-1)^2(a(1-\delta)-2)$.

The value of k_{min} depends on the unknown parameter σ^2. An estimate $\hat{k}_{min}$ of k_{min} may be obtained by replacing σ^2 by its unbiased estimate s^2. Hence the improved shrinkage estimator is

$$Y_1 = \sigma_0^2 + \hat{k}_{min}\left(s^2 - \sigma_0^2\right). \tag{2.1}$$

The expressions of risk for improved shrinkage estimator Y_1 under the invariant form of the LINEX loss and the squared error loss, are thus obtained as

$$R(Y_1) = \frac{1}{\Gamma\left(\frac{v}{2}\right)}\left\{e^{a(\delta-1)}\int_0^\infty e^{f_1 - w}\, w^{\frac{v}{2}-1}dw - \int_0^\infty f_1\, e^{-w}\, w^{\frac{v}{2}-1}dw - (a(\delta-1)+1)\Gamma\left(\frac{v}{2}\right)\right\}$$

and

$$MSE(Y_1) = \frac{\sigma^4}{\Gamma\left(\frac{v}{2}\right)}\left\{\int_0^\infty f_2^2\, e^{-w} w^{\frac{v}{2}-1}dw - 2(\delta-1)\int_0^\infty f_2\, e^{-w}\, w^{\frac{v}{2}-1}dw - (1-\delta)^2\Gamma\left(\frac{v}{2}\right)\right\},$$

where $f_1 = a\hat{k}_{min}\left(\frac{2w}{v} - \delta\right)$ and $f_2 = \hat{k}_{min}\left(\frac{2w}{v} - \delta\right)$.

The relative efficiency of the shrinkage testimator Y_1 with respect to the improved estimator P_1 is defined as

$$RE_{(L)}(Y_1, P_1) = \frac{R(P_1)}{R(Y_1)} \quad \text{and} \quad RE_{(M)}(Y_1, P_1) = \frac{MSE(P_1)}{MSE(Y_1)}.$$

The suffix L and M stands respectively for the LINEX loss and squared error loss. Both the expressions of relative efficiency are the function of δ, a and n. To guard against large risk, the smaller values of $a(\leq 1)$ are considered. Hence, for the selected values of $a = 0.25, 0.50, 1.00$; $\delta = 0.20(0.20)1.60$; $n = 05, 10, 15, 25$, the relative efficiencies have been calculated and presented them in Tables 1 - 2 and in order to save space graphs are presented here only for $a = 1.00$ in Figs $-1-2$.

From these tables, we can say that the estimator Y_1 performs well with respect to P_1 in the range $0.40 \leq \delta \leq 1.20$ for a fixed a and n and attains maximum efficiency at $\delta = 1$. The effective intervals of estimator Y_1 decrease as n increases. The relative efficiency increases as a increases for all the considered values of δ under square error loss whereas; it first increases for small $\delta < 1.00$ and then decreases as a increases when the loss is LINEX. As for as gain in efficiency is considered, the LINEX loss looks good for the small δ and a .

As $c_1 \in [0,1]$ from (1.4), it may be a choice for the shrinkage factor. Based on this shrinkage factor, the shrinkage estimator is defined as

$$Y_2 = \sigma_0^2 + c_1\left(s^2 - \sigma_0^2\right). \tag{2.2}$$

The risk for Y_2 under the invariant form of the LINEX loss and the squared error loss are given as

$$R\left(Y_2\right) = \exp\left(a\left[\delta\left(1-c_1\right) - \frac{2}{v+2}\right]\right) - a\left(\delta-1\right)\left(1-c_1\right) - 1$$

and

$$MSE\left(Y_2\right) = \sigma^4\left\{c_1^2\left(\frac{v+2}{v} + \delta\left(\delta-2\right)\right) + \left(1-\delta\right)^2\left(1-2c_1\right)\right\}.$$

Similarly, the relative efficiency for Y_2 with respect to P_1 is given by

$$RE_{(L)}\left(Y_2, P_1\right) = \frac{R\left(P_1\right)}{R\left(Y_2\right)} \quad \text{and} \quad RE_{(M)}\left(Y_2, P_1\right) = \frac{MSE\left(P_1\right)}{MSE\left(Y_2\right)}.$$

The expression of $RE_{(L)}\left(Y_2, P_1\right)$ involves δ, a and n . For the same set of values as considered earlier, we have calculated the relative efficiency and presented them in Table 3 and graph in Fig - 3 (for $a = 1.00$ only). This table shows that the estimator Y_2 performs well than P_1 for all the considered values of δ, a and n, and the relative efficiency decreases as sample size n increases. The effect of the increase of a on the relative efficiency is similar to Y_1. Now,

$$RE_{(M)}\left(Y_2, P_1\right) > 1 \Rightarrow \delta\left(2-\delta\right)\left(1-c_1\right)^2 > 0. \tag{2.3}$$

It is clear from equation (2.3) that the shrinkage testimator Y_2 always performs better than P_1 when $\delta < 2$ under the squared error loss.

3 Different Shrinkage Testimators and their Properties

The above result suggests that the shrinkage estimators Y_1 and Y_2 have smaller risk than the improved estimator P_1 when the guess value σ_0^2 is in the vicinity of the true value σ^2. Thus, we propose the shrinkage testimator for the testing of the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$ on the basis of a given set of data as follows:

$$T_1 = \begin{cases} \sigma_0^2 + \hat{k}_{min}\left(s^2 - \sigma_0^2\right) & \text{if } H_0 : \sigma^2 = \sigma_0^2 \text{ is accepted} \\ P_1 & \text{otherwise.} \end{cases}$$

For testing $H_0 \ : \ \sigma^2 = \sigma_0^2$ we have a test statistic $\frac{v\, s^2}{\sigma_0^2}$, which follows chi - square distribution with v degrees of freedom. The hypothesis H_0 is accepted when $l_1 \leq \left(\frac{v\, s^2}{\sigma_0^2}\right) \leq l_2$, where l_1 and l_2 being the values of the lower and upper $100\left(\frac{\alpha}{2}\right)\%$ points of the chi - square distribution with v degrees of freedom. Hence,

$$T_1 = \begin{cases} \sigma_0^2 + \hat{k}_{min}\left(s^2 - \sigma_0^2\right) & \text{if} \frac{\sigma_0^2\, l_1}{v} \leq s^2 \leq \frac{\sigma_0^2\, l_2}{v} \\ P_1 & \text{otherwise.} \end{cases}$$

The risk for the testimator T_1 under the invariant form of the LINEX loss is given by

$$\begin{aligned} R(T_1) = \frac{1}{\Gamma\left(\frac{v}{2}\right)} \Bigg\{ & e^{a(\delta-1)} \int_{w_1}^{w_2} e^{f_1 - w}\; w^{\frac{v}{2}-1}\, dw - e^{-a} \int_{w_1}^{w_2} e^{f_0 - w}\; w^{\frac{v}{2}-1}\, dw \\ & + e^{-a} \int_0^{\infty} e^{f_0 - w}\; w^{\frac{v}{2}-1}\, dw - \int_{w_1}^{w_2} f_1\, e^{-w}\; w^{\frac{v}{2}-1}\, dw + \int_{w_1}^{w_2} f_0\, e^{-w}\; w^{\frac{v}{2}-1}\, dw \\ & - a\,\delta \int_{w_1}^{w_2} e^{-w}\; w^{\frac{v}{2}-1}\, dw \; + \; (a - 1 - a\, c_1)\, \Gamma\left(\frac{v}{2}\right) \Bigg\}, \end{aligned} \tag{3.1}$$

where $w_1 = \frac{l_1 \delta}{2}$, $w_2 = \frac{l_2 \delta}{2}$, and $f_0 = \frac{2 a c_1 w}{v}$.

Also, the mean square error for the shrinkage testimator T_1 is obtained as

$$\begin{aligned} MSE(T_1) = \frac{\sigma^4}{\Gamma\left(\frac{v}{2}\right)} \Bigg\{ & \int_{w_1}^{w_2} (f_2 + \delta)^2\; e^{-w}\; w^{\frac{v}{2}-1}\, dw - 2 \int_{w_1}^{w_2} (f_2 + \delta)\; e^{-w}\; w^{\frac{v}{2}-1}\, dw \\ & - \int_{w_1}^{w_2} \acute{f_0}\left(\acute{f_0} - 2\right)\, e^{-w}\; w^{\frac{v}{2}-1}\, dw + \frac{v+2}{v}\, c_1^2 - 2\, c_1 + 1 \Bigg\}, \end{aligned} \tag{3.2}$$

where $\acute{f_0} = \frac{2\, c_1\, w}{v}$.

Waikar et al. (1984) have suggested an idea of taking the shrinkage factor, which is the function of test statistic. Under $H_0 \ : \ \sigma^2 = \sigma_0^2$,

$$\frac{\sigma_0^2\, l_1}{v} \leq s^2 \leq \frac{\sigma_0^2\, l_2}{v} \;\Leftrightarrow\; 0 \leq \frac{v}{l_2 - l_1}\left(\frac{s^2}{\sigma_0^2} - \frac{l_1}{v}\right) = k_1(say) \leq 1.$$

The shrinkage testimator may be defined as

$$T_2 = \begin{cases} \sigma_0^2 + k_1\left(s^2 - \sigma_0^2\right) & \text{if} \frac{\sigma_0^2\, l_1}{v} \leq s^2 \leq \frac{\sigma_0^2\, l_2}{v} \\ P_1 & \text{otherwise.} \end{cases}$$

When $H_0 \ : \ \sigma^2 = \sigma_0^2$ is accepted, $l_1 \leq v \leq l_2 \Rightarrow \frac{l_1}{v} \leq 1$. If one is interested to take smaller values of the shrinkage factor, he can take $\frac{l_1}{v} \cong 1$. Then the shrinkage factor becomes $\frac{v}{l_2 - l_1}\left|\frac{s^2}{\sigma_0^2} - 1\right| = k_2$ (say), it may possible that the value of the shrinkage factor is negative so we make it positive. Adke et al. (1987) and Pandey et al. (1988) considered this type of shrinkage factor. Based on this shrinkage factor the shrinkage testimator is proposed as

$$T_3 = \begin{cases} \sigma_0^2 + k_2\left(s^2 - \sigma_0^2\right) & \text{if} \frac{\sigma_0^2\, l_1}{v} \leq s^2 \leq \frac{\sigma_0^2\, l_2}{v} \\ P_1 & \text{otherwise.} \end{cases}$$

In addition, based on c_1, the proposed shrinkage testimator is

$$T_4 = \begin{cases} \sigma_0^2 + c_1\left(s^2 - \sigma_0^2\right) & \text{if } \frac{\sigma_0^2\, l_1}{v} \le s^2 \le \frac{\sigma_0^2\, l_2}{v} \\ P_1 & \text{otherwise.} \end{cases}$$

The expressions for the risk of the given testimators under the invariant form of the LINEX loss and squared error loss can be easily obtained by making some modifications in (3.1) and (3.2) respectively. The relative efficiencies of T_i ; $1, \ldots, 4$, with respect to the improved estimator P_1 are given as

$$RE_{(L)}\left(T_i, P_1\right) = \frac{R\left(P_1\right)}{R\left(T_i\right)}$$

and

$$RE_{(M)}\left(T_i, P_1\right) = \frac{MSE\left(P_1\right)}{MSE\left(T_i\right)} \ ; i = 1, \ldots, 4.$$

The expressions of relative efficiencies are the functions of δ, a, n and α. For the same set of values as considered in previous section with $\alpha = 0.01$ and 0.05, the relative efficiency have been calculated and presented in Tables $4 - 11$ for $n = 10$ and 25. Also, the graphs are presented in Figs $-4 - 11$ for $n = 10$.

When the risk criterion is LINEX, we can say that the shrinkage testimators $T_i(i = 1, 2, 3)$ perform better than the improved estimator P_1 in the range $0.20 \le \delta \le 1.40$ for fixed n, a and α and the effective interval decreases as sample size n increases; whereas the testimator T_4 is efficient then P_1 for all the considered values of δ, n, a and α.

On the other hand, under the square error risk criterion, testimators $T_i(i = 1, 2)$ are efficient than the improved estimator P_1 in the range $0.40 \le \delta \le 1.20$ and the testimator T_3 is efficient than P_1 in the range $0.40 \le \delta \le 1.40$ for fixed n, a and α and the effective interval decreases as sample size n increases. The performance of T_4 in this risk criterion is the same as that of the LINEX loss.

As the level of significance α increases, the relative efficiency of $T_i(i = 1, 2, 3)$ decreases when δ is near to one and the relative efficiency of T_4 decreases when $\delta \ge 1.00$, in both risk criteria. The relative efficiency increases as a increases for all δ when the risk criterion is square error loss; whereas, when risk criterion is the LINEX loss it increases with the increase of a for small $\delta \le 0.60$. The maximum value of the relative efficiency occurs at $\delta = 1$ for $T_i(i = 1, 2, 3)$ under square error loss and $T_i(i = 1, 3)$ under the LINEX loss. For other testimators T_4 (LINEX loss) and T_2 , T_4 (square error loss) the values of maximum relative efficiency are near at $\delta = 1$.

Under the LINEX loss, in the common effective interval i.e., $0.20 \le \delta \le 1.20$, $n \le 10$ and for all the considered values of level of significance α, $\overset{Max}{i}\ \overset{Min}{\delta}\ \left(RE\left(T_i, P_1\right)\right)$ corresponds to the shrinkage testimator T_2.

Similarly, under the mean square error criterion and in the common effective interval $0.40 \le \delta \le 1.20$ with $n > 05$, $\overset{Max}{i}\ \overset{Min}{\delta}\ \left(RE\left(T_i, P_1\right)\right)$ corresponds to the shrinkage testimator T_4 for small level of significance $\alpha(= 0.01)$ and the shrinkage testimator T_2 for large $\alpha(= 0.05)$.

Acknowledgments

Thanks are due to Dr. D. C. Singh (Head, Department of Statistics, Harishchandra P. G. College, Varanasi, India) for his keen interest. The authors also acknowledge helpful suggestions by the editor of the journal which have considerable improved the earlier version.

References

[1] Adke, S. R., Waikar, V. B., and Schuurmann, F. J. (1987). A two stage shrinkage testimator for the mean of an Exponential distribution. *Communication in Statistics - Theory and Methods,* **16***, 1821 - 1834.*

[2] Ahmadi, J., Doostparast, M., and Parsian, A. (2005). Estimation and prediction in a two–parameter Exponential distribution based on k - record values under LINEX loss function. *Communications in Statistics - Theory and Methods,* **34***, 795 - 805.*

[3] Basu, A. P. and Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. *Journal of Statistical Planning and Inferences,* **29***, 21 - 31.*

[4] Misra, N. and Meulen, E. C. V. D. (2003). On estimating the mean of the selected Normal population under the LINEX loss function. *Metrika, 173 - 184.*

[5] Pandey, B. N., Malik, H. J. and Srivastava, R. (1988). Shrinkage testimator for the variance of a Normal distribution at single and double stages. *Microelectron Reliability,* **28 (6)***, 929 - 944.*

[6] Pandey, B. N. and Singh, J. (1977). Estimation of variance of Normal population using apriori information. *Journal of Indian Statistical Association,* **15***, 141 - 150.*

[7] Pandey, B. N., Srivastava, A. K. and Mishra, G. C. (2004). Invariant version of LINEX loss function and its applications in Exponential type II censored data. *Aligarh Journal of Statistics,* **24***, 1 - 22.*

[8] Parsian, A. and Farsipour, N. S. (1999). Estimation of the mean of the selected population under asymmetric loss function. *Metrika, 89 - 107.*

[9] Prakash, G. and Singh, D. C. (2006). Shrinkage testimators for the inverse dispersion of the inverse Gaussian distribution under the LINEX loss function. *Austrian Journal of Statistics,* **35 (4)***, 463 - 470.*

[10] Singh, D. C., Prakash, G. and Singh, P. (2007). Shrinkage testimators for the shape parameter of Pareto distribution using the LINEX loss function. *Communication in Statistics - Theory and Methods,* **36 (4)***, 741 - 753.*

[11] Singh, U., Gupta, P. K. and Upadhyay, S. K. (2002). Estimation of exponentiated Weibull shape parameters under LINEX loss function. *Communication in statistics - Simulation and Computation,* **31(4)***, 523 - 537.*

[12] Thompson, J. R. (1968). Some shrinkage techniques for estimating the mean. *Journal of the American Statistical Association,* **63***, 113 - 122.*

[13] Varian, H. R (1975). A Bayesian approach to real estate assessment. In studies in Bayesian econometrics and statistics in honor of L.J. Savage, Eds S.E. Feinberge and A, Zellner, *Amsterdam North Holland, 195 - 208.*

[14] Waikar, V. B., Schuurmann, F. J. and Raghunathan, T. E. (1984). On a two stage shrinkage testimator of the mean of a Normal distribution. *Communications in Statistics - Theory and Methods,* **13**, *1901 - 1913.*

[15] Xiao, Y., Takada, Y. and Shi, N. (2005). Minimax confidence bound of the Normal mean under an Asymmetric loss function. *The Annals of Statistical Mathematics,* **57 (1)**, *167 - 182.*

[16] Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss function. *Journal of the American Statistical Association,* **81**, *446 - 451.*

Appendix

Table 1: ► $RE_{(L)}\,(Y_1, P_1)$

n	a	δ							
↓	↓	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
	0.25	1.0963	2.0672	3.2094	4.9242	5.2614	3.7417	2.2267	1.3195
05	0.50	1.1039	2.0972	3.2574	4.9973	5.2573	3.6460	2.1184	1.2301
	1.00	1.1176	2.1489	3.3375	5.0917	5.1877	3.3994	1.8859	1.0522
	0.25	0.9990	1.7415	2.5321	4.1624	5.0574	3.1087	1.4371	0.7595
10	0.50	1.0033	1.7670	2.5769	4.2207	5.0305	3.0229	1.3723	0.7133
	1.00	1.0115	1.8143	2.6602	4.3231	4.9481	2.8409	1.2443	0.6257
	0.25	0.9756	1.5875	2.1319	3.3814	4.4145	2.7613	1.1974	0.5771
15	0.50	0.9771	1.6080	2.1685	3.4261	4.3910	2.6841	1.1458	0.5438
	1.00	0.9798	1.6465	2.2386	3.5068	4.3268	2.5285	1.0465	0.4809
	0.25	0.9603	1.4756	1.7844	2.5670	3.7449	2.3990	0.8710	0.4234
25	0.50	0.9651	1.4889	1.8119	2.6000	3.7212	2.3373	0.8381	0.4025
	1.00	0.9674	1.5144	1.8652	2.6623	3.6650	2.2143	0.7745	0.3629

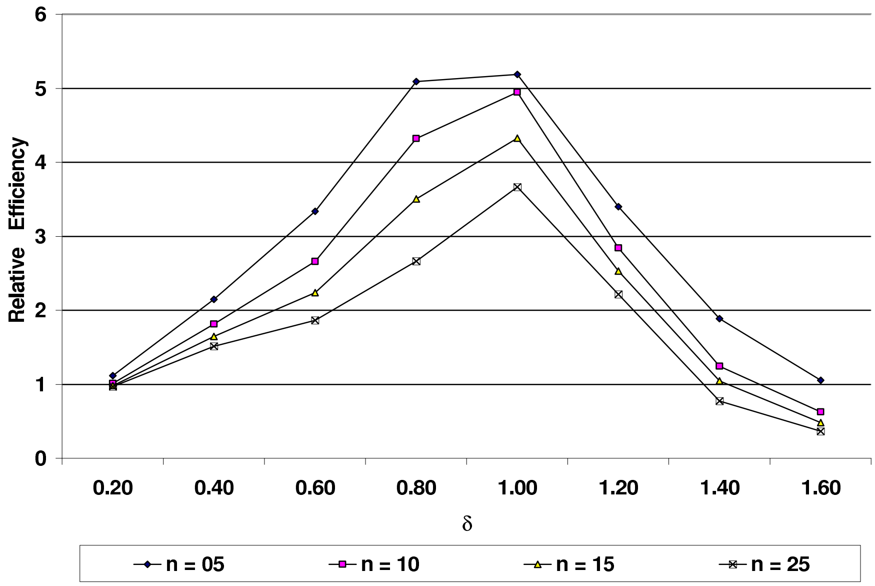

Figure 1: $RE_{(L)}\,(Y_1, P_1)$ :: $a = 1.00$

Table 2: ► $RE_{(M)}\,(Y_1, P_1)$

n	a	δ							
↓	↓	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
05	0.25	1.0776	1.8958	2.9880	4.6600	5.0478	3.6312	2.1889	1.3168
	0.50	1.0714	1.9187	3.0664	4.8541	5.2634	3.7366	2.2318	1.3376
	1.00	1.0704	1.9765	3.2298	5.2057	5.7612	3.8995	2.3039	1.3782
10	0.25	0.9801	1.5920	2.3533	3.9670	4.9208	3.0225	1.4022	0.7491
	0.50	0.9678	1.6033	2.4157	4.1614	5.1748	3.1087	1.4225	0.7562
	1.00	0.9497	1.6318	2.5489	4.5823	5.7298	3.3011	1.4749	0.7781
15	0.25	0.9589	1.4499	1.9728	3.2026	4.2713	2.6654	1.1591	0.5658
	0.50	0.9447	1.4548	2.0144	3.3355	4.4661	2.7181	1.1678	0.5690
	1.00	0.9201	1.4679	2.1032	3.6232	4.8998	2.8425	1.1940	0.5799
25	0.25	0.9553	1.3488	1.6445	2.4058	3.5927	2.2993	0.8366	0.4106
	0.50	0.9411	1.3482	1.6687	2.4774	3.7178	2.3316	0.8401	0.4113
	1.00	0.9134	1.3480	1.7198	2.6317	3.9929	2.4082	0.8520	0.4155

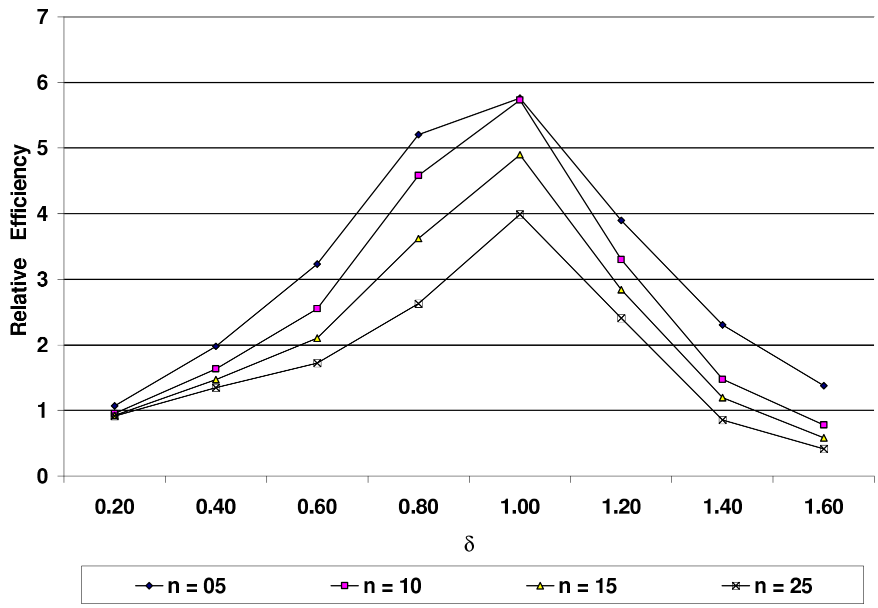

Figure 2: $RE_{(M)}(Y_1, P_1)$ :: $a = 1.00$

Table 3: ► $RE_{(L)}(Y_2, P_1)$

n	a	δ							
↓	↓	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
05	0.25	1.1459	1.7994	1.9689	2.0733	2.0856	2.0010	1.8397	1.6358
	0.50	1.1548	1.8232	1.9960	2.0867	2.0653	1.9344	1.7283	1.4919
	1.00	1.1704	1.8652	2.0388	2.0908	1.9900	1.7653	1.4842	1.4072
10	0.25	1.0763	1.5935	1.6621	1.6997	1.7012	1.6662	1.5987	1.5060
	0.50	1.0822	1.6069	1.6751	1.7045	1.6898	1.6324	1.5397	1.4230
	1.00	1.0932	1.6307	1.6939	1.7104	1.6461	1.5392	1.3969	1.3388
15	0.25	1.0516	1.5265	1.5691	1.5914	1.5917	1.5697	1.5270	1.4668
	0.50	1.0559	1.5358	1.5775	1.5942	1.5837	1.5468	1.4866	1.4082
	1.00	1.0640	1.5520	1.5892	1.5980	1.5529	1.4819	1.3847	1.2714
25	0.25	1.0314	1.4735	1.4976	1.5099	1.5097	1.4970	1.4724	1.4370
	0.50	1.0341	1.4792	1.5026	1.5113	1.5047	1.4831	1.4476	1.4000
	1.00	1.0393	1.4891	1.5092	1.5180	1.4854	1.4425	1.3826	1.2396

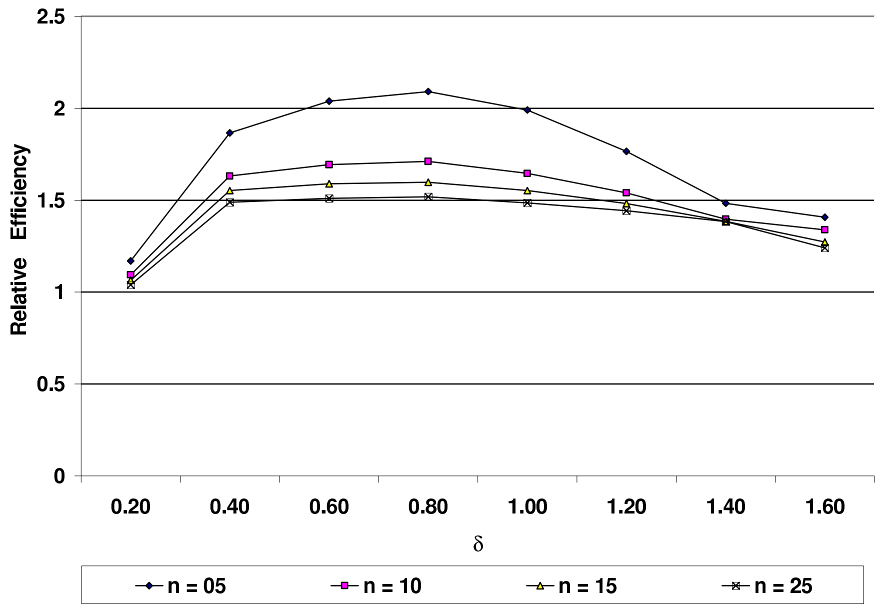

Figure 3: $RE_{(L)}\,(Y_2, P_1)$ :: $a = 1.00$

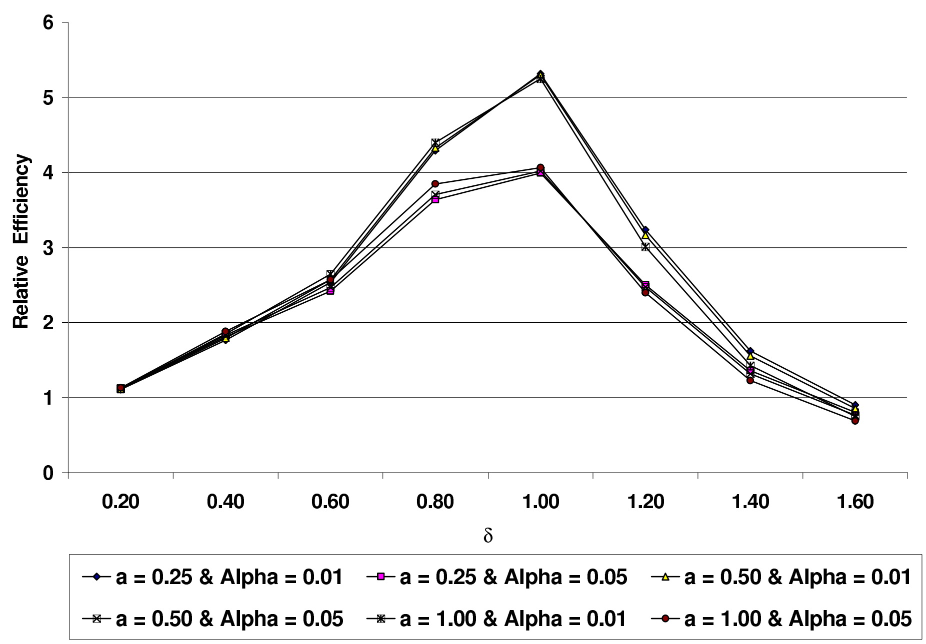

Figure 4: $RE_{(L)}\,(T_1, P_1)$ :: $n = 10$

Table 04										
$RE_{(L)}(T_1, P_1)$			δ							
n	a	α	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
10	0.25	0.01	1.1086	1.7670	2.5338	4.2924	5.3141	3.2371	1.6235	0.9034
		0.05	1.1240	1.8081	2.4215	3.6400	3.9953	2.5049	1.3593	0.8052
	0.50	0.01	1.1137	1.7946	2.5700	4.3293	5.3037	3.1674	1.5586	0.8521
		0.05	1.1266	1.8331	2.4701	3.7037	4.0195	2.4726	1.3164	0.7657
	1.00	0.01	1.1226	1.8500	2.6485	4.4009	5.2497	3.0122	1.4285	0.7537
		0.05	1.1309	1.8795	2.5716	3.8455	4.0648	2.3989	1.2283	0.6880
25	0.25	0.01	1.1212	1.6280	1.8470	2.7948	3.9895	2.3953	0.9813	0.4654
		0.05	1.1216	1.6618	1.8447	2.5614	3.1236	1.9430	0.9457	0.5972
	0.50	0.01	1.1213	1.6376	1.8705	2.8228	3.9768	2.3522	0.9805	0.4445
		0.05	1.1217	1.6678	1.8701	2.5985	3.1324	1.9200	0.9307	0.5734
	1.00	0.01	1.1215	1.6560	1.9390	2.8785	3.9405	2.2618	0.8749	0.4039
		0.05	1.1217	1.6786	1.9207	2.6763	3.1463	1.8708	0.8107	0.5271

Table 05										
$RE_{(M)}(T_1, P_1)$			δ							
n	a	α	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
10	0.25	0.01	0.9877	1.4870	2.1687	3.7491	4.6750	2.8252	1.4224	0.8005
		0.05	1.0023	1.5219	2.0701	3.1599	3.4535	2.1482	1.1745	0.7057
	0.50	0.01	0.9924	1.5215	2.2536	3.9578	4.9289	2.8996	1.4388	0.8052
		0.05	1.0052	1.5548	2.1604	3.3412	3.6010	2.1847	1.1819	0.7078
	1.00	0.01	1.0018	1.5958	2.4407	4.4242	5.4993	3.0743	1.4861	0.8236
		0.05	1.0110	1.6220	2.3563	3.7496	3.9333	2.2772	1.2091	0.7198
25	0.25	0.01	0.9995	1.3608	1.5540	2.3830	3.4455	2.0534	0.8276	0.6063
		0.05	0.9999	1.3901	1.5518	2.1796	2.6641	1.6452	0.8206	0.5125
	0.50	0.01	0.9996	1.3690	1.5848	2.4574	3.5545	2.0726	0.8389	0.6066
		0.05	1.0000	1.3960	1.5839	2.2527	2.7295	1.6502	0.8378	0.5146
	1.00	0.01	0.9997	1.3867	1.6538	2.6214	3.7950	2.1214	0.8464	0.6096
		0.05	1.0000	1.4083	1.6529	2.4147	2.8750	1.6684	0.8462	0.5147

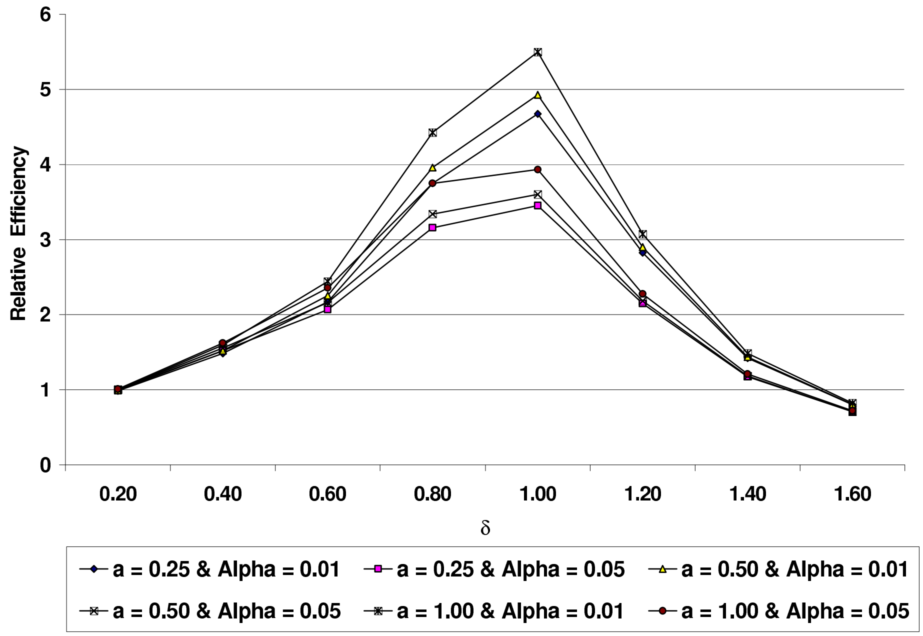

Figure 5: $RE_{(M)}\,(T_1, P_1) \quad :: n = 10$

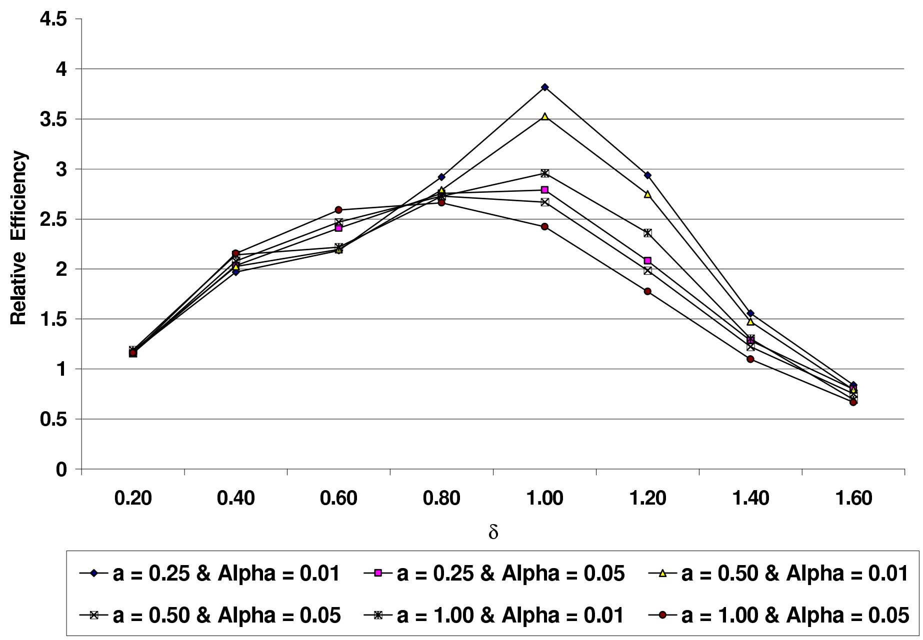

Figure 6: $RE_{(L)}\,(T_2, P_1) \quad :: n = 10$

Table 06										
$RE_{(L)}(T_2, P_1)$			δ							
n	a	α	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
10	0.25	0.01	1.1754	1.9669	2.1845	2.9182	3.8172	2.9358	1.5571	0.8419
		0.05	1.1561	2.0358	2.4096	2.7568	2.7874	2.0812	1.2835	0.7960
	0.50	0.01	1.1820	2.0258	2.1962	2.7890	3.5273	2.7500	1.4750	0.7923
		0.05	1.1585	2.0786	2.4680	2.7272	2.6697	1.9842	1.2242	0.7530
	1.00	0.01	1.1922	2.1420	2.2188	2.7267	2.9560	2.3604	1.3059	0.6953
		0.05	1.1620	2.1561	2.5893	2.6627	2.4215	1.7736	1.0980	0.6665
25	0.25	0.01	1.1225	1.7155	1.7860	2.4963	4.1273	2.2784	0.8630	0.4253
		0.05	1.1220	1.7140	1.9231	2.4881	2.9553	1.7155	0.8214	0.4939
	0.50	0.01	1.1226	1.7360	1.8237	2.4707	3.9716	2.2184	0.8354	0.4055
		0.05	1.1220	1.7232	1.9646	2.4575	2.8963	1.6795	0.7968	0.4715
	1.00	0.01	1.1228	1.7732	1.9015	2.4575	3.6576	2.0930	0.7805	0.3670
		0.05	1.1221	1.7389	2.0476	2.4433	2.7727	1.6031	0.7476	0.4281
Table 07										
$RE_{(M)}(T_2, P_1)$			δ							
10	0.25	0.01	1.0509	1.6594	1.8803	2.5974	3.4616	2.6093	1.3708	0.7468
		0.05	1.0328	1.7206	2.0637	2.3116	2.4597	1.8184	1.1181	0.6995
	0.50	0.01	1.0604	1.7270	1.9507	2.6574	3.5024	2.6251	1.3778	0.7507
		0.05	1.0375	1.7778	2.1675	2.5035	2.8033	1.8280	1.1203	0.7006
	1.00	0.01	1.0782	1.8717	2.1112	2.8041	3.6205	2.6856	1.4073	0.7668
		0.05	1.0459	1.8931	2.3971	2.7135	2.9145	1.8665	1.1372	0.7108
25	0.25	0.01	1.0008	1.4362	1.5019	2.1404	3.5943	1.9517	0.7439	0.4721
		0.05	1.0003	1.4357	1.6190	2.0861	2.5370	1.4527	0.6938	0.4229
	0.50	0.01	1.0009	1.4566	1.5441	2.1768	3.6156	1.9537	0.7445	0.4724
		0.05	1.0003	1.4465	1.6671	2.1452	2.5597	1.4537	0.7014	0.4287
	1.00	0.01	1.0013	1.4981	1.6379	2.2825	3.6793	1.9697	0.7504	0.4753
		0.05	1.0004	1.4679	1.7709	2.2783	2.6192	1.4556	0.7016	0.4289

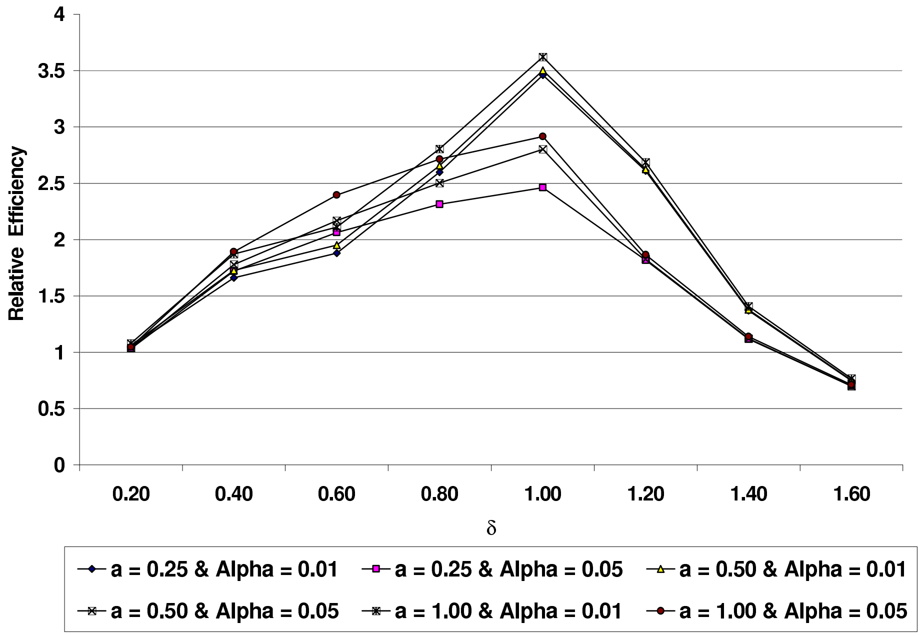

Figure 7: $RE_{(M)}\,(T_2, P_1)$ $::\, n = 10$

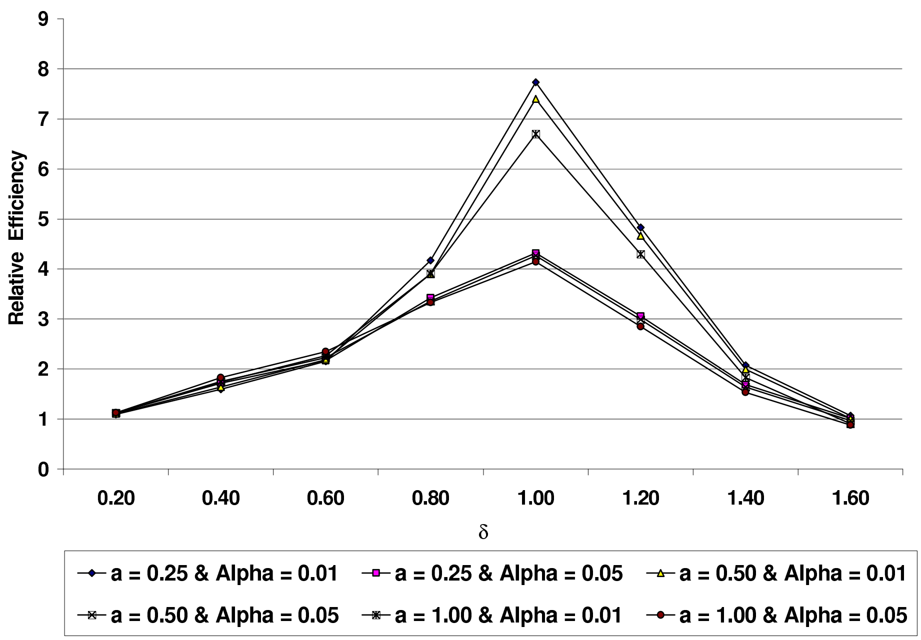

Figure 8: $RE_{(L)}\,(T_3, P_1)$ $::\, n = 10$

Table 08										
$RE_{(L)}(T_3, P_1)$			δ							
n	a	α	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
10	0.25	0.01	1.0952	1.5959	2.1556	4.1678	7.7343	4.8301	2.0737	1.0657
		0.05	1.1184	1.7168	2.1705	3.4202	4.3142	3.0525	1.6909	1.0108
	0.50	0.01	1.1053	1.6443	2.1910	3.9012	7.3983	4.6619	1.9934	1.0116
		0.05	1.1230	1.7566	2.2269	3.3565	4.2618	2.9920	1.6413	0.9678
	1.00	0.01	1.1221	1.7401	2.2683	3.9001	6.6988	4.2936	1.8296	0.9057
		0.05	1.1304	1.8302	2.3454	3.3298	4.1447	2.8516	1.5352	0.8809
25	0.25	0.01	1.1191	1.3980	1.3558	2.9173	8.9746	3.1383	1.0804	0.5625
		0.05	1.1212	1.5582	1.5003	2.5509	4.5076	2.2711	1.0272	0.6377
	0.50	0.01	1.1195	1.4231	1.3860	2.9038	8.8263	3.0919	1.0507	0.5396
		0.05	1.1212	1.5728	1.5338	2.4917	4.4921	2.2481	1.0023	0.6132
	1.00	0.01	1.1201	1.4700	1.4482	2.8979	8.5034	2.9932	1.0012	0.4949
		0.05	1.1214	1.5987	1.6009	2.4509	4.4559	2.1982	1.0005	0.5651
Table 09										
$RE_{(M)}(T_3, P_1)$			δ							
10	0.25	0.01	0.9749	1.3363	1.8352	3.6360	6.8262	4.1764	1.8010	0.9375
		0.05	0.9969	1.4409	1.8471	2.8809	3.7220	2.5993	1.4481	0.8775
	0.50	0.01	0.9836	1.3816	1.9025	3.7448	6.9420	4.2005	1.8095	0.9421
		0.05	1.0015	1.4824	1.9312	3.0091	3.8098	2.6116	1.4490	0.8778
	1.00	0.01	1.0006	1.4801	2.0562	4.0013	7.2435	4.2949	1.8467	0.9617
		0.05	1.0101	1.5682	2.1173	3.3005	4.0208	2.6634	1.4671	0.8885
25	0.25	0.01	0.9975	1.1605	1.1316	2.4605	7.6743	2.6564	1.0766	0.5887
		0.05	0.9995	1.2990	1.2545	2.0936	3.8193	1.9057	1.0638	0.5505
	0.50	0.01	0.9977	1.1745	1.1564	2.5075	7.7435	2.6572	1.0769	0.5888
		0.05	0.9995	1.3082	1.2841	2.1531	3.8647	1.9898	1.0698	0.5548
	1.00	0.01	0.9980	1.2043	1.2121	2.6168	7.9259	2.6748	1.0831	0.5919
		0.05	0.9996	1.3273	1.3489	2.2871	3.9759	1.9943	1.0771	0.5664

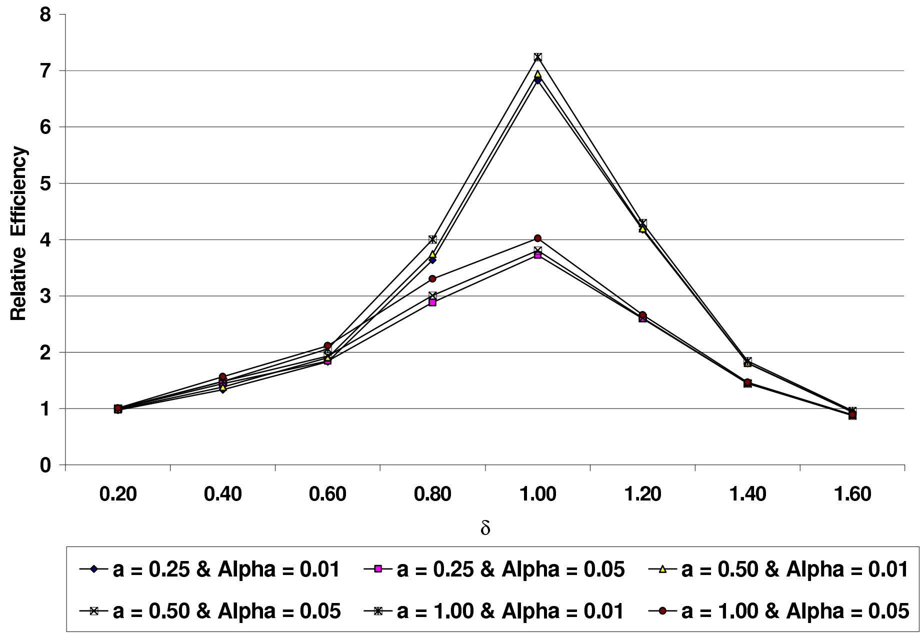

Figure 9: $RE_{(M)}\,(T_3, P_1)$ $::\, n = 10$

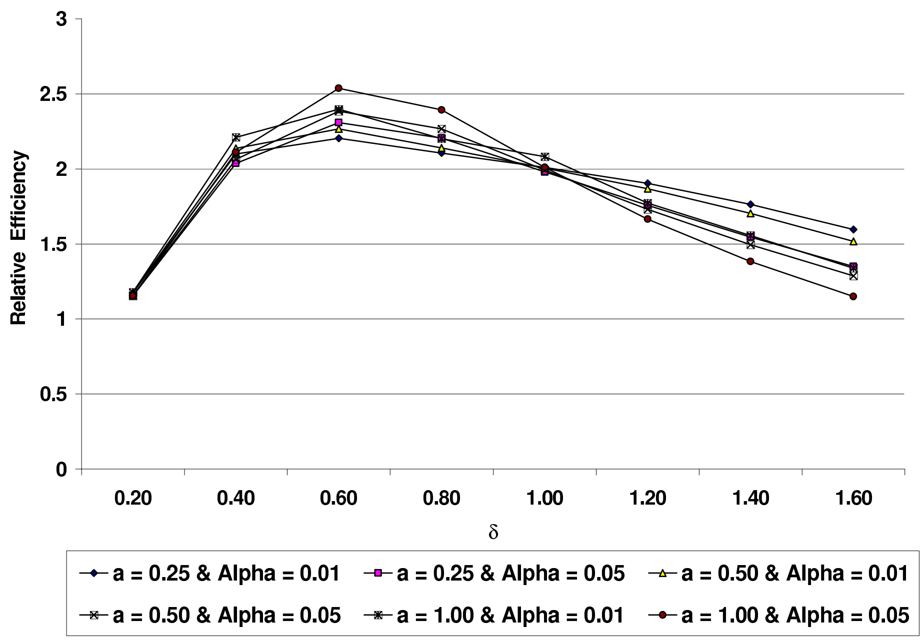

Figure 10: $RE_{(L)}\,(T_4, P_1)$ $::\, n = 10$

Table 10

$RE_{(L)}(T_4, P_1)$			δ							
n	a	α	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
10	0.25	0.01	1.1745	2.0984	2.2031	2.1068	2.0108	1.9041	1.7644	1.5978
		0.05	1.1532	2.0380	2.3080	2.2058	1.9797	1.7565	1.5450	1.3501
	0.50	0.01	1.1762	2.1378	2.2657	2.1387	2.0065	1.8688	1.7032	1.5169
		0.05	1.1539	2.0646	2.3834	2.2657	1.9921	1.7309	1.4953	1.2866
	1.00	0.01	1.1782	2.2102	2.3980	2.2012	2.0812	1.7734	1.5564	1.3377
		0.05	1.1546	2.1095	2.5372	2.3933	2.0102	1.6652	1.3822	1.1505
25	0.25	0.01	1.1227	1.7991	2.0057	1.8872	1.7925	1.7135	1.5893	1.4288
		0.05	1.1220	1.7407	2.0184	1.9871	1.7835	1.5964	1.4126	1.2596
	0.50	0.01	1.1228	1.8090	2.0461	1.9072	1.7899	1.6963	1.5587	1.3867
		0.05	1.1221	1.7454	2.0563	2.0249	1.7875	1.5771	1.3785	1.2158
	1.00	0.01	1.1228	1.8262	2.1286	1.9461	1.7976	1.6495	1.4862	1.2951
		0.05	1.1221	1.7530	2.1302	2.1031	1.7908	1.5319	1.3063	1.1279

Table 11

$RE_{(M)}(T_4, P_1)$			δ							
10	0.25	0.01	1.0503	1.7742	1.8776	1.8199	1.7584	1.6787	1.5628	1.4183
		0.05	1.0302	1.7222	1.9616	1.8902	1.7127	1.5288	1.3479	1.1786
	0.50	0.01	1.0552	1.8296	1.9717	1.9167	1.8479	1.7525	1.6134	1.4436
		0.05	1.0329	1.7647	2.0623	1.9953	1.7918	1.5784	1.3714	1.1822
	1.00	0.01	1.0638	1.9415	2.1833	2.1440	2.0600	1.9260	1.7308	1.5035
		0.05	1.0378	1.8471	2.2828	2.2390	1.9774	1.6957	1.4303	1.1992
25	0.25	0.01	1.0010	1.5100	1.6893	1.5991	1.5284	1.4650	1.3586	1.2196
		0.05	1.0003	1.4593	1.6998	1.6787	1.5141	1.3573	1.1200	1.0699
	0.50	0.01	1.0011	1.5247	1.7379	1.6413	1.5621	1.4893	1.3682	1.2236
		0.05	1.0004	1.4674	1.7459	1.7311	1.5450	1.3681	1.1938	1.0726
	1.00	0.01	1.0013	1.5538	1.8440	1.7381	1.6410	1.5473	1.3943	1.2273
		0.05	1.0004	1.4830	1.8440	1.8494	1.6173	1.3971	1.2188	1.0755

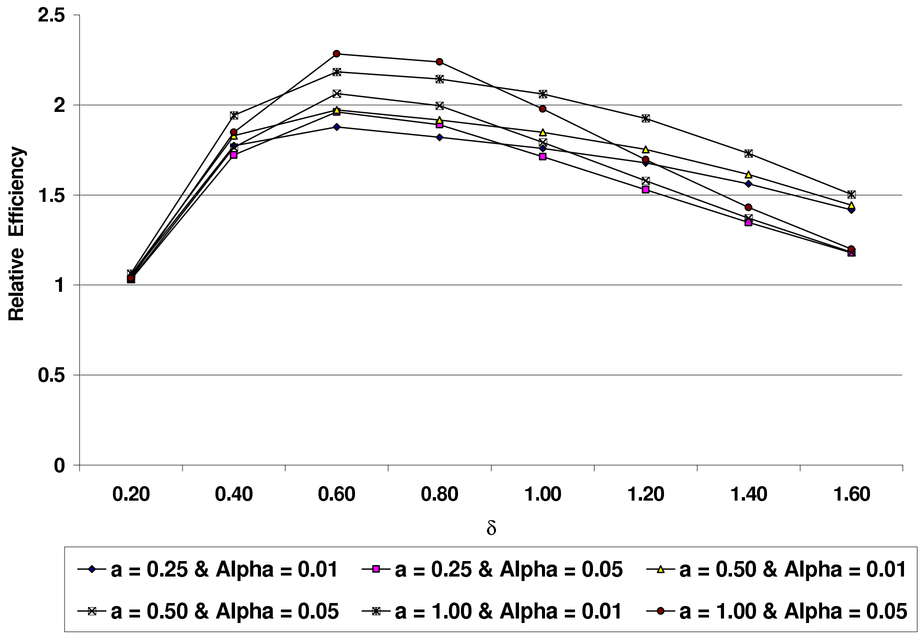

Figure 11: $RE_{(M)}\,(T_4, P_1)\quad :: n = 10$