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USING A TWO–STAGE MINIMUM ABERRATION CRITERION TO SELECT OPTIMAL TWO–LEVEL FRACTIONAL FACTORIAL DESIGNS

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summary

In selecting 2^{m-p} designs when some of the two-factor interactions are important, the key issues are to permit estimation of the main effects and important two-factor interactions in a postulated model and to minimize the bias caused by the other effects not included in the model. If the main effects need more protection than the important two-factor interactions, we should first minimize the bias of the main effects, and then minimize the bias of the important two-factor interactions. In this paper, a two-stage minimum aberration criterion is proposed to minimize the bias of the main effects and that of the important two-factor interactions sequentially. Searching for the best designs according to this criterion is discussed and some results for designs of 16 and 32 runs are presented.

Keywords and phrases: defining words; design resolution; fractional factorial design; non-isomorphic model; requirement set; confounding pattern.

AMS Classification: Primary 62K05; secondary 62K10, 62K15.

1 Introduction

A regular two-level fractional factorial design is commonly referred to as a 2^{m-p} design. It has m two-level factors with 2^{m-p} runs, and is completely determined by p independent defining words. This design allows us to study many factors with relatively small run size. It is very useful for identifying important factors and is commonly used in industrial experiments and other areas of scientific investigation. Since a 2^{m-p} design can be chosen in many different ways, a key question is how to choose a fraction of the full factorial design for a given run size and number of factors. The most commonly used criterion for 2^{m-p} design selection is the minimum aberration criterion proposed by Fries and Hunter (1980). This criterion selects designs by sequentially minimizing the number of defining words of length j in the defining relation where the length of a defining word is the number of factors it contains. It includes the resolution criterion of Box and Hunter (1961) as a special case. The robust properties of this criterion were discussed by Tang and Deng (1999) and Cheng, Steinberg, and Sun (1999). For small number of factors, Box, Hunter, and Hunter (1978) provided a useful catalogue of 2^{m-p} designs with minimum aberration. Franklin (1984) constructed more minimum aberration designs. A more complete catalogue of 2^{m-p} designs ranked by the minimum aberration criterion was provided by Chen, Sun, and Wu (1993).

In this article we consider how to select 2^{m-p} designs when some two-factor interactions (2fi's) are presumably important. One way of solving this design problem is to find designs that allow joint estimation of all main effects and these presumably important 2fi's under the assumption that all other effects are negligible. Much work has been done on finding a design allowing estimation of a set of specified effects, often referred to as a *requirement set* in the literature. This includes Addelman (1962), Greenfield (1976), Franklin (1985), Wu and Chen (1992), and Ke, Tang, and Wu (2005).

When some 2fi's are important, the postulated model should consist of all main effects and these important 2fi's. If the effects not in the postulated model cannot be completely ignored, they will bias the estimates of the effects in the model. To solve this problem, Ke and Tang (2003) propose a minimum N-aberration criterion that systematically minimizes the bias of all effects in the model caused by the other effects. In some situations, main effects need more protection than the important 2fi's. In such case, we should first minimize the bias of the main effects, and then minimize the bias of the important 2fi's. In this paper, we propose a two-stage minimum N-aberration criterion to minimize the bias of the main effects and that of the important 2fi's sequentially.

Section 2 of the paper introduces and studies this two-stage minimum N-aberration criterion. Section 3 examines how to search for designs that are best according to the criterion and present some results for designs of 16 and 32 runs. Section 4 concludes the paper with a discussion.

2 Two–Stage Minimum N-Aberration Criterion

2.1 A General Criterion of Aberration: Minimum N–Aberration

Suppose that we are interested in estimating all main effects and a set of important 2fi's. Then the fitted model is given by

$$Y = \beta_0 I + W_1\gamma_1 + \epsilon \tag{2.1}$$

where Y denotes the vector of n observations, β_0 is the grand mean, I denotes the vector of n ones, γ_1 is the vector of parameters containing all main effects and the important 2fi's, W_1 is the corresponding matrix, and ϵ is the vector of uncorrelated random errors, assumed to have zero mean and a constant variance. Since other interactions not in the model may not be negligible, the true model can be written as

$$Y = \beta_0 I + W_1\gamma_1 + W_2\gamma_2 + X_3\beta_3 + \ldots + X_m\beta_m + \epsilon \tag{2.2}$$

where γ_2 is the vector of remaining 2fi's, W_2 is the corresponding matrix, and β_j is the vector of $\binom{m}{j}$ interactions involving j factors and X_j is the corresponding matrix. The least square estimator $\hat{\gamma}_1 = (W_1^T W_1)^{-1} W_1^T Y = n^{-1} W_1^T Y$ from the fitted model in (2.1) has expectation, taken under the true model in (2.2), $E(\hat{\gamma}_1) = \gamma_1 + P_2\gamma_2 + P_3\beta_3 + \ldots + P_m\beta_m$, where $P_2 = n^{-1} W_1^T W_2$ and $P_j = n^{-1} W_1^T X_j$ for $j \geq 3$. So the bias of $\hat{\gamma}_1$ for estimating γ_1 is given by

$$\text{Bias}(\hat{\gamma}_1, \gamma_1) = P_2\gamma_2 + P_3\beta_3 + \ldots + P_m\beta_m \tag{2.3}$$

Note that $P_2\gamma_2$ is the contribution of γ_2 to the bias, and $P_j\beta_j$ is the contribution of β_j to the bias. As γ_2 and β_j are unknown, we will have to work with P_j. One size measure for a matrix $P = (p_{ij})$ is given by $||P||^2 \stackrel{\text{def}}{=} \text{trace}(P^T P) = \sum_{i,j} p_{ij}^2$. Under the hierarchical assumption that lower order effects are more important than higher order effects, to minimize the bias of $\hat{\gamma}_1$ we should sequentially minimize $||P_2||^2, \ldots, ||P_m||^2$. For regular designs, the entries of P_j are 0 or 1, and thus $||P_j||^2$ is simply the number of j-factor interactions aliased with the effects in the postulated model in (2.1). Now let $N_j = ||P_j||^2$. Based on the above results, we can select designs using a minimum aberration criterion, defined as the one that sequentially minimizes $N_2, \ldots, N_m$. We call this criterion as the minimum N-aberration criterion. For further discussion about this issue, the reader is referred to Tang and Deng (1999, 2003), Tang (2001, 2006), Ke and Tang (2003), and Cheng and Tang (2005).

2.2 A Two–Stage Minimum N–Aberration Criterion

Sometimes in practice, we feel that the main effects need more protection than the important 2fi's, then the bias of the main effects should be minimized first. This suggests a variation of the minimum N-aberration criterion, which sequentially minimizes $N_{21}, N_{22}, N_{31}, N_{32}, \ldots, N_{m1}, N_{m2}$, where N_{j1} is the number of j-factor interactions aliased with the main effects, and N_{j2} the number of j-factor interactions aliased with the important 2fi's. We call this criterion as the two-stage minimum N-aberration criterion. Clearly, $N_j = N_{j1} + N_{j2}$, where N_j is the number of j-factor interactions aliased with the effects in the postulated model.

To gain further insight into the two-stage minimum N-aberration criterion, we now examine the criterion in detail. The model in (2.1) consists of all the main effects and all the important 2fi's. For the effects in the requirement set to be estimable, the important 2fi's in the requirement set cannot be aliased with each and with main effects. In general, a design allows estimation of the model in (2.1) if and only if none of its length 3 words contains an important 2fi and none of its length 4 words contains two important 2fi's that have no letter in common. The two-stage minimum N-aberration criterion selects a design, from among all designs satisfying the above condition for estimation, by sequentially minimizing $N_{21}, N_{22}, N_{31}, N_{32}, \ldots, N_{m1}, N_{m2}$. A general expression for N_{j1} and N_{j2} can be derived.

Let A_j be the number of defining words of length j in the defining relation. Let $A_j^{(2)} = \sum_{i=1}^k a_i$, where k denotes the number of important 2fi's, and a_i is the number of length j words containing the i-th important 2fi, $A_j^{(1)} = \sum_{i=1}^k b_i$, where b_i is the number of length j words containing one and only one letter in the i-th important 2fi, and $A_j^{(0)} = \sum_{i=1}^k c_i$, where c_i is the number of length j words without containing any letter in the i-th important 2fi. A general expression of N_{j1} and N_{j2} is given by

$$N_{j1} = (j+1)A_{j+1} + (m-j+1)A_{j-1}, \tag{2.4}$$

$$N_{j2} = A_{j+2}^{(2)} + A_j^{(1)} + A_{j-2}^{(0)}, \tag{2.5}$$

where we define $A_j = 0$ for $j < 3$ or $j > m$. Hence we have that

$$N_{21} = 3A_3, N_{22} = A_4^{(2)}, \tag{2.6}$$

$$N_{31} = 4A_4, N_{32} = A_5^{(2)} + A_3^{(1)}. \tag{2.7}$$

It should be noted that $A_j^{(2)}$ does not represent the number of length j words containing an important 2fi. This is because if a length j word contains more than one important 2fi, it is counted more than once in calculating $A_j^{(2)}$. In fact, $A_j^{(2)}$ corresponds to the total number of times a length j word contains an important 2fi. A similar interpretation holds for $A_j^{(1)}$.

The requirement set of model in (2.1) consists of all the main effects and all the important 2fi's. The 2fi's that are not in the requirement set generally cause a bias on the estimation of the effects in the requirement set. The measure of this bias are given by N_{21} and N_{22} where N_{21} is the number of those 2fi's outside the requirement set that are aliased with the main effects and N_{22} represents the number of those 2fi's outside the requirement set that are aliased with the important 2fi's. If a design has resolution IV, then $A_3 = 0$, in which case, $N_{21} = 3A_3 = 0$. If all the important 2fi's in the requirement set are clear, meaning that they are not aliased with any main effects and any other 2fi's (Wu and Hamada 2000), then $A_4^{(2)} = 0$, in which case, $N_{22} = A_4^{(2)} = 0$. The two-stage minimum N-aberration criterion goes on to minimize N_{31} and N_{32} once N_{21} and N_{22} are minimized. Note that N_{31} is the number of 3fi's that are aliased with the main effects and N_{32} is the number of 3fi's that are aliased with the important 2fi's in the requirement set.

Suppose that we want to study six factors $1, 2, \ldots, 6$ by using a design of 16 runs. In addition to the main effects of the six factors, we also want to estimate the three 2fi's that are between factor 1 and factor 2, denoted by 12, between factor 1 and factor 3, denoted by 13, and between factor 1 and factor 4, denoted by 14, respectively. Considering two designs, D_1, and D_2, where D_1 is given by $I = 1235 = 2346 = 1456$, and D_2 by $I = 12345 = 2346 = 156$. For D_1, we have $A_3 = A_5 = 0, A_4 = A_4^{(2)} = 3$. Then we have $N_{21} = 3A_3 = 0, N_{22} = A_4^{(2)} = 3, N_{31} = 4A_4 = 12$, and $N_{32} = A_5^{(2)} + A_3^{(1)} = 0$. For D_2, we have $A_3 = A_4 = A_5 = 1, A_3^{(1)} = A_5^{(2)} = 3$ and $A_4^{(2)} = 0$. Then we have $N_{21} = 3A_3 = 3, N_{22} = A_4^{(2)} = 0, N_{31} = 4A_4 = 4$, and $N_{32} = A_5^{(2)} + A_3^{(1)} = 6$. If the main effects need more protection than the important two-factor interactions, we should use the two-stage minimum N-aberration criterion to select the good design. Let $(N_{21}, N_{22}, N_{31}, N_{32}, \ldots)$ denote the confounding pattern of two-stage minimum N-aberration. The confounding patterns of the two designs D_1, and D_2 are given by

$$\begin{aligned} D_1 &: (0, 3, 12, 0) \\ D_2 &: (3, 0, 4, 6) \end{aligned}$$

Based on the confounding patterns, D_1 is better than D_2 because $N_{21}(D_1) < N_{21}(D_2)$ where $N_{21}(D_1)$ and $N_{21}(D_2)$ denote the N_{21} for D_1 and D_2 respectively. If all the effects in the requirement set, consisting of the main effects and important 2fi's, are equally important, we may choose the best design by using the minimum N-aberration criterion proposed by Ke and Tang (2003). The best design under the two-stage minimum N-aberration criterion may be not the best under the minimum N-aberration criterion. In the above example, we have $N_2 = N_{21} + N_{22} = 3$ and $N_3 = N_{31} + N_{32} = 12$ for D_1, $N_2 = 3$ and $N_3 = 10$ for D_2. So D_2 is better than D_1 under the usual minimum N-aberration criterion because $N_2(D_1) = N_2(D_2)$ and $N_3(D_1) > N_3(D_2)$.

In practice, we are often quite confident that interactions involving three or more factors are negligible. In this case, we are satisfied with only minimizing N_{21} and N_{22} in finding the best designs. This gives a weak version of the two-stage minimum N-aberration criterion. For the above example, D_1 is still better than D_2 under the weak version of the two-stage minimum N-aberration criterion because $N_{21}(D_1) < N_{21}(D_2)$. But the two designs are the same under the weak version of the usual minimum N-aberration criterion because $N_2(D_1) = N_2(D_2)$.

3 Searching for Two–Stage Minimum N–Aberration Designs

3.1 Search Method

In this paper, we consider all non-isomorphic models containing up to three 2fi's. Let k be the number of important 2fi's. For $k = 1$, there is only one non-isomorphic model, as represented by Figure 1. For $k = 2, 3$, the number of non-isomorphic model is 2 and 5 respectively, and the graphs for these non-isomorphic models are given in Figures 2 and 3 respectively.

Chen, Sun, and Wu (1993) gave a complete catalog of all non-isomorphic designs of 16 and 32 runs, and all resolution IV non-isomorphic designs of 64 runs. This catalog can be used to generate all possible designs of 16 and 32 runs containing some important 2fi's. For a given model containing specified 2fi's, we select the best design by sequentially minimizing $N_{21}, N_{22}, N_{31}, N_{32}, \ldots, N_{m1}, N_{m2}$, where N_{j1} is the number of j-factor interactions aliased with the main effects, and N_{j2} the number of j-factor interactions aliased with the important 2fi's. Through our search effort, we have found all two-stage minimum N-aberration designs of 16 runs for $1 \leq k \leq 3$, and almost all two-stage minimum N-aberration designs of 32 runs for $1 \leq k \leq 3$, where k is the number of important 2fi's. In our search effort, we have used $(N_{21}, N_{22}, N_{31}, N_{32})$ instead of the entire vector $(N_{21}, N_{22}, \ldots, N_{m1}, N_{m2})$ to reduce the computing burden.

From (2.6) in Section 2.2, we notice that $N_{21} = 3A_3$. So the two-stage minimum N-aberration criterion firstly minimizes A_3 as the usual minimum aberration does. If a model containing specified 2fi's exists in the designs that have the smallest A_3, the best designs under the two-stage minimum N-aberration criterion must come from these designs. Hence we firstly search for the best design from the designs in the catalog with the smallest A_3. If the given model does not exist in these designs, we search for the best design from the designs in the catalog with the second smallest A_3, and so on. By this way the computing burden can be further reduced.

3.2 Two–Stage Minimum N–Aberration Designs of 16 and 32 Runs

Tables 1, 2, and 3 present two-stage minimum N-aberration designs of 16 runs for models with one, two, and three important 2fi's respectively. In Tables 2 and 3, the entries under "model" indicate which model is under consideration; for example, an entry of 2(a) denotes the model represented by Figure 2(a). The entries under "parent design" give the designs from which two-stage minimum N-aberration designs are found; the design labels from Chen, Sun, and Wu (1993) are used here. These parent designs can be reconstructed based on the information in Table 4, which provides the design columns for each design. Column j in Table 4 denotes the j-th column in the 16 run saturated design with its columns arranged in Yates order, which can be written as

$$(a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, a_1a_2a_3, a_4, a_1a_4, a_2a_4, a_1a_2a_4, a_3a_4, a_1a_3a_4, a_2a_3a_4, a_1a_2a_3a_4), \tag{3.1}$$

where a_1, a_2, a_3, and a_4 are four independent columns. The entries under "2fi's" tell how to assign the factors involved in the important 2fi's. The last column in these tables gives the confounding pattern $(N_{21}, N_{22}, N_{31}, N_{32})$. Tables 5, 6, and 7 present two-stage minimum N-aberration designs of 32 runs for models with one, two, and three important 2fi's respectively. The parent designs of 32 runs are given in Table 8.

3.3 An Illustrative Example

Suppose that in an experiment, the experimenter want to study six factors: *time, temperature, moisture, pressure, weight,* and *size.* She would like to use a two-level fractional factorial design of 16 runs. In addition to the main effects of these factors, she also wants to estimate the three 2fi's that are between *time* and *temperature*, between *time* and *moisture*, and between *time* and *pressure.* The graph for this model is 3(c) as in Figure 3. The two-stage minimum N-aberration design for this model can be found in Table 3 at the row for $m = 6$ and model 3(c). From this row in Table 3, we see that the parent design is design 6-2.1, which, according to Table 4, collects columns 1, 2, 4, 8, 7, and 11. To complete the specification of the two-stage minimum N-aberration design, the six factors need to be appropriately assigned to the six columns. The 2fi's column in Table 3 says that we should assign *time* to column 4, and assign *temperature, moisture,* and *pressure* to column 1, 2, and 8. Other factors can be arbitrarily assigned to the remaining columns. This design has $N_{21} = 0, N_{22} = 3$, meaning that no 2fi's outside the model are aliased with the main effects, but three 2fi's not in the model are aliased with the important 2fi's in the model.

4 Discussion

In selecting 2^{m-p} designs when some 2fi's are presumably important, the effort is focused on estimating the main effects and important 2fi's in the postulated model and on minimizing the bias caused by the other effects not included in the model. If we have taken the view that all the effects in the requirement set, consisting of the main effects and important 2fi's, are equally important, the minimum N-aberration criterion proposed by Ke and Tang (2003) can be used to sequentially minimize the bias. If we consider that the main effects need more protection than the important 2fi's, then the two-stage minimum N-aberration designs can be used to solve this problem. From (2.6) in Section 2.2, we know that $N_{21} = 3A_3$. Hence if the minimum aberration design proposed by Fries and Hunter (1980) is estimable under the given model and is the only one that minimizes A_3, the best design under the two-stage minimum N-aberration criterion must come from the minimum aberration design. Another interesting fact is that if the minimum N-aberration design comes from the minimum aberration design, then the two-stage minimum N-aberration design and the minimum N-aberration design is exactly the same. This can be explained as follows. In a usual minimum aberration design, $A_3, A_4, \ldots, A_m$ are minimized sequentially. Then $N_{21} = 3A_3, N_{31} = 4A_4, \ldots, N_{j1} = (j+1)A_{j+1} + (m-j+1)A_{j-1}, \ldots, N_{m1} = A_{m-1}$ are also minimized sequentially. If N_2 is minimized, then $N_{22} = N_2 - N_{21}$ can also be minimized given N_{21}. The same way we know that minimizing N_j means minimizing $N_{j2} = N_j - N_{j1}$ given N_{j1}. Hence the minimum N-aberration criterion and the two-stage minimum aberration criterion can be satisfied simultaneously.

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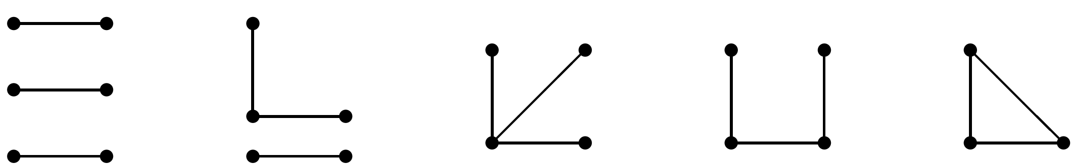
Figure 1. Graph for model with one 2-factor interaction.

(a) (b)

Figure 2. Graphs for models with two 2-factor interactions.



(a) (b) (c) (d) (e)

Figure 3. Graphs for models with three 2-factor interactions.

Table 1: Two-stage minimum N-aberration designs of 16 runs for the model containing one 2fi, as in Figure 1.

m	parent design	2fi	$(N_{21}, N_{22}, N_{31}, N_{32})$
5	5-1.1	$(1, 2)$	$(0, 0, 0, 1)$
6	6-2.1	$(1, 4)$	$(0, 1, 12, 0)$
7	7-3.1	$(1, 2)$	$(0, 2, 28, 0)$
8	8-4.1	$(1, 2)$	$(0, 3, 56, 0)$
9	9-5.1	$(2, 4)$	$(12, 3, 56, 4)$
10	10-6.1	$(2, 8)$	$(24, 3, 72, 8)$
11	11-7.1	$(2, 14)$	$(36, 3, 104, 13)$
12	12-8.1	$(1, 6)$	$(48, 5, 156, 16)$
13	13-9.1	$(2, 12)$	$(66, 5, 220, 22)$
14	14-10.1	$(1, 14)$	$(84, 6, 308, 28)$

Table 2: Two-stage minimum N-aberration designs of 16 runs for the models containing two 2fi's, as in Figure 2.

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
5	2(a)	5-1.1	$(1,2)(4,8)$	$(0,0,0,2)$
	2(b)	5-1.1	$(1,2)(1,4)$	$(0,0,0,2)$
6	2(a)	6-2.1	$(1,4)(2,8)$	$(0,2,12,0)$
	2(b)	6-2.1	$(1,4)(2,4)$	$(0,2,12,0)$
7	2(a)	7-3.1	$(1,2)(4,8)$	$(0,4,28,0)$
	2(b)	7-3.1	$(1,2)(1,4)$	$(0,4,28,0)$
8	2(a)	8-4.1	$(1,2)(4,8)$	$(0,6,56,0)$
	2(b)	8-4.1	$(1,2)(1,4)$	$(0,6,56,0)$
9	2(a)	9-5.1	$(2,4)(3,8)$	$(12,6,56,8)$
	2(b)	9-5.1	$(2,4)(3,4)$	$(12,6,56,8)$
10	2(a)	10-6.1	$(2,8)(3,14)$	$(24,6,72,16)$
	2(b)	10-6.1	$(2,8)(3,8)$	$(24,6,72,16)$
11	2(a)	11-7.1	$(1,14)(2,5)$	$(36,7,104,25)$
	2(b)	11-7.1	$(1,13)(2,13)$	$(36,7,104,25)$
12	2(a)	12-8.1	$(1,10)(2,5)$	$(48,10,156,32)$
	2(b)	12-8.1	$(1,6)(1,10)$	$(48,10,156,32)$
13	2(a)	13-9.1	$(2,13)(4,10)$	$(66,10,220,44)$
	2(b)	13-9.1	$(2,12)(3,12)$	$(66,10,220,44)$

Table 3: Two-stage minimum N-aberration designs of 16 runs for the models containing three 2fi's, as in Figure 3. (A row having entries "–" indicates the situation where the specified model does not exist for the given number of m factors.)

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
5	3(a)	–	–	–
	3(b)	5-1.1	$(1,2)(4,8)(4,15)$	$(0,0,0,3)$
	3(c)	5-1.1	$(1,2)(1,4)(1,8)$	$(0,0,0,3)$
	3(d)	5-1.1	$(1,4)(4,8)(8,2)$	$(0,0,0,3)$
	3(e)	5-1.1	$(1,2)(1,4)(2,4)$	$(0,0,0,3)$
6	3(a)	6-2.1	$(1,4)(2,8)(7,11)$	$(0,3,12,0)$
	3(b)	6-2.1	$(1,4)(2,8)(7,8)$	$(0,3,12,0)$
	3(c)	6-2.1	$(1,4)(2,4)(4,8)$	$(0,3,12,0)$
	3(d)	6-2.1	$(1,4)(4,8)(8,2)$	$(0,3,12,0)$
	3(e)	6-2.1	$(1,4)(1,8)(4,8)$	$(0,3,12,0)$
7	3(a)	7-3.1	$(1,4)(2,8)(7,11)$	$(0,6,28,0)$
	3(b)	7-3.1	$(1,2)(4,8)(7,8)$	$(0,6,28,0)$
	3(c)	7-3.1	$(1,2)(1,4)(1,7)$	$(0,6,28,0)$
	3(d)	7-3.1	$(1,4)(4,8)(8,2)$	$(0,6,28,0)$
	3(e)	7-3.1	$(1,2)(1,4)(2,4)$	$(0,6,28,0)$
8	3(a)	8-4.1	$(1,4)(2,8)(7,11)$	$(0,9,56,0)$
	3(b)	8-4.1	$(1,2)(4,8)(7,8)$	$(0,9,56,0)$
	3(c)	8-4.1	$(1,2)(1,4)(1,7)$	$(0,9,56,0)$
	3(d)	8-4.1	$(1,4)(4,8)(8,2)$	$(0,9,56,0)$
	3(e)	8-4.1	$(1,2)(1,4)(2,4)$	$(0,9,56,0)$
9	3(a)	9-5.1	$(2,4)(3,8)(5,9)$	$(12,9,56,12)$
	3(b)	9-5.1	$(2,4)(3,8)(5,8)$	$(12,9,56,12)$
	3(c)	9-5.1	$(2,4)(3,4)(4,8)$	$(12,9,56,12)$
	3(d)	9-5.1	$(2,4)(4,8)(8,3)$	$(12,9,56,12)$
	3(e)	9-5.1	$(2,4)(2,8)(4,8)$	$(12,9,56,12)$

Table 3 (*Continued*)

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
10	3(a)	10-6.1	$(2,9)(3,14)(4,8)$	$(24,9,72,24)$
	3(b)	10-6.1	$(4,8)(2,9)(3,9)$	$(24,9,72,24)$
	3(c)	10-6.1	$(2,8)(3,8)(4,8)$	$(24,9,72,24)$
	3(d)	10-6.1	$(2,14)(14,3)(3,8)$	$(24,9,72,24)$
	3(e)	10-6.1	$(2,5)(2,8)(5,8)$	$(24,10,72,24)$
11	3(a)	11-7.1	$(1,14)(2,9)(3,4)$	$(36,11,104,37)$
	3(b)	11-7.1	$(3,4)(1,13)(2,13)$	$(36,11,104,37)$
	3(c)	11-7.1	$(1,14)(2,14)(5,14)$	$(36,11,104,37)$
	3(d)	11-7.1	$(1,10)(10,5)(5,2)$	$(36,11,104,37)$
	3(e)	11-7.1	$(1,6)(1,10)(6,10)$	$(36,12,104,36)$
12	3(a)	12-8.1	$(1,13)(2,9)(3,4)$	$(48,15,156,48)$
	3(b)	12-8.1	$(1,6)(2,9)(5,9)$	$(48,15,156,48)$
	3(c)	12-8.1	$(1,6)(1,10)(1,13)$	$(48,15,156,48)$
	3(d)	12-8.2	$(4,10)(10,5)(5,8)$	$(51,12,152,51)$
	3(e)	12-8.1	$(1,6)(1,10)(6,10)$	$(48,15,156,48)$

Table 4: The parent designs in Tables 1, 2, and 3. Here each design contains the four independent columns 1, 2, 4, and 8 besides those additional columns.

m	parent design	additional columns
5	5-1.1	15
6	6-2.1	7 11
7	7-3.1	7 11 13
8	8-4.1	7 11 13 14
9	9-5.1	3 5 9 14 15
10	10-6.1	3 5 6 9 14 15
11	11-7.1	3 5 6 9 10 13 14
	11-7.3	3 5 6 7 9 10 11
12	12-8.1	3 5 6 9 10 13 14 15
	12-8.2	3 5 6 7 9 10 11 12
13	13-9.1	3 5 6 7 9 10 11 12 13
14	14-10.1	3 5 6 7 9 10 11 12 13 14

Table 5: Two-stage minimum N-aberration designs of 32 runs for the model containing one 2fi, as in Figure 1.

m	parent design	2fi	$(N_{21}, N_{22}, N_{31}, N_{32})$
6	6-1.1	$(1, 2)$	$(0, 0, 0, 0)$
7	7-2.1	$(1, 8)$	$(0, 0, 4, 1)$
8	8-3.1	$(1, 16)$	$(0, 0, 12, 2)$
9	9-4.1	$(1, 29)$	$(0, 0, 24, 4)$
10	10-5.1	$(1, 4)$	$(0, 1, 40, 4)$
11	11-6.1	$(2, 13)$	$(0, 2, 100, 0)$
12	12-7.1	$(1, 16)$	$(0, 3, 152, 0)$
13	13-8.1	$(1, 8)$	$(0, 4, 220, 0)$
14	14-9.1	$(1, 4)$	$(0, 5, 308, 0)$
15	15-10.1	$(1, 2)$	$(0, 6, 420, 0)$
16	16-11.1	$(1, 2)$	$(0, 7, 560, 0)$
17	17-12.1	$(2, 4)$	$(24, 7, 560, 8)$
18	18-13.1	$(2, 8)$	$(48, 7, 592, 16)$
19	19-14.1	$(2, 16)$	$(72, 7, 656, 24)$
20	20-15.1	$(2, 29)$	$(96, 7, 752, 32)$
21	21-16.2	$(2, 23)$	$(120, 7, 884, 42)$
22	22-17.1	$(1, 26)$	$(144, 7, 1052, 52)$
23	23-18.1	$(2, 28)$	$(168, 7, 1260, 63)$
24	24-19.1	$(1, 6)$	$(192, 11, 1512, 64)$
25	25-20.1	$(2, 12)$	$(228, 11, 1768, 76)$
26	26-21.1	$(2, 20)$	$(264, 11, 2072, 88)$
27	27-22.1	$(1, 30)$	$(300, 11, 2424, 101)$
28	28-23.1	$(1, 14)$	$(336, 13, 2828, 112)$
29	29-24.1	$(2, 28)$	$(378, 13, 3276, 126)$
30	30-25.1	$(1, 30)$	$(420, 14, 3780, 140)$

Table 6: Two-stage minimum N-aberration designs of 32 runs for the models containing two 2fi's, as in Figure 2.

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
6	2(a)	6-1.1	$(1,2)(4,8)$	$(0,0,0,0)$
	2(b)	6-1.1	$(1,2)(1,4)$	$(0,0,0,0)$
7	2(a)	7-2.1	$(1,8)(2,16)$	$(0,0,4,2)$
	2(b)	7-2.1	$(1,8)(2,8)$	$(0,0,4,2)$
8	2(a)	8-3.1	$(1,16)(2,29)$	$(0,0,12,4)$
	2(b)	8-3.1	$(1,16)(2,16)$	$(0,0,12,4)$
9	2(a)	9-4.2	$(1,16)(2,30)$	$(0,0,28,6)$
	2(b)	9-4.1	$(1,29)(2,29)$	$(0,0,24,8)$
10	2(a)	10-5.1	$(1,4)(2,8)$	$(0,2,40,8)$
	2(b)	10-5.1	$(1,4)(2,4)$	$(0,2,40,8)$
11	2(a)	11-6.1	$(2,21)(4,11)$	$(0,4,100,0)$
	2(b)	11-6.1	$(2,25)(4,25)$	$(0,4,100,0)$
12	2(a)	12-7.1	$(1,16)(2,21)$	$(0,6,152,0)$
	2(b)	12-7.1	$(1,16)(2,16)$	$(0,6,152,0)$
13	2(a)	13-8.1	$(1,8)(2,13)$	$(0,8,220,0)$
	2(b)	13-8.1	$(1,8)(2,8)$	$(0,8,220,0)$
14	2(a)	14-9.1	$(1,4)(2,8)$	$(0,10,308,0)$
	2(b)	14-9.1	$(1,4)(2,4)$	$(0,10,308,0)$
15	2(a)	15-10.1	$(1,2)(4,8)$	$(0,12,420,0)$
	2(b)	15-10.1	$(1,2)(1,4)$	$(0,12,420,0)$
16	2(a)	16-11.1	$(1,2)(4,8)$	$(0,14,560,0)$
	2(b)	16-11.1	$(1,2)(1,4)$	$(0,14,560,0)$
17	2(a)	17-12.1	$(2,4)(3,8)$	$(24,14,560,16)$
	2(b)	17-12.1	$(2,4)(3,4)$	$(24,14,560,16)$
18	2(a)	18-13.1	$(2,8)(3,14)$	$(48,14,592,32)$
	2(b)	18-13.1	$(2,8)(3,8)$	$(48,14,592,32)$

Table 6 (*Continued*)

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
19	2(a)	19-14.1	$(2,16)(3,22)$	$(72,14,656,48)$
	2(b)	19-14.1	$(2,16)(3,16)$	$(72,14,656,48)$
20	2(a)	20-15.1	$(2,29)(3,14)$	$(96,14,752,65)$
	2(b)	20-15.1	$(2,28)(3,28)$	$(96,14,752,65)$
21	2(a)	21-16.2	$(2,27)(3,22)$	$(120,14,884,84)$
	2(b)	21-16.2	$(2,27)(5,27)$	$(120,14,884,84)$
22	2(a)	22-17.1	$(1,29)(2,25)$	$(144,14,1052,104)$
	2(b)	22-17.1	$(1,26)(6,26)$	$(144,14,1052,104)$
23	2(a)	23-18.1	$(1,6)(2,28)$	$(168,17,1260,119)$
	2(b)	23-18.1	$(2,28)(3,28)$	$(168,17,1260,119)$
24	2(a)	24-19.1	$(1,10)(2,5)$	$(192,22,1512,128)$
	2(b)	24-19.1	$(1,6)(1,10)$	$(192,22,1512,128)$
25	2(a)	25-20.1	$(2,12)(3,20)$	$(228,22,1768,152)$
	2(b)	25-20.1	$(2,12)(3,12)$	$(228,22,1768,152)$
26	2(a)	26-21.1	$(2,20)(3,26)$	$(264,22,2072,176)$
	2(b)	26-21.1	$(2,20)(3,20)$	$(264,22,2072,176)$
27	2(a)	27-22.1	$(1,30)(2,13)$	$(300,23,2424,201)$
	2(b)	27-22.1	$(1,25)(6,25)$	$(300,23,2424,201)$
28	2(a)	28-23.1	$(1,22)(2,13)$	$(336,26,2828,224)$
	2(b)	28-23.1	$(1,14)(1,12)$	$(336,26,2828,224)$
29	2(a)	29-24.1	$(2,29)(4,26)$	$(378,26,3276,252)$
	2(b)	29-24.1	$(2,28)(3,28)$	$(378,26,3276,252)$

Table 7: Two-stage minimum N-aberration designs of 32 runs for the models containing three 2fi's, as in Figure 3.

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
6	3(a)	6-1.1	$(1,2)(4,8)(16,31)$	$(0,0,0,0)$
	3(b)	6-1.1	$(1,2)(4,8)(4,16)$	$(0,0,0,0)$
	3(c)	6-1.1	$(1,2)(1,4)(1,8)$	$(0,0,0,0)$
	3(d)	6-1.1	$(1,4)(4,8)(8,2)$	$(0,0,0,0)$
	3(e)	6-1.1	$(1,2)(1,4)(2,4)$	$(0,0,0,0)$
7	3(a)	7-2.1	$(1,8)(2,16)(4,27)$	$(0,0,4,3)$
	3(b)	7-2.1	$(1,8)(2,16)(4,16)$	$(0,0,4,3)$
	3(c)	7-2.1	$(1,8)(2,8)(4,8)$	$(0,0,4,3)$
	3(d)	7-2.1	$(1,16)(16,2)(2,8)$	$(0,0,4,3)$
	3(e)	7-2.1	$(1,8)(1,16)(8,16)$	$(0,0,4,4)$
8	3(a)	8-3.1	$(1,4)(2,16)(7,29)$	$(0,1,12,5)$
	3(b)	8-3.1	$(1,16)(2,29)(4,29)$	$(0,0,12,6)$
	3(c)	8-3.1	$(1,16)(2,16)(4,16)$	$(0,0,12,6)$
	3(d)	8-3.1	$(1,29)(29,2)(2,16)$	$(0,0,12,6)$
	3(e)	8-3.1	$(1,16)(1,29)(16,29)$	$(0,0,12,8)$
9	3(a)	9-4.1	$(1,4)(2,8)(7,29)$	$(0,2,24,8)$
	3(b)	9-4.2	$(1,16)(2,30)(4,30)$	$(0,0,28,9)$
	3(c)	9-4.1	$(1,29)(2,29)(4,29)$	$(0,0,24,12)$
	3(d)	9-4.2	$(1,30)(30,2)(2,16)$	$(0,0,28,9)$
	3(e)	9-4.2	$(1,16)(1,30)(16,30)$	$(0,0,28,13)$
10	3(a)	10-5.1	$(1,4)(2,8)(7,11)$	$(0,3,40,12)$
	3(b)	10-5.1	$(1,4)(2,8)(7,8)$	$(0,3,40,12)$
	3(c)	10-5.1	$(1,4)(2,4)(4,8)$	$(0,3,40,12)$
	3(d)	10-5.1	$(1,4)(4,8)(8,2)$	$(0,3,40,12)$
	3(e)	10-5.1	$(1,4)(1,8)(4,8)$	$(0,3,40,12)$

Table 7 (*Continued*)

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
11	3(a)	11-6.1	$(2,25)(4,19)(7,8)$	$(0,6,100,0)$
	3(b)	11-6.1	$(7,8)(2,25)(4,25)$	$(0,6,100,0)$
	3(c)	11-6.1	$(2,25)(4,25)(7,25)$	$(0,6,100,0)$
	3(d)	11-6.1	$(2,21)(21,11)(11,4)$	$(0,6,100,0)$
	3(e)	11-6.1	$(2,4)(2,25)(4,25)$	$(0,7,100,0)$
12	3(a)	12-7.1	$(1,19)(2,21)(4,16)$	$(0,9,152,0)$
	3(b)	12-7.1	$(4,11)(1,16)(2,16)$	$(0,9,152,0)$
	3(c)	12-7.1	$(1,16)(2,16)(4,16)$	$(0,9,152,0)$
	3(d)	12-7.1	$(1,21)(21,2)(2,16)$	$(0,9,152,0)$
	3(e)	12-7.1	$(1,14)(1,16)(14,16)$	$(0,9,152,0)$
13	3(a)	13-8.1	$(1,11)(2,13)(4,8)$	$(0,12,220,0)$
	3(b)	13-8.1	$(4,8)(1,11)(2,11)$	$(0,12,220,0)$
	3(c)	13-8.1	$(1,8)(2,8)(4,8)$	$(0,12,220,0)$
	3(d)	13-8.1	$(1,13)(13,2)(2,8)$	$(0,12,220,0)$
	3(e)	13-8.1	$(1,8)(1,16)(8,16)$	$(0,12,220,0)$
14	3(a)	14-9.1	$(1,4)(2,8)(7,11)$	$(0,15,308,0)$
	3(b)	14-9.1	$(1,4)(2,8)(7,8)$	$(0,15,308,0)$
	3(c)	14-9.1	$(1,4)(2,4)(4,8)$	$(0,15,308,0)$
	3(d)	14-9.1	$(1,4)(4,8)(8,2)$	$(0,15,308,0)$
	3(e)	14-9.1	$(1,4)(1,8)(4,8)$	$(0,15,308,0)$
15	3(a)	15-10.1	$(1,4)(2,8)(7,11)$	$(0,18,420,0)$
	3(b)	15-10.1	$(1,2)(4,8)(7,8)$	$(0,18,420,0)$
	3(c)	15-10.1	$(1,2)(1,4)(1,7)$	$(0,18,420,0)$
	3(d)	15-10.1	$(1,4)(4,8)(8,2)$	$(0,18,420,0)$
	3(e)	15-10.1	$(1,2)(1,4)(2,4)$	$(0,18,420,0)$

Table 7 (*Continued*)

m	model	parent design	2fi's	$(N_{21}, N_{22}, N_{31}, N_{32})$
16	3(a)	16-11.1	$(1,4)(2,8)(7,11)$	$(0,21,560,0)$
	3(b)	16-11.1	$(1,2)(4,8)(7,8)$	$(0,21,560,0)$
	3(c)	16-11.1	$(1,2)(1,4)(1,7)$	$(0,21,560,0)$
	3(d)	16-11.1	$(1,4)(4,8)(8,2)$	$(0,21,560,0)$
	3(e)	16-11.1	$(1,2)(1,4)(2,4)$	$(0,21,560,0)$
17	3(a)	17-12.1	$(2,8)(3,14)(4,15)$	$(24,21,560,24)$
	3(b)	17-12.1	$(4,16)(2,8)(3,8)$	$(24,21,560,24)$
	3(c)	17-12.1	$(2,4)(3,4)(4,8)$	$(24,21,560,24)$
	3(d)	17-12.1	$(2,4)(4,8(8,3)$	$(24,21,560,24)$
	3(e)	17-12.1	$(2,4)(2,8)(4,8)$	$(24,21,560,24)$
18	3(a)	18-13.1	$(1,6)(2,8)(3,14)$	$(48,22,592,48)$
	3(b)	18-13.1	$(1,6)(2,8)(3,8)$	$(48,22,592,48)$
	3(c)	18-13.1	$(2,8)(3,8)(4,8)$	$(48,21,592,48)$
	3(d)	18-13.1	$(2,14)(14,3)(3,8)$	$(48,21,592,48)$
	3(e)	18-13.1	$(2,8)(2,16)(8,16)$	$(48,21,592,48)$
19	3(a)	19-14.1	$(1,6)(2,16)(3,22)$	$(72,22,656,72)$
	3(b)	19-14.1	$(1,6)(2,16)(3,16)$	$(72,22,656,72)$
	3(c)	19-14.1	$(2,16)(3,16)(4,16)$	$(72,21,656,72)$
	3(d)	19-14.1	$(2,22)(22,3)(3,16)$	$(72,21,656,72)$
	3(e)	19-14.1	$(2,15)(2,16)(15,16)$	$(72,21,656,73)$
20	3(a)	20-15.1	$(1,6)(2,29)(3,14)$	$(96,22,752,97)$
	3(b)	20-15.1	$(1,6)(2,28)(3,28)$	$(96,22,752,97)$
	3(c)	20-15.1	$(2,29)(3,29)(4,29)$	$(96,21,752,98)$
	3(d)	20-15.1	$(2,28)(28,3)(3,14)$	$(96,21,752,98)$
	3(e)	20-15.1	$(2,5)(2,27)(5,27)$	$(96,22,752,98)$

Table 8: The parent designs in Tables 5, 6 and 7. Here each design contains the five independent columns 1, 2, 4, 8 and 16 besides those additional columns.

m	parent design	additional columns
6	6-1.1	31
7	7-2.1	7 27
8	8-3.1	7 11 29
9	9-4.1	7 11 19 29
	9-4.2	7 11 13 30
10	10-5.1	7 11 19 29 30
11	11-6.1	7 11 13 19 21 25
12	12-7.1	7 11 13 14 19 21 25
13	13-8.1	7 11 13 14 19 21 22 25
14	14-9.1	7 11 13 14 19 21 22 25 26
15	15-10.1	7 11 13 14 19 21 22 25 26 28
16	16-11.1	7 11 13 14 19 21 22 25 26 28 31
17	17-12.1	3 5 9 14 15 17 22 23 26 27 28 29
18	18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29
19	19-14.1	3 5 6 9 10 14 15 17 22 23 26 27 28 29
20	20-15.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29
21	21-16.1	3 5 6 9 10 13 14 15 17 18 22 23 26 27 28 29
22	22-17.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 29 30
23	23-18.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 27 28 29
24	24-19.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 27 28 29 30
25	25-20.1	3 5 6 7 9 10 11 12 13 17 18 19 20 21 26 27 28 29 30 31
26	26-21.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 26 27 28 29 30 31
27	27-22.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 22 25 26 27 28 29 30
28	28-23.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 22 25 26 27 28 29 30 31
29	29-24.1	3 5 6 7 9 10 11 12 13 14 15 17 18 19 20 21 22 23 24 25 26 27 28 29
30	30-25.1	3 5 6 7 9 10 11 12 13 14 15 17 18 19 20 21 22 23 24 25 26 27 28 29 30