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EXPECTATION IDENTITIES BASED ON RECURRENCE RELATIONS OF FUNCTIONS OF GENERALIZED ORDER STATISTICS

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SUMMARY

Some recurrence relations between expectation of function on single and joint generalized order statistics for a general class of distribution

$$1 - F(x) = \exp\Big[-\frac{1}{c}\{h(x) - h(\alpha)\}\Big], \quad \alpha < x < \beta$$

are obtained. Further, various deductions and particular cases are also discussed.

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1 Introduction

The concept of generalized order statistics (*gos*) was given by Kamps (1995), which is given as below:

Let $F(\cdot)$ be an absolutely continuous distribution function (*df*) with probability density function (*pdf*) $f(\cdot)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathcal{R}^{n-1}$, $M_r = \sum\limits_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called *gos* if their joint *pdf* is given by

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[1 - F(x_i)]^{m_i} f(x_i)\Big)[1 - F(x_n)]^{k-1} f(x_n)$$

on the cone $F^{-1}(0+) < x_1 \le x_2 \ldots \le x_n \le F^{-1}(1)$ of $\mathcal{R}^n$.

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n-i+1$; $i = 1, 2, \cdots, n$ i.e. $m_1 = m_2 = \cdots = m_{n-1} = 0$, $k = 1$), k^{th} record values ($\gamma_i = k$ i.e. $m_1 = m_2 = \cdots = m_{n-1} = -1$, $k \in N$), sequential order statistics ($\gamma_i = (n-i+1)\alpha_i$; $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = \alpha - i + 1$, $\alpha > 0$). Pfeifer's record values ($\gamma_i = \beta_i$; $\beta_1, \beta_2, \ldots, \beta_n > 0$) and progressive type II censored order statistics ($m_i \in N_0$, $k \in N$) are obtained [Kamps (1995), Kamps and Cramer (2001)].

Here we may consider two cases, which are:

$$\textbf{Case I} \quad : \quad m_i = m_j = m;\ i, j = 1, 2, \ldots, n-1,$$
$$\textbf{Case II} \quad : \quad \gamma_i \ne \gamma_j,\ i, j = 1, 2, \ldots, n-1.$$

For **Case I**, generalized order statistics (*gos*) will be denoted as $X(r, n, m, k)$ and its *pdf* is given by [Kamps, 1995]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [1-F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \le r < s \le n$, is

$$\begin{aligned} f_{X(r,n,m,k),\, X(s,n,m,k)}(x, y) &= \frac{C_{s-1}}{(r-1)!\ (s-r-1)!} [1-F(x)]^m\, g_m^{r-1}(F(x)) \\ &\quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [1-F(y)]^{\gamma_s - 1} f(x)\, f(y), \end{aligned}$$

where

$$\begin{aligned} C_{r-1} &= \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1) \\ h_m(x) &= \begin{cases} -\dfrac{1}{m+1} (1-x)^{m+1} & , \quad m \ne -1 \\ -\log(1-x) & , \quad m = -1 \end{cases} \\ g_m(x) &= h_m(x) - h_m(0), \quad x \in [0, 1). \end{aligned}$$

For **case II**, the *pdf* of $X(r, n, \tilde{m}, k)$ is [Kamps and Cramer, 2001]

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}\, f(x) \sum_{i=1}^{r} a_i(r) [1-F(x)]^{\gamma_i - 1}$$

and the joint *pdf* of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \le r < s \le n$ is

$$\begin{aligned} f_{X(r,n,\tilde{m},k),\, X(s,n,\tilde{m},k)}(x, y) &= C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i} \Big[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\Big] \\ &\quad \times \frac{f(x)}{(1-F(x))} \frac{f(y)}{(1-F(y))}, \end{aligned}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k+n-i+M_i,$$

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \le i \le s \le n.$$

Let the *df* of general form of distribution be

$$1 - F(x) = \exp\left[-\frac{1}{c}\{h(x) - h(\alpha)\}\right], \quad x \in (\alpha, \beta) \tag{1.1}$$

and corresponding *pdf* is

$$f(x) = \frac{h'(x)}{c} \exp\left[-\frac{1}{c}\{h(x) - h(\alpha)\}\right], \quad x \in (\alpha, \beta). \tag{1.2}$$

where c is a nonzero real constant, $h(x)$ is a monotonic and differentiable function of x in the interval (α, β) such that $F(\alpha) = 0$ and $F(\beta) = 1$. Thus in view of (1.1) and (1.2), we have

$$1 - F(x) = \frac{c}{h'(x)} f(x). \tag{1.3}$$

Here an attempt is made to unify earlier results for different distributions, which are discussed in the form of examples at the end.

2 Recurrence Relations for Function of GOS

Case I: $m_i = m_j = m;\ i \neq j = 1, 2 \ldots = n-1.$

Theorem 1. *For distribution given in (1.1) and* $n \in N$, $m = (m_1 = \cdots = m_{n-1}) \in \mathcal{R}$, $2 \le r \le n$, $k = 1, 2, \ldots,$

$$E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}] = \frac{c}{\gamma_r} E[\phi\{X(r,n,m,k)\}], \tag{2.1}$$

where $\phi(x) = \dfrac{\xi'(x)}{h'(x)}$.

Proof. From Athar and Islam (2004), we have

$$E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}] = \frac{C_{r-2}}{(r-1)!} \int_{\alpha}^{\beta} \xi'(x)[1-F(x)]^{\gamma_r}\, g_m^{r-1}(F(x))dx.$$

Now in view of (1.3), we get

$$E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n,m,k)\}]$$

$$= \frac{C_{r-2}}{(r-1)!}\int_{\alpha}^{\beta} \xi'(x)[1-F(x)]^{\gamma_r-1}\left\{\frac{c}{h'(x)}\,f(x)\right\} g_m^{r-1}(F(x))dx$$
$$= \frac{c}{\gamma_r}\,\frac{C_{r-2}}{(r-1)!}\int_{\alpha}^{\beta} \phi(x)[1-F(x)]^{\gamma_r-1}\, g_m^{r-1}(F(x))f(x)dx$$

and hence the result. □

Remark 2.1: Recurrence relation for single moments of order statistics (at $m=0$, $k=1$) is

$$E[\xi(X_{r:n})] \;=\; E[\xi(X_{r-1:n})] + \frac{c}{(n-r+1)}\,E[\phi(X_{r:n})]$$

as obtained by Ali and Khan (1997).

Remark 2.2: The recurrence relation for single moments of k^{th} record values will be

$$E[\xi(X_r^{(k)})] \;=\; E[\xi(X_{r-1}^{(k)})] + \frac{c}{k}\,E[\phi(X_r^{(k)})],$$

where $X_r^{(k)}$, $r=1,2,\ldots$ is r^{th} k records.

Remark 2.3: For $m=0$ and $k=\alpha-n+1$, $\alpha\in\mathcal{R}_+$, we obtain the recurrence relation for single moments of order statistics with non-integral sample size as

$$E[\xi(X_{r:\alpha})] \;=\; E[\xi(X_{r-1:\alpha})] + \frac{c}{(\alpha-r+1)}\,E[\phi(X_{r:\alpha})]$$

Remark 2.4: For $m=\alpha-1$, $k=\alpha$, the recurrence relation for sequential order statistics is

$$E[\xi\{X(r,n,\alpha-1,\alpha)\}]$$
$$= E[\xi\{X(r-1,n,\alpha-1,\alpha)\}] + \frac{c}{\alpha(n-r+1)}\,E[\phi\{X(r-1,n,\alpha-1,\alpha)\}].$$

Theorem 2. *Under the conditions as stated in Theorem 1*

(i) $E[\xi\{X(r-1,n,m,k)\}] - E[\xi\{X(r-1,n-1,m,k)\}]$
$$= -\frac{c(m+1)(r-1)}{\gamma_1\gamma_r}\,E[\phi\{X(r,n,m,k)\}].$$

(ii) $E[\xi\{X(r,n,m,k)\}] - E[\xi\{X(r-1,n-1,m,k)\}]$
$$= \frac{c}{\gamma_1}\,E[\phi\{X(r,n,m,k)\}].$$

Proof. Results can be established in view of Athar and Islam (2004) and (1.3).

Case II: $\gamma_i \neq \gamma_j$; $i\neq j = 1,2,\ldots,n-1$. □

Theorem 3. *For distribution given in (1.1) and* $n\in N$, $\tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in\mathcal{R}^{n-1}$, $\tilde{m}^*=(m_2,m_3,\ldots,m_{n-1})\in\mathcal{R}^{n-2}$, $2\le r\le n$, $k=1,2,\ldots,$

(i) $E[\xi\{X(r,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n,\tilde{m},k)\}] \;=\; \frac{c}{\gamma_r}\,E[\phi\{X(r,n,\tilde{m},k)\}].$

(ii) $E[\xi\{X(r-1,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}]$

$$= \; -\frac{c}{\gamma_1\gamma_r}\left\{(r-1)+\sum_{j=1}^{r-1} m_j\right\} E[\phi\{X(r,n,\tilde{m},k)\}].$$

(iii) $E[\xi\{X(r,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n-1,\tilde{m}^*,k)\}] \; = \; \frac{c}{\gamma_1} E[\phi\{X(r,n,\tilde{m},k)\}].$

Proof. Proof is easy. □

Remark 2.5: Theorems 1 and 2 can be deduced from Theorem 3 by replacing $\tilde{m}$ with m, $m \neq -1$.

2.1 Examples

Recurrence relations for moments of generalized order statistics for some selected distributions are provided below:

2.1.1 Burr Distribution

$$\bar{F}(x) \; = \; \left(\frac{\beta}{\beta+x^\tau}\right)^\lambda, \quad 0 < x < \infty.$$

Here we have

$$h(x) \; = \; ln(\beta+x^\tau), \quad c = \frac{1}{\lambda} > 0, \quad \xi(x) = x^{j+\tau}.$$

Then $\phi(x) \; = \; \frac{(j+\tau)}{\tau}(\beta x^j + x^{j+\tau}).$

Therefore from (2.1), we have

$$E[X^{j+\tau}(r,n,m,k)] \; = \; \frac{\beta(j+\tau)}{\lambda\gamma_r\tau-(j+\tau)} E[X^j(r,n,m,k)] + \frac{\lambda\gamma_r\tau}{\lambda\gamma_r\tau-(j+\tau)} E[X^{j+\tau}(r-1,n,m,k)]$$

as obtained by Pawlas and Szynal (2001).

2.1.2 Power Function Distribution

$$F(x) \; = \; x^p, \quad 0 < x \leq 1.$$

We have

$$h(x) \; = \; -\,ln(1-x^p), \quad c = 1, \quad \xi(x) = x^{j+1},$$

then $\phi(x) \; = \; \frac{(j+1)}{p}(x^{j+1-p} - x^{j+1}).$

Therefore from (2.1), we get

$$E[X^{j+1}(r,n,m,k)] \; = \; \frac{(j+1)}{p\gamma_r+(j+1)} E[X^{j+1-p}(r,n,m,k)] + \frac{p\gamma_r}{p\gamma_r+(j+1)} E[X^{j+1}(r-1,n,m,k)]. \tag{2.2}$$

At $p = 1$ (2.2) reduces to

$$E[X^{j+1}(r,n,m,k)] = \frac{(j+1)}{\gamma_r + (j+1)} E[X^j(r,n,m,k)] + \frac{\gamma_r}{\gamma_r + (j+1)} E[X^{j+1}(r-1,n,m,k)],$$

which is recurrence relation for uniform distribution on (0,1) as obtained by Pawlas and Szynal (2001).

2.1.3 Pareto Distribution

$$F(x) \;=\; 1 - a^p x^{-p}, \quad a \le x < \infty.$$

We have

$$h(x) \;=\; -\ln x, \quad c = \frac{1}{p}, \quad \xi(x) = x^{j+1},$$

then $\phi(x) \;=\; (j+1)x^{j+1}$.

Therefore from (2.1), we have

$$E[X^{j+1}(r,n,m,k)] \;=\; \frac{p\gamma_r}{p\gamma_r - (j+1)} E[X^{j+1}(r-1,n,m,k)].$$

2.1.4 Weibull Distribution

$$F(x) \;=\; 1 - e^{-\theta x^p}, \quad 0 \le x < \infty, \;\; p, \theta > 0.$$

We have $h(x) \;=\; x^p$, $c \;=\; \frac{1}{\theta}$, $\xi(x) = x^{j+1}$, then $\phi(x) \;=\; \frac{(j+1)}{p} x^{j+1-p}$.

Now in view of (2.1), we get

$$E[X^{j+1}(r,n,m,k)] - E[X^{j+1}(r-1,n,m,k)] \;=\; \frac{(j+1)}{p\theta\gamma_r} E[X^{j+1-p}(r,n,m,k)]. \qquad (2.3)$$

Further at $p = 1$ and $\theta = 1$, (2.3) becomes

$$E[X^{j+1}(r,n,m,k)] - E[X^{j+1}(r-1,n,m,k)] \;=\; \frac{(j+1)}{\gamma_r} E[X^j(r,n,m,k)].$$

which is the recurrence relation for moments of generalized order statistics from standard exponential distribution as given by Pawlas and Szynal (2001). Similarly recurrence relations for moments of generalized order statistics for some other distributions may be obtained with proper choice of $h(x)$ and c.

3 Recurrence Relations for Functions of Two GOS

Case I: $m_i = m_j = m$; $i \neq j = 1, 2, \ldots = n-1$.

Theorem 4. *For distribution given in (1.1) and* $n \in N$, $m = (m_1 = m_2 = \cdots = m_{n-1}) \in \mathcal{R}$, $1 \le r < s \le n-1$, $k = 1, 2, \ldots$,

$$E[\xi\{X(r,n,m,k),X(s,n,m,k)\}]-E[\xi\{X(r,n,m,k),X(s-1,n,m,k)\}]$$
$$=\frac{c}{\gamma_s}E[\phi\{X(r,n,m,k),X(s,n,m,k)\}], \quad (3.1)$$

where $\phi(x,y)=\dfrac{\frac{\partial}{\partial y}\xi(x,y)}{h'(y)}$, $\xi(x,y)=\xi_1(x)\xi_2(y)$.

Proof. can be established in view of Athar and Islam (2004) and (1.3). □

Remark 3.1: Under the assumption given in Theorem 4 with $k=1$, $m=0$, we get the recurrence relations for product moments of order statistics

$$E[\xi(X_{r:n},X_{s:n})]-E[\xi(X_{r:n},X_{s-1:n})] = \frac{c}{(n-s+1)}E[\phi(X_{r:n},X_{s:n})]$$

as obtained by Ali and Khan (1998).

Remark 3.2: At $m=-1$; we have the recurrence relation for product moments of k^{th} record values

$$E[\xi\{X(r,n,-1,k),X(s,n,-1,k)\}]-E[\xi\{X(r,n,-1,k),X(s-1,n,-1,k)\}]$$

$$=\frac{c}{k}E[\phi\{X(r,n,-1,k),X(s,n,-1,k)\}].$$

Case II: $\gamma_i\neq\gamma_j$; $i\neq j=1,2,\ldots,n-1$.

Theorem 5. *For distribution given in (1.1) and for* $n\in N$, $\tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in\mathcal{R}$, $1\leq r<s\leq n-1$, $k=1,2,\ldots$,

$$E[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}]-E[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}]$$

$$=\frac{c}{\gamma_s}E[\phi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}].$$

Proof. Proof follows on the lines of Theorem 4. □

3.1 Examples

Recurrence relations for product moments of generalized order statistics for some selected distributions are provided below:

3.1.1 Burr Distribution

$$\bar{F}(x) = \left(\frac{\beta}{\beta + x^{\tau}}\right)^{\lambda}, \quad 0 < x < \infty.$$

We have,

$$h(x) = ln(\beta + x^{\tau}), \quad c = \frac{1}{\lambda} > 0, \quad \xi_1(x) = x^i, \quad \xi_2(y) = y^{j+\tau}, \quad s = r + 1.$$

Thus $\phi(x, y) = \dfrac{\beta(j+\tau)}{\tau} x^i y^j + \dfrac{(j+\tau)}{\tau} x^i y^{j+\tau}$.
Therefore from (3.1), we have

$$E[X^i(r,n,m,k)X^{j+\tau}(r+1,n,m,k)] = \frac{(j+\tau)\beta}{\lambda\gamma_{r+1}\tau - (j+\tau)} E[X^i(r,n,m,k)X^j(r+1,n,m,k)]$$

$$+ \frac{\lambda\gamma_{r+1}\tau}{\lambda\gamma_{r+1}\tau - (j+r)} E[X^{i+j+\tau}(r,n,m,k)]$$

as obtained by Pawlas and Szynal (2001).

3.1.2 Power Function Distribution

$$F(x) = x^p, \quad 0 < x \le 1.$$

We have

$$h(x) = -ln(1 - x^p), \quad c = 1, \quad \xi_1(x) = x^i, \quad \xi_2(y) = y^{j+1},$$

then $\phi(x, y) = \dfrac{(j+1)}{p} (x^i y^{j+1-p} - x^i y^{j+1})$.

Therefore from (3.1), we get

$$E[X^i(r,n,m,k)X^{j+1}(r+1,n,m,k)] = \frac{(j+1)}{p\gamma_{r+1} + (j+1)} E[X^i(r,n,m,k)X^{j+1-p}(r+1,n,m,k)]$$

$$+ \frac{p\gamma_{r+1}}{p\gamma_{r+1} + (j+1)} E[X^{i+j+1}(r,n,m,k)].$$

3.1.3 Pareto Distribution

$$F(x) = 1 - a^p x^{-p}, \quad a \le x < \infty.$$

We have

$$h(x) = ln\, x, \quad c = \frac{1}{p}, \quad \xi_1(x) = x^i, \quad \xi_2(y) = y^j.$$

Then $\phi(x, y) = j\, x^i\, y^j$.

Thus in view of (3.1)

$$E[X^i(r,n,m,k)X^j(r+1,n,m,k)] = \frac{p\,\gamma_{r+1}}{(p\,\gamma_{r+1} - j)} E[X^{i+j}(r,n,m,k)].$$

3.1.4 Weibull Distribution

$$F(x) \;=\; 1 - e^{-\theta x^p}, \quad 0 \le x < \infty, \;\; p, \theta > 0.$$

We have $h(x) = x^p$, $c = \dfrac{1}{\theta}$, $\xi_1(x) = x^i$, $\xi_2(y) = y^{j+1}$ and $\phi(x,y) = \dfrac{(j+1)}{p}\, x^i\, y^{j+1-p}$.
Therefore

$$E[X^i(r,n,m,k)X^{j+1}(r+1,n,m,k)] = \frac{(j+1)}{p\,\theta\,\gamma_{r+1}}\, E[X^i(r,n,m,k)X^{j+1-p}(r+1,n,m,k)] + E[X^{i+j+1}(r,n,m,k)].$$

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References

[1] Ali, M.A. and Khan, A.H. (1997): Recurrence relations for the expectations of a function of single order statistics from general class of distributions. *J. Indian Statist. Assoc.*, **35**, 1–9.

[2] Ali, M.A. and Khan, A.H. (1998): Recurrence relations for expected values of certain functions of two order statistics. *Metron*, **LVII(2)**, 107–119.

[3] Athar, Haseeb and Islam, H.M. (2004): Recurrence relations between single and product moments of generalized order statistics from a general class of distributions. *Metron*, **LXII(3)**, 327–337.

[4] Kamps, U. (1995): A concept of generalized order statistics. *J. Statist. Plann. Inference*, **48**, 1–23.

[5] Kamps, U. and Cramer, E. (2001): On distribution of generalized order statistics. *Statistics*, **35**, 269–280.

[6] Pawlas, P. and Szynal, D. (2001): Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions. *Commun. Statist.-Theory Meth.*, **30(4)**, 739–746.