Journal of Statistical Research *ISSN 0256 - 422 X*
2008, Vol. 42, No. 1, pp. 71-74
Bangladesh

EXPLICIT EXPRESSIONS FOR MOMENTS OF GENERALIZED GAMMA ORDER STATISTICS

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SUMMARY

Explicit closed form expressions for moments of order statistics from the generalized gamma distribution are derived using Lauricella function of type A.

Keywords and phrases: Gamma distribution; Generalized gamma distribution; Lauricella function of type A; Order statistics.

AMS Classification: 33C90; 62E99.

1 Introduction

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the generalized gamma distribution given by the probability density function (pdf):

$$f(x) = \frac{c x^{c\alpha - 1} \exp\left(-x^c\right)}{\Gamma(\alpha)} \tag{1.1}$$

for $x > 0$, $\alpha > 0$ and $c > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. There has been some work relating to moments of the generalized gamma order statistics $X_{r:n}$, see e.g. Khan and Khan (1983) and Thomas (1996). However, all of the work that we are aware of express $E(X_{r:n}^k)$ in terms of recurrence relations, numerical tables or multiple infinite series of the form

$$E\left(X_{r:n}^k\right) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} A\left(m_1, \ldots, m_j, r, n, k\right) \tag{1.2}$$

for some function $A(\cdot)$ and $j = j(r, n, k)$. These representations have little practical appeal because, for example, the computation of the j–fold infinite sum in (1.2) will become prohibitive as j gets large. Even for moderate j, one would have to code in (1.2) and this will be a waste of man power as well as computer time.

In this note, for the first time, we derive expressions for $E(X_{r:n}^k)$ that are finite sums of a well known special function – namely, the Lauricella function of type A (Exton, 1978) defined by

$$
\begin{aligned}
& F_A^{(n)}\left(a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right) \\
= & \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{a_{m_1+\cdots+m_n}\left(b_1\right)_{m_1} \cdots \left(b_n\right)_{m_n}}{\left(c_1\right)_{m_1} \cdots \left(c_n\right)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!},
\end{aligned} \tag{1.3}
$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Numerical routines for the direct computation of (1.3) are widely available, see e.g. Mathematica and Exton (1978).

2 Explicit Expression for $E(X_{r:n}^k)$

If $X_1, X_2, \ldots, X_n$ is a random sample from (1.1) then it is well known that the pdf of $Y = X_{r:n}$ is given by

$$
f_Y(y) = \frac{n!}{(r-1)!(n-r)!} \left\{F(y)\right\}^{r-1} \left\{1 - F(y)\right\}^{n-r} f(y)
$$

for $r = 1, 2, \ldots, n$, where $F(\cdot)$ is the cumulative distribution function (cdf) corresponding to (1.1) given by

$$
F(y) = \frac{\gamma\left(\alpha, y^c\right)}{\Gamma(\alpha)},
$$

where $\gamma(\cdot, \cdot)$ denotes the incomplete gamma function defined by

$$
\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \exp\left(-t\right) dt.
$$

Thus, the kth moment of $X_{r:n}$ can be expressed as

$$\begin{aligned}
E\left(X_{r:n}^{k}\right) &= \frac{cn!}{(r-1)!(n-r)!\left\{\Gamma(\alpha)\right\}^{n}} \\
&\quad \times \int_{0}^{\infty} y^{k+c\alpha-1}\exp\left(-y^{c}\right)\left\{\gamma\left(\alpha,y^{c}\right)\right\}^{r-1}\left\{\Gamma(\alpha)-\gamma\left(\alpha,y^{c}\right)\right\}^{n-r}dy \\
&= \frac{cn!}{(r-1)!(n-r)!\left\{\Gamma(\alpha)\right\}^{n}}\int_{0}^{\infty} y^{k+c\alpha-1}\exp\left(-y^{c}\right) \\
&\quad \times \sum_{l=0}^{n-r}\binom{n-r}{l}\left\{\Gamma(\alpha)\right\}^{n-r-l}(-1)^{l}\left\{\gamma\left(\alpha,y^{c}\right)\right\}^{r+l-1}dy \\
&= \frac{cn!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}\left\{\Gamma(\alpha)\right\}^{-r-l} \\
&\quad \times \int_{0}^{\infty} y^{k+c\alpha-1}\exp\left(-y^{c}\right)\left\{\gamma\left(\alpha,y^{c}\right)\right\}^{r+l-1}dy \\
&= \frac{cn!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}\left\{\Gamma(\alpha)\right\}^{-r-l}I(l). \qquad (2.1)
\end{aligned}$$

Using the series expansion

$$\gamma(\alpha,x) = x^{\alpha}\sum_{m=0}^{\infty}\frac{(-x)^{m}}{(\alpha+m)m!},$$

the integral $I(l)$ in (2.1) can be expressed as

$$\begin{aligned}
I(l) &= \int_{0}^{\infty} y^{k+c\alpha-1}\exp\left(-y^{c}\right)\left\{y^{c\alpha}\sum_{m=0}^{\infty}\frac{\left(-y^{c}\right)^{m}}{(\alpha+m)m!}\right\}^{r+l-1}dy \\
&= \int_{0}^{\infty}\sum_{m_{1}=0}^{\infty}\cdots\sum_{m_{r+l-1}=0}^{\infty}\frac{(-1)^{m_{1}+\cdots+m_{r+l-1}}y^{k+c\alpha(r+l)+c(m_{1}+\cdots+m_{r+l-1})-1}\exp\left(-y^{c}\right)}{(\alpha+m_{1})\cdots(\alpha+m_{r+l-1})m_{1}!\cdots m_{r+l-1}!}dy \\
&= \sum_{m_{1}=0}^{\infty}\cdots\sum_{m_{r+l-1}=0}^{\infty}\frac{(-1)^{m_{1}+\cdots+m_{r+l-1}}}{(\alpha+m_{1})\cdots(\alpha+m_{r+l-1})m_{1}!\cdots m_{r+l-1}!} \\
&\quad \times \int_{0}^{\infty} y^{k+c\alpha(r+l)+c(m_{1}+\cdots+m_{r+l-1})-1}\exp\left(-y^{c}\right)dy \\
&= \sum_{m_{1}=0}^{\infty}\cdots\sum_{m_{r+l-1}=0}^{\infty}\frac{(-1)^{m_{1}+\cdots+m_{r+l-1}}\Gamma\left(k/c+\alpha(r+l)+m_{1}+\cdots+m_{r+l-1}\right)}{c(\alpha+m_{1})\cdots(\alpha+m_{r+l-1})m_{1}!\cdots m_{r+l-1}!}. \qquad (2.2)
\end{aligned}$$

Using the fact $(f)_k = \Gamma(f+k)/\Gamma(f)$ and the definition in (1.3), one can reexpress (2.2) as

$$\begin{aligned}
I(l) &= c^{-1}\alpha^{1-r-l}\Gamma\left(k/c+\alpha(r+l)\right) \\
&\quad \times F_{A}^{(r+l-1)}\left(k/c+\alpha(r+l),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right). \qquad (2.3)
\end{aligned}$$

Combining (2.1) and (2.3), we obtain the expression

$$\begin{aligned} E\left(X_{r:n}^{k}\right) &= \frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}\left\{\Gamma(\alpha)\right\}^{-r-l}\alpha^{1-r-l}\Gamma\left(k/c+\alpha(r+l)\right) \\ &\quad \times F_{A}^{(r+l-1)}\left(k/c+\alpha(r+l),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right) \end{aligned} \tag{2.4}$$

Note that (2.4) is a finite sum of the Lauricella function of type A.

3 Conclusions

We have derived an expression for moments of the generalized gamma order statistics as a finite sum of a well known special function. This expression is simpler than previously known work.

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