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MOMENTS OF ORDER STATISTICS FROM DOUBLY TRUNCATED BURR XII DISTRIBUTION: A COMPLEMENTARY NOTE WITH APPLICATIONS

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summary

By using some distributional properties, we obtain some results on recurrence relations for single and product moments of order statistics from doubly truncated Burr XII distribution. These results complement earlier results of Begum and Parvin [2002], as well as, generalize results obtained by Balakrishnan and Gupta [1998], Balakrishnan et al. [1994], and Saran and Pushkarna [1999]. Simulation results are consistent with those obtained by Begum and Parvin [2002] and are given for single and product moments in Tables 1 and 2. Applications to least squares estimation of the Best Linear Unbiased Estimates of location-scale parameters involving singly and doubly censored life-testing data are considered. The estimation results compare favorably with those by Balakrishnan and Gupta [1998] in estimating the scale parameter of the censored data using the exponential distribution.

Keywords and phrases: order statistics; Burr XII distribution; single and product moments; truncation.

1 Introduction

The Burr distribution is very important in modelling of finance and insurance data. Experience has shown that the Pareto formula is often an appropriate model for claim size distribution, particularly where exceptionally large claims may occur. However, there is sometimes a need to find heavy tailed distributions which offer greater flexibility than the Pareto law. Such flexibility is provided by the Burr distribution with distribution function given by

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$$F(x) = 1 - (\frac{\lambda}{\lambda + x^{\theta}})^{\alpha}; \quad x > 0 \text{ where } \alpha, \lambda, \theta > 0. \tag{1.1}$$

Its mean exists for $\alpha, \theta > 1$.

The distribution has been used extensively to model franchise deductible premium, fixed amount deductible premium, proportional deductible premium, limited proportional deductible premium and disappearing deductible premium (see Burnecki et al. [2004]).

The distribution (1.1) reduces to the Lomax distribution for $\alpha = 1 = \theta$. The Lomax distribution has been used in connection with studies in income, size of cities and reliability modelling. The Lomax distribution , a subclass of the Burr distribution, is also known as the Pareto II distribution (see Arnold [1983]). Lomax (1954) used the distribution in the analysis of business failure data. In this paper, we consider a truncated version of the Burr XII distribution with probability function given by, Begum and Parvin [2002],

$$F(x) = \begin{cases} 0; & x < Q_1 \\ \frac{1-Q-(1+\theta x^{\alpha})^{-\lambda}}{P-Q} & Q_1 \le x \le P_1, \lambda, \theta, \alpha > 0 \\ 1; & x > P_1. \end{cases} \tag{1.2}$$

and probability density function (pdf)

$$f(x) = \frac{\lambda\theta\alpha x^{\alpha-1}(1+\theta x^{\alpha})^{-(\lambda+1)}}{P-Q}; Q_1 \le x \le P_1, \lambda, \theta, \alpha > 0 \tag{1.3}$$

where Q and $(1-P), (Q < P)$ are the proportions of truncation on the left and right of the distribution respectively and

$$Q_1 = \left[\frac{(1-Q)^{-\frac{1}{\lambda}} - 1}{\theta}\right]^{\frac{1}{\alpha}} \quad \text{and} \quad P_1 = \left[\frac{(1-P)^{-\frac{1}{\lambda}} - 1}{\theta}\right]^{\frac{1}{\alpha}}$$

The quantities Q and P are assumed to be known. Denote $Q_2 = \frac{1-Q}{P-Q}$ and $P_2 = \frac{1-P}{P-Q}$, it is easy to see that (Begum and Parvin [2002]),

$$(1+\theta x^{\alpha})f(x) = \lambda\theta\alpha x^{\alpha-1}[P_2 + (1 - F(x))] \tag{1.4}$$

or equivalently

$$(1+\theta x^{\alpha})f(x) = \lambda\theta\alpha x^{\alpha-1}[Q_2 - F(x))] \tag{1.5}$$

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be order statistics from a continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Let

$$\mu_{r:n}^{(i)} = E[X_{r:n}^{i}] \quad 1 \le r \le n$$

and

$$\mu_{r,s:n}^{(i,j)} = E[X_{r:n}^{i} X_{s:n}^{j}] \quad 1 \le r < s \le n.$$

David [1970] give the density function of $X_{r:n}(1 \le r \le n)$ as

$$f_{r:n}(x) = C_{r:n}[f(x)]^{r-1}[1 - F(x)]^{n-r} f(x) \quad -\infty < x < \infty \tag{1.6}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and the joint density function of $x = X_{r:n}$ and $y = X_{s:n}$ as

$$f_{r,s:n}(x, y) = C_{r,s:n}[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(x)f(y) - \infty < x < y < \infty \tag{1.7}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

In this paper, we present some results on recurrence relations for single and product moments of order statistics from doubly truncated Burr XII distribution. These results complement those recently obtained by Begum and Parvin [2002]. Simulation results show consistency with those obtained by the authors.

2 Single Moments

Single moments of order statistics from the doubly truncated Burr XII distribution obeys the recurrence relations presented below.

Relation 2.1 For $\alpha, \lambda > 0$

$$\mu_{1:1}^{(\alpha+1)} = \frac{\lambda\alpha}{2\theta}\left\{P_1^2 P_2 - Q_2^2(P_2 + 1) + \mu_{1:1}^{(2)}\right\} - \frac{1}{\theta}\mu_{1:1} \tag{2.1}$$

Proof. From (1.6) for $n = r = 1$, we have

$$\mu_{1:1} + \theta\mu_{1:1}^{(\alpha+1)} = \lambda\theta\alpha \int_{Q_1}^{P_1} x[P_2 + (1 - F(x))]dx \tag{2.2}$$

having used (1.4). Integrating (2.2) by parts and simplifying the resulting expressions, we have

$$\frac{2}{\lambda\alpha}[\mu_{1:1} + \theta\mu_{1:1}^{(\alpha+1)}] = P_2(P_1^2 - Q_1^2) - Q_1^2 + \mu_{1:1}^{(2)} \quad (2.3).$$

By rewriting (2.3), we obtain the relation (2.1) □

Relation 2.2 For $\lambda \neq \frac{\alpha+1}{n\alpha}$ and $n \ge 2$

$$\mu_{1:n}^{(\alpha+1)} = \frac{\lambda[(n-1)P_2\mu_{1:n-1}^{(\alpha+1)} + nQ_1^{\alpha+1}(1 - P_2)]}{(1 - n\lambda) + \frac{1}{\alpha}} - \frac{\alpha + 1}{\theta[(\alpha + 1) - n\lambda\alpha]} \tag{2.4}$$

Proof. From (1.6) for $r = 1$, we have

$$\mu_{1:n} + \theta\mu_{1:n}^{(\alpha+1)} = n\lambda\theta\alpha P_2 \int_{Q_1}^{P_1} x^\alpha[1 - F(x)]^{n-1}dx + \lambda\theta\alpha \int_{Q_1}^{P_1} x^\alpha[1 - F(x)]^n dx \tag{2.5}$$

having used (1.4). By integrating (2.5) by parts and simplifying the resulting expression, we get

$$\mu_{1:n} + \theta\mu_{1:n}^{(\alpha+1)} = \frac{(n-1)\lambda P_2}{[(1-n\lambda)+\frac{1}{\alpha}]}\mu_{1:n-1}^{(\alpha+1)} - \frac{\alpha+1}{\theta[(\alpha+1)-n\lambda\alpha]}\mu_{1:n} + \frac{n\lambda\alpha Q_1^{\alpha+1}(1+P_2)}{(\alpha+1)-n\lambda\alpha}$$

By rewriting the above expression, we obtain the relation (2.4) □

Remark The moments of the first order statistics based on a random sample of size n. is given by (2.5).

Relation 2.3 For $\lambda \neq \frac{\alpha+1}{\alpha(n-r+1)}$ $\lambda, \alpha, \theta > 0$ and $1 \leq r \leq n-1$

$$\mu_{r:n}^{(\alpha+1)} = \frac{n\lambda\alpha P_2}{(\alpha+1)-\lambda\alpha(n-r+1)}[\mu_{r:n-1}^{(\alpha+1)} - \mu_{r-1:n-1}^{(\alpha+1)}] - \frac{(\alpha+1)\mu_{r:n}}{\theta[(\alpha+1)-\lambda\alpha(n-r+1)]}$$
$$-\frac{\lambda\alpha(n-r+1)}{(\alpha+1)-\lambda\alpha(n-r+1)}\mu_{r-1:n}^{(\alpha+1)} \tag{2.6}$$

Proof. From (1.6), we have

$$\mu_{r:n} + \theta\mu_{r:n}^{(\alpha+1)} = C_{r:n}\lambda\alpha\theta\{P_2\int_{Q_1}^{P_1} x^{\alpha}[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$
$$+\int_{Q_1}^{P_1} x^{\alpha}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx\} \tag{2.7}$$

having used (1.4). Upon integrating (2.7) by parts and simplifying the resulting expressions, we obtain

$$\mu_{r:n} + \theta\mu_{r:n}^{(\alpha+1)} = \frac{n\lambda\alpha\theta P_2}{\alpha+1}\mu_{r:n-1}^{(\alpha+1)} - \frac{n\lambda\theta\alpha P_2}{\alpha+1}\mu_{r-1:n-1}^{(\alpha+1)}$$
$$+\frac{\lambda\theta\alpha(n-r+1)}{\alpha+1}\mu_{r:n}^{(\alpha+1)} - \frac{\lambda\theta\alpha(n-r+1)}{\alpha+1}\mu_{r-1:n}^{(\alpha+1)} \tag{2.8}$$

By rewriting (2.8), we have the relation (2.6) □

Corollary 2.1. By replacing $(\alpha+1)$ with k, any positive constant, we have the result of Begum and Parvin [2002 p. 183].

Relation 2.4 For $\lambda \neq \frac{\alpha+i}{\alpha(n-r+1)}$, $\lambda, \alpha, \theta > 0$ and $1 \leq r \leq n$, $i = 0, 1, 2, ...$

$$\mu_{r:n}^{(\alpha+i)} = \frac{n\lambda\alpha P_2}{(\alpha+i)-\lambda\alpha(n-r+1)}[\mu_{r:n-1}^{(\alpha+i)} - \mu_{r-1:n-1}^{(\alpha+i)}]$$
$$-\frac{\alpha+i}{\theta[(\alpha+i)-\lambda(n-r+1)]}\mu_{r:n}^{(i)} - \frac{\lambda\alpha(n-r+1)}{(\alpha+i)-\lambda\alpha(n-r+1)}\mu_{r-1:n}^{(\alpha+i)}. \tag{2.9}$$

Proof. From (1.6) for $1 \le r \le n$, $i = 0, 1, 2, ...$

$$\mu_{r:n}^{(i)} + \theta\mu_{r:n}^{(\alpha+i)} = \lambda\theta\alpha C_{r:n}\{P_2 \int x^{\alpha+i}[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

$$+ \int x^{\alpha+i-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx\}. \tag{2.10}$$

having used (1.4). By integrating (2.10) by parts, we obtain

$$\mu_{r:n}^{(i)} + \theta\mu_{r:n}^{(\alpha+i)} = \frac{C_{r:n}\lambda\theta\alpha P_2}{\alpha+i}\{(n-r)\int x^{\alpha+i}[F(x)]^{r-1}[1-F(x)]^{n-r-1}f(x)dx$$

$$-(r-1)\int x^{\alpha+i}[F(x)]^{r-2}[1-F(x)]^{n-r}f(x)dx\}$$

$$+\frac{C_{r:n}\lambda\theta\alpha}{\alpha+i}\{(n-r+1)\int x^{\alpha+i}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

$$-(r-1)\int x^{\alpha+i}[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)dx\}. \tag{2.11}$$

By simplifying (2.11) and rewriting the resulting expression, we get the result as in (2.9). □

Corollary 2.2. For $\lambda \neq 1, r = n$ and $i = 0$,

$$\mu_{n:n}^{(\alpha)} = \frac{n\lambda P_2}{1-\lambda}[P_1^{\alpha} - \mu_{n-1:n-1}^{(\alpha)}] - \frac{1}{1-\lambda}[1 + \lambda\mu_{n-1:n}^{(\alpha)}]. \tag{2.12}$$

3 Product Moments

Product moments of order statistics from the doubly truncated Burr XII distribution obeys the recurrence relations presented below.

Relation 3.1 For $\lambda, \theta, \alpha > 0$ and $1 \le r < s \le n$,

$$\mu_{r,s:n}^{(\alpha)} = n\lambda\alpha\{P_2[\mu_{r,s:n-1} - \mu_{r,s-1:n-1}] - \frac{n-s+1}{n}[\mu_{r,s-1:n} - \mu_{r,s:n}] - \frac{1}{\theta}\mu_{r:n}\}. \tag{3.1}$$

Proof. From (1.7), we have

$$\mu_{r:n} + \theta\mu_{r,s:n}^{(\alpha)} = \lambda\theta\alpha C_{r,s:n}$$

$$\{P_2\int_{Q_1}^{P_1} x[F(x)]^{r-1}f(x)I_1(x)dx + \int_{Q_1}^{P_1} x[F(x)]^{r-1}f(x)I_2(x)dx\}. \tag{3.2}$$

having used (1.4), where

$$I_1(x) = \int_x^{P_1}[F(y) - F(x)]^{s-r-1}[1-F(y)]^{n-s}dy \tag{3.3}$$

and

$$I_2(x) = \int_x^{P_1}[F(y) - F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy \tag{3.4}$$

Upon integrating (3.3) and (3.4) by parts and putting in (3.2), we have

$$\mu_{r:n} + \theta\mu^{(\alpha)}_{r,s:n} = \lambda\theta\alpha C_{r,s:n} P_2$$

$$\times\{(n-s)\int_{Q_1}^{P_1}\int_{x}^{P_1} xy[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1} f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} xy[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)dxdy\}$$

$$+\lambda\theta\alpha C_{r,s:n}\{(n-s+1)\int_{Q_1}^{P_1}\int_{x}^{P_1} xy[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} xy[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(x)f(y)dxdy\}. \quad (3.5)$$

By simplifying (3.5) and rewriting the resulting expression, we have the relation (3.1).

□

Corollary 3.1. For $s = r + 1$, λ, α, $\theta > 0$

$$\mu^{(\alpha)}_{r,r+1:n} = n\lambda\alpha[\mu_{r,r+1:n-1} + \frac{n-r}{n}\mu_{r,r+1:n}] - \frac{1}{\theta}\mu_{r:n}. \quad (3.6)$$

Relation 3.2 For λ, θ, $\alpha > 0$ and $1 \le r < s \le n-1$,

$$\mu^{(\alpha,\alpha)}_{r,s:n} = n\lambda\alpha P_2[\mu^{(\alpha)}_{r,s:n-1} - \mu^{(\alpha)}_{r,s-1:n-1} - \mu^{(\alpha)}_{r,s-1:n}]$$

$$+\frac{\lambda\theta\alpha(n-s+1)}{\theta}\mu^{(\alpha)}_{r,s:n}. \quad (3.7)$$

and for $s = r + 1$

$$\mu^{(\alpha,\alpha)}_{r,r+1:n} = n\lambda\alpha P_2\mu^{(\alpha)}_{r,r+1:n-1} + \frac{n\lambda\alpha(n-r)-1}{\theta}\mu^{(\alpha)}_{r,r+1:n}. \quad (3.8)$$

Proof.

$$\mu^{(\alpha)}_{r,s:n} + \theta\mu^{(\alpha,\alpha)}_{r,s:n} = C_{r,s:n}\{P_2\int_{Q_1}^{P_1} x^{\alpha}[F(x)]^{r-1} f(x)H_1(x)dx$$

$$+\int_{Q_1}^{P_1} x^{\alpha}[F(x)]^{r-1} f(x)H_2(x)dx\}. \quad (3.9)$$

having used (1.7), where

$$H_1(x) = \int_{x}^{P_1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy \quad (3.10)$$

and

$$H_2(x) = \int_{x}^{P_1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy \quad (3.11)$$

By integrating (3.10) and (3.11) by parts and substituting into (3.9), we have

$$\mu^{(\alpha)}_{r,s:n} + \theta\mu^{(\alpha,\alpha)}_{r,s:n} = \lambda\theta\alpha C_{r,s:n} P_2$$

$$\times\{(n-s)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^{\alpha} y [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1} f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^{\alpha} y [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s} f(x)f(y)dxdy\}$$

$$+\lambda\theta\alpha C_{r,s:n}\{(n-s+1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^{\alpha} y [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^{\alpha} y [F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1} f(x)f(y)dxdy\}. \quad (3.12)$$

By simplifying (3.12), we have the relation (3.7). For $s = r+1$ in (3.12), we have the equation (3.8). □

Relation 3.3 For $\lambda, \ \theta, \ \alpha > 0$ and $1 \le r < s \le n-1$, and $i, j = 0, 1, 2, ...$

$$\mu^{(i,j+\alpha)}_{r,s:n} = \frac{n\lambda\alpha P_2}{j+1}[\mu^{(i,j+1)}_{r,s:n-1} - \mu^{(i,j+1)}_{r,s-1:n-1}]$$

$$-\frac{\lambda\alpha(n-s+1)}{j+1}[\mu^{(i,j+1)}_{r,s:n} - \mu^{i,j+1}_{r,s-1:n}] - \frac{\mu^{i,j+1}_{r,s:n}}{\theta} \quad (3.13)$$

and for $s = r+1$

$$\mu^{(i,j+\alpha)}_{r,r+1:n} = \frac{n\lambda\alpha P_2}{j+1}\mu^{(i,j+\alpha)}_{r,r+1:n-1} + \frac{\lambda\alpha(n-r)}{j+1}\mu^{(i,j+1)}_{r,r+1:n} - \frac{1}{\theta}\mu^{i,j}_{r,r+1:n}. \quad (3.14)$$

Proof. From (1.7)

$$\mu^{(i,j+\alpha)}_{r,s:n} + \theta\mu^{(i,j+\alpha)}_{r,s:n} = C_{r,s:n}\lambda\theta$$

$$\times\{P_2\int_{Q_1}^{P_1} x^i [F(x)]^{r-1} f(x) J_1(x)dx + \int_{Q_1}^{P_1} x^i [F(x)]^{r-1} f(x) J_1(x)dx\}. \quad (3.15)$$

having used (1.7), where

$$J_1(x) = \int_x^{P_1} y^i [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy \quad (3.16)$$

and

$$J_1(x) = \int_x^{P_1} y^i [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy \quad (3.17)$$

By integrating (3.16) and (3.17) by parts and putting into (3.15), we have

$$\mu^{(i,j)}_{r,s:n} + \theta\mu^{(i,j+\alpha)}_{r,s:n} = \frac{\lambda\theta\alpha C_{r,s:n} P_2}{j+1}$$

$$\times\{(n-s)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j+1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1}f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j+1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(x)f(y)dxdy\}$$

$$+\frac{\lambda\theta\alpha C_{r,s:n}}{j+1}\{(n-s+1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j+1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dxdy$$

$$-(s-r-1)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j+1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s+1}f(x)f(y)dxdy\}. \tag{3.18}$$

By simplifying (3.18) and rewriting the resulting expression, we have the relation (3.13). By setting $s = r + 1$ in (3.18), and simplifying the resulting expression, we have the relation (3.14) □

Remark

1. For non-truncated i.e.,$Q = 0$ and $P = 1$, Balakrishnan *et al.*[*1994*, *1998*] considered Lomax distribution which follows as a special case for the recurrence relations of Burr XII distribution when $\alpha = 1, \quad \theta = 1$ and $\lambda = \alpha$.

2. By using the relations (1.4) and (1.5) for $\alpha = 0, \quad \theta = 1$ and $\lambda = \alpha$, Saran and Pushkarna [1999] have established the recurrence relations for the single and product moments of order statistics from doubly truncated Lomax distribution.

3. Tables 1 and 2 give the mean of single and product moments of order statistics from the doubly truncated Burr XII distribution for $\alpha = 1, \quad \theta = 1, \quad \lambda = 3, \quad P = 0.95$, and $Q = 0.05$.

4 Least Squares Estimation of Location and Scale Parameters.

In this section, we consider least squares estimation of scale-location parameters based on doubly Type II censored samples. Let $Y = \mu + X\sigma$ be a random variable with probability density function given by (1.2), then Y is the location-scale form of the model (1.2), where μ, and σ are the location and scale parameters respectively. Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from the distribution of Y and $Y_{1:n} \le Y_{2:n} \le .... \le Y_{n:n}$ be its order statistics. The ordered X and Y variates (in random sample of n) are linked by

$$X_{(i)} = \frac{Y_{(i)} - \mu}{\sigma}, \quad i = 1, 2, ...n.$$

The moments of the $Y_{(i)}$ depend only on the form of the distribution of Y and not on μ and σ.

Let

$$EX_{(i)} = \alpha_i, cov(X_{(i)}, X_{(j)}) = \beta_{ij}, \quad j = 1, 2, ..., n. \tag{4.1}$$

Then

$$EY_{(i)} = \mu + \sigma\alpha_i, \quad cov(X_{(i)}, Y_{(j)}) = \sigma^2\beta_{ij}. \tag{4.2}$$

where the α_i and β_{ij} are easily evaluated by using the results in the previous sections with σ^2 unknown.

We write

$$\mathbf{EY} = \mu\mathbf{1} + \sigma\alpha$$

$$\mathbf{EY} = \mathbf{A\Theta} \tag{4.3}$$

where $\mathbf{Y}, \alpha$ are, respectively, the column vectors of the $Y_{(i)}, \alpha_i$ and $\mathbf{1}$ is a column vector of $n \;\; 1's$ and

$$\mathbf{A} = (\mathbf{1}; \alpha), \quad \Theta' = (\mu, \sigma)$$

Therefore, the best linear unbiased estimators BLUE's of μ and σ are then given, [David 1970 p. 103] and Balakrishnan and Gupta [1998], by

$$\mu^\star = -\mu'\mathbf{\Gamma X} = \mathbf{a'X} = \sum_{i=1}^{n} a_i X_{(i)} \tag{4.4}$$

and

$$\sigma^\star = \mathbf{1'\Gamma X} = \mathbf{b'X} = \sum_{i=1}^{n} b_i X_{(i)} \tag{4.5}$$

where a and b are vectors of coefficients for BLUE's of the location and scale parameters respectively.

$$\mathbf{\Gamma} = \frac{\mathbf{1}}{\mathbf{\Delta}}[\Sigma^{-1}(1\mu' - \mu 1)\Sigma^{-1}]$$

$$\mathbf{\Delta} = (\mathbf{1\Sigma^{-1}1})(\mu'\mathbf{\Sigma^{-1}}\mu) - (\mu'\mathbf{\Sigma^{-1}1})^2 \mathbf{1} = [1, 1, .., 1]_{n\times 1} \quad \mu = [\mu_{1:n}, \mu_{2:n}, ..., \mu_{n:n}]'_{n\times 1} \tag{4.6}$$

and

$$\mathbf{\Sigma} = \sum\sum\nolimits_{i,j} \sigma_{ij:n}. \tag{4.7}$$

The variances and covariances of the BLUE's $\mu^\star$ and $\sigma^\star$ are given by

$$Var(\mu^\star) = \frac{\sigma^2\mu'\mathbf{\Sigma}^{-1}\mu}{\mathbf{\Delta}} \qquad Var(\sigma^\star) = \frac{\sigma^2\mathbf{1'\Sigma^{-1}1}}{\mathbf{\Delta}}$$

$$cov(\mu^\star, \sigma^\star) = \frac{(\sigma^2 - \mathbf{1\Sigma^{-1}}\mu)}{\mathbf{\Delta}}. \tag{4.8}$$

We present numerical results for the vectors a and b. Because many tables are involved for the various values of α, λ, θ, we present tables for $\alpha, \lambda, \theta = 1(0.5)2.5$ considered in the following examples. These examples are those considered by Balakrishnan and Gupta [1998] for the exponential distribution.

Example 1. Twelve components were placed on a life-test and the time-to-fail of the first eight components that failed were recorded and the experiment was terminated as soon as the eight failure occurred. The Type-II right-censored sample resulting from the experiment is given below

$$31, 58, 157, 185, 300, 470, 497, 673, \ldots.$$

We have $n = 12, r = 0$ and $s = 4$ corresponding to Table 3.

We have $\mu^\star = 191.5449$, $var(\mu^\star) = 3.64626$ and $\sigma^\star = 631.68824$, $var(\sigma^\star) = 1.4084$.

Example 2. The following data represent failure times, in minutes, for a specific type of electrical insulation in an experiment in which the insulation was subjected to a continuously increasing voltage stress:

$$\ldots, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, \ldots.$$

In this case we have $n = 12, r = 2$ and $s = 1$ corresponding to Table 4.

We have, in this case, $\mu^\star = 51.3288$, $var(\mu^\star) = 0.1991$ and $\sigma^\star = 71.3848$, $var(\sigma^\star) = 0.08985$.

Example 3. This data arise from an experiment on insulting fluid breakdowns. Among the $n = 12$ specimens tested at $45kV$, 3 failed before 1 second and the times to breakdown (in seconds) of the remaining 9 specimens were

$$2, 2, 3, 9, 13, 47, 50, 55, 71.$$

We assume the data to be doubly Type-II censored with $n = 12, r = 3$ and $s = 0$. This corresponds to Table 1, we have $\mu^\star = 20.1893$, $var(\mu^\star) = 0.6572$ and $\sigma^\star = 20.7864$, $var(\sigma^\star) = 0.2551$.

Remark For the same set of examples considered, Balakrishnan and Gupta [1998] obtained estimates of the scale parameter using the exponential distribution. Our results compare favorably well with their estimates for this (scale) parameter:

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Appendix

Table 1: The Means of the r-th order statistics from doubly truncated Burr XII distribution
$\alpha = 1$, $\theta = 1$, $\lambda = 3$, P=0.95 and Q=0.05

r=0, s=4, k=.1667		r=0, s=4, k=.25		r=0, s=4, k=.50						
n/r	1	2	3	4	5	6	7	8	9	10
1	0.38444									
2	0.19621	0.57266								
3	0.13320	0.32222	0.69788							
4	0.10270	0.22478	0.41965	0.79062						
5	0.08482	0.17420	0.30059	0.49897	0.86354					
6	0.07311	0.14337	0.23582	0.36551	0.56570	0.92311				
7	0.06485	0.12266	0.19515	0.29005	0.42210	0.32313	0.97300			
8	0.05871	0.10779	0.16727	0.24161	0.33848	0.47228	0.67341	1.01591		
9	0.05398	0.09661	0.14697	0.20788	0.28378	0.38224	0.51731	0.71802	1.05315	
10	0.05021	0.08787	0.13152	0.18302	0.24517	0.32239	0.42214	0.55809	0.75800	1.08595

Table 2: Product moments of order statistics from doubly truncated Burr XII distribution $\alpha = 1$, $\theta = 1$, $\lambda = 3$, P=0.95 and Q=0.05

r=0, s=4, k=.1667		r=0, s=4, k=.25		r=0, s=4, k=.50		
n	s/r	1	2	3	4	5
1	1	0.27993				
2	1	0.07525				
2	2	0.14779	0.48464			
3	1	0.03309				
3	2	0.05910	0.15957			
3	3	0.10810	0.27618	0.64717		
4	1	0.01871				
4	2	0.03172	0.07623			
4	3	0.05207	0.12090	0.24290		
4	4	0.08935	0.20164	0.39108	0.78192	
5	1	0.01220				
5	2	0.02005	0.04470			
5	3	0.03104	0.06741	0.12353		
5	4	0.04800	0.10227	0.18368	0.32250	
5	5	0.07827	0.16395	0.28903	0.49240	0.89678

Table 3: $\alpha = 1 = \lambda = \theta$ Coefficients of BLUE

r=3, s=0, k=.125		r=3, s=0, k=.25		r=3, s=0		r=3, s=0, k=.50	
a	b	a	b	a	b	a	b
0.0028	-1.441	0.0125	-0.5320	0.0127	-0.9067	0.0119	0.0386
0.0302	1.2633	0.0195	1.0519	0.0199	1.1878	0.0183	0.7569
0.0191	0.1598	0.0280	0.1615	0.0287	0.1608	0.0258	0.1556
0.0558	1.1263	0.0435	0.0716	0.0450	1.0723	0.0394	0.7367
0.0941	0.0688	0.0685	0.2082	0.0822	0.1245	0.0689	0.3124
0.0892	0.0093	0.0707	0.5093	0.0928	0.0093	0.0504	0.0093
0.2129	0.0136	0.2240	0.0135	0.2115	0.0136	0.2003	0.0134
0.0077	0.0186	0.0172	0.4184	0.0175	0.0186	0.0163	0.0181
0.0426	0.0272	0.0288	0.0267	0.0306	0.0270	0.0267	0.0258

Table 4: Coefficients of BLUE

r=2, s=1, k=.125		r=2, s=1, k=.25		r=2, s=1		r=2, s=1, k=.50	
a	b	a	b	a	b	a	b
0.0114	0.8807	0.0114	0.7868	0.0114	0.8491	0.0115	0.7054
0.0174	0.3261	0.0173	0.3352	0.0174	0.3313	0.0171	0.3178
0.0251	0.0143	0.0247	0.0144	0.0249	0.0144	0.0241	0.0144
0.0393	0.0231	0.0384	0.0229	0.0390	0.0230	0.0367	0.0225
0.0722	0.0355	0.0693	0.0348	0.0712	0.0353	0.0641	0.0337
0.1048	0.0617	0.1028	0.0597	0.1021	0.0610	0.0934	0.0561
0.3114	0.1348	0.3002	0.1270	0.3108	0.1322	0.2488	0.1137
0.0151	0.1337	0.0147	0.1094	0.0149	0.1184	0.0140	0.0920
0.0246	0.0413	0.0238	0.0303	0.0243	0.0387	0.0222	0.0219