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A NOTE ON DOUBLE K-CLASS ESTIMATORS UNDER ELLIPTICAL SYMMETRY

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SUMMARY

In this paper, estimation of the regression vector parameter in the multiple regression model $y = \boldsymbol{X}\beta + \epsilon$ is considered, when the error term belongs to the class of elliptically contoured distributions (ECD), say, $\epsilon \sim EC_n(0, \sigma^2\boldsymbol{V}, \psi)$, where σ^2 is unknown and $\boldsymbol{V}$ is a symmetric p.d known matrix with the characteristic generator ψ. It is well-known that UMVU estimator of β has the form $(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}y$. In this paper using integral series representation of ECDs, the dominance conditions of double k-class estimators given by

$$\hat{\beta}_{k_1,k_2} = \left[1 - \frac{k_1\hat{\epsilon}'\boldsymbol{V}^{-1}\hat{\epsilon}}{y'y - k_2\hat{\epsilon}'\boldsymbol{V}^{-1}\hat{\epsilon}}\right]\hat{\beta}$$

over UMVUE, have been derived under weighted quadratic loss function.

Keywords and phrases: Double k-class estimators, Elliptically contoured distributions, Inverse Laplace transform, Weight function, Quadratic loss function.

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1 Introduction

Following Stein (1956) and James and Stein (1961), constructing improved estimators over minimum variance unbiased estimators (MVUEs) under normal theory, there are a lot of works involving biased estimators to make different types of dominant estimators under many sorts of statistical models. For the review on the recent results, see Brandwein and Strawderman (1991), Lehmann and Casella (1998), and Saleh (2006).

Ullah and Ullah (1978) proposed double k-class (DKC) estimators that under a mild condition, dominated the MVUE of a regression vector parameter in a multiple regression model, under normal assumptions. After that, Singh (1991) provided some conditions for demonstrating the superiority of DKC estimators upon MVUE of the regression vector parameter using multivariate Student's t distribution with the scale matrix $\sigma^2 I_n$. No work

has been done so far under the assumption of elliptical symmetry.
To deal with such situations, consider the multiple regression model given by

$$y = \mathbf{X}\beta + \epsilon \tag{1.1}$$

where y is an n-vector of responses, $\mathbf{X}$ is an $n \times p$ matrix with full rank p, $\beta = (\beta_1, \cdots, \beta_p)'$ is p-vector of regression coefficients and $\epsilon = (\epsilon_1, \cdots, \epsilon_n)'$ is the n-vector of errors distributed according to the law belonging to the class of elliptically contoured distributions (ECDs), denoted by $EC_n(0, \sigma^2\boldsymbol{V}, \psi)$ for a symmetric p.d. known matrix $\mathbf{V}$ with the following characteristic function

$$\phi_\epsilon(t) = \psi\left(\frac{\sigma^2}{2}t'\boldsymbol{V}t\right) \tag{1.2}$$

for some functions $\psi : [0, \infty) \to \Re$ say characteristic generator (Fang et al., 1990).

If ϵ has a density, then it can be represented as

$$f(\epsilon) = d_n|\sigma^2\mathbf{V}|^{-\frac{1}{2}}\boldsymbol{g}_n\left[\frac{1}{2\sigma^2}\epsilon'\boldsymbol{V}^{-1}\epsilon\right]$$

for a normalizing constant d_n and for some function $\boldsymbol{g}_n(.)$ say density generator satisfying the following condition

$$\int_0^\infty x^{\frac{n}{2}-1}\boldsymbol{g}_n(x)dx < \infty$$

Then we will use the notation $\epsilon \sim EC_n(0, \sigma^2\boldsymbol{V}, \boldsymbol{g}_n)$. In addition, if $\boldsymbol{g}_n$ does not depend on n, we use the notation $\boldsymbol{g}$ instead.
Then for $\sigma_e^2 = -2\sigma^2\psi'(0)$ we have

$$E(\epsilon) = 0, \; E(\epsilon'\epsilon) = -2\sigma^2\psi'(0)\boldsymbol{V} = \sigma_e^2\mathbf{V},$$

provided $|\psi'(0)| < \infty$.

Some of the well-known members of the class of ECDs are the multivariate normal, Pearson Type VII, multivariate Student's t, multivariate Cauchy, Pearson Type II, multivariate logistic distribution, multivariate Laplace distribution, multivariate Kotz type distribution and Generalized Slash. For more details on ECDs see Fang et al. (1990) and Gupta and Varga (1993).

We need the following essential Lemma for the proofs of the main results of this approach.

Lemma 1.1 (Chu, 1973). *If z is a n-dimensional elliptically contoured random vector with mean equal to θ and scale matrix Σ and density function $h(z)$, then, under some regularity conditions, there exists a scalar function $w(t)$ defined on $(0, \infty)$ such that*

$$h(z) \;\; = \;\; \int_0^\infty w(t)\phi_N(z)dt,$$

where $\phi_N(z)$ denotes the density function of $N_n(\theta, t^{-1}\Sigma)$, and

$$w(t) \;\; = \;\; (2\pi)^{n/2}|\Sigma|^{1/2}t^{-n/2}\mathcal{L}^{-1}(f(s)),$$

$\mathcal{L}^{-1}(f(s))$ denotes the inverse Laplace transform of $f(s)$ with $f(s) = h(z)$ when $s = \frac{z'\Sigma^{-1}z}{2}$.

Distribution	$h(s)$	$w(t)$
Multivariate Normal	$\frac{\|V\|^{-1/2}e^{-s}}{(2\pi)^{n/2}}$	$\delta(t)$
Multivariate Pearson Type VII	$\frac{\Gamma(m)\|V\|^{-1/2}}{(q\pi)^{n/2}\Gamma(m-n/2)}$ $\times(1+2s/q)^{-m}$	$\frac{t^{m-n/2-1}e^{-qt/2}}{(q/2)^{n/2-m}\Gamma(m-n/2)}$
Multivariate Exponential Power	$\frac{k\Gamma(n/2)\|V\|^{-1/2}r^{\frac{n}{2k}}e^{-rs}}{(2\pi)^{n/2}\Gamma(n/2k)}$	$\delta(t-r)$
Multivariate Student-t with ν d.f.	$\frac{\nu^{\nu/2}\Gamma((\nu+n)/2)\|V\|^{-1/2}}{\pi^{n/2}\Gamma(\nu/2)}$ $\times(\nu+2s)^{-(\nu+n)/2}$	$\frac{\nu(\nu t/2)^{\nu/2-1}e^{-\nu t/2}}{2\Gamma(\nu/2)}$
Multivariate Laplace	$\frac{\Gamma(n/2)\|V\|^{-1/2}e^{-\sqrt{2}s}}{2\pi^{n/2}\Gamma(n)}$	$\delta(t-\sqrt{2})$
Generalized Slash	$\frac{\nu s^{-n/2-\nu}\|V\|^{-1/2}}{(2\pi)^{n/2}}$ $\times[\Gamma(n/2+\nu)-\Gamma(n/2+\nu,s)]$	$\nu t^{\nu-1}$
Multivariate Kotz type	$\frac{q^{m-1+n/2}\Gamma(n/2)\|V\|^{-1/2}}{\pi^{n/2}\Gamma(m-1+n/2)}$ $(2s)^{m-1}e^{-2qs}$	$\frac{(2q)^{m-1+n/2}\|V\|^{-1/2}\Gamma(n/2)}{\Gamma(m-1+n/2)}$ $t^{-n/2}\delta^{(m-1)}(t-2q)$

Table 1: Some mixtures by weighting function

For details on the properties of Laplace transform and its inverse see Debnath and Bhatta (2007).

On integrating $h(z)$ over $\Re^n$, $w(t)$ integrates to 1. Thus for nonnegative function $w(t)$, it is a density. Some explicit representations of $h(.)$ and $w(.)$ for $s = z'\Sigma^{-1}z/2$ are given in Table 1. Here $\delta(.)$ is the unit impulse function or the Dirac delta function with the following properties

1. $\int_0^\infty \delta(t)dt = 1$
2. $\int_{-\infty}^\infty f(x)\delta(x)dx = f(0)$ for every Borel measurable function $f(\cdot)$.

Also $\delta^{(m)}(t)$ denotes the mth derivative of $\delta(t)$ w.r.t t. See Gupta and Varga (1995) and Cheong (1999) for some applications of Lemma 1.1.

2 DKC Estimators

In this section, we define the double k-class estimators.

By direct computations, it can be obtained that the MVUE of β is given by

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}y = G_1\boldsymbol{X}'y, \qquad G_1 = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1} \tag{2.1}$$

Also the LSE of σ^2 is

$$\tilde{\sigma}^2 = \frac{1}{n}(y-\boldsymbol{X}\tilde{\beta})'\boldsymbol{V}^{-1}(y-\boldsymbol{X}\tilde{\beta})$$

Under the assumption $\epsilon \sim EC_n(0, \sigma^2 \boldsymbol{V}, \boldsymbol{g})$, using 2.1 we get $\hat{\beta} \sim EC_n(\beta, \sigma^2 G_1, \boldsymbol{g})$ which leads

$$E(\hat{\beta} - \beta)'(\hat{\beta} - \beta) = -2\sigma^2 \Psi'(0) tr(G_1) = \sigma_\epsilon^2 tr(G_1).$$

Now consider the DKC estimators with the following structure

$$\hat{\beta}_{k_1,k_2} = \left[1 - \frac{k_1 \hat{\epsilon}' \boldsymbol{V}^{-1} \hat{\epsilon}}{y'y - k_2 \hat{\epsilon}' \boldsymbol{V}^{-1} \hat{\epsilon}}\right] \hat{\beta}. \tag{2.2}$$

Ullah and Ullah (1978) utilized such estimators for the model (1.1) assuming $\epsilon \sim N_n(o, \sigma^2 I_n)$. Theil (1971), Bock (1975) and Judge and Bock (1976) considered the DKC estimators taking $k_2 = 1$ under normal theory. Singh (1991) extended the results by taking $\epsilon \sim Mt(0, \sigma^2 I_n, \nu)$, multivariate Student's t distribution with mean vector 0 and variance-covariance structure $\frac{\sigma^2 \nu}{\nu - 2} I_n$ for known degrees of freedom ν. Considering DKC estimators given by (2.2) by the assumption $\epsilon \sim EC(0, \sigma^2 \boldsymbol{V}, \boldsymbol{g})$ in the model (1.1), has the following advantages comparing to all the works have done yet.

1. The broader class of distributions is considered which contains heavier/lighter tail alternatives to the multivariate normal model such as multivariate normal and Student's t types.
2. Based on the proof of Theorem 1, we can conclude that the dominance conditions of DKCEs over MVUE, is robust under normal theory and it does not change by departure from normality.
3. Comparing to the method used in Singh (1991), he applied the scale mixture representation of multivariate Student's t distribution, but we use the integral of a set of multivariate normal densities to obtain the results.
4. Taking the general structure $\boldsymbol{V}$ in variance-covariance matrix, enables to consider correlated variables.

3 Main Results

In this section, using weighted quadratic loss function, we provide conditions on k_1 and k_2 under which $\hat{\beta}_{k_1,k_2}$ given by (2.2) dominate $\hat{\beta}$ in the sense that $R(\hat{\beta}_{k_1,k_2}; \beta) < R(\hat{\beta}; \beta)$ denoted by $\hat{\beta}_{k_1,k_2} \succ \hat{\beta}$.

In this approach, we use $R(\hat{\beta}; \beta) = E_\beta[L(\hat{\beta}; \beta)]$ for the weighted quadratic loss given by

$$L(\hat{\beta}; \beta) = (\hat{\beta} - \beta)' \boldsymbol{W} (\hat{\beta} - \beta), \tag{3.1}$$

where $\boldsymbol{W}$ is a symmetric p.d. known matrix.

Theorem 1. *Suppose in the multiple regression model (1.1), $\epsilon \sim EC_n(0, \sigma^2 \boldsymbol{V}, \boldsymbol{g})$, where σ^2 is unknown and $\boldsymbol{V}$ is a p.d. known matrix. Also assume $\mathcal{L}^{-1}(f(s))$ for $s = \frac{\epsilon' \boldsymbol{V}^{-1} \epsilon}{2\sigma^2}$*

is nonnegative. Then $\hat{\beta}_{k_1,k_2} \succ \hat{\beta}$ *w.r.t the weighted quadratic loss function given by (3.1) provided* $k_2 \leq 1$ *and*

$$0 < k_1 \leq \frac{2(\mathcal{F}-2)}{m+2}, \quad \mathcal{F} = \frac{tr[G_1\boldsymbol{W}]}{\lambda_1[G_1\boldsymbol{W}]}, \; m = n-p,$$

where $\lambda_1(G_1\boldsymbol{W})$ *is the largest eigenvalue of* $G_1\boldsymbol{W}$.

Proof. Consider in the model (1.1), $\epsilon \sim EC_n(0, \sigma^2\boldsymbol{V}, \boldsymbol{g})$, then using Lemma 1.1, the density of ϵ can be expressed as

$$h_\epsilon(z) \quad = \quad \int_0^\infty w_*(t)\phi_{N^*}(z)dt,$$

where ϕ_{N^*} denotes the density of $N(0, \sigma^2 t^{-1}\boldsymbol{V})$ and

$$w_*(t) = (2\pi)^{n/2}\sigma^n|V|^{1/2}t^{-n/2}\mathcal{L}^{-1}(f(s)), \tag{3.2}$$

with $f(s) = h(z)$ when $s = \frac{z'\boldsymbol{V}^{-1}z}{2\sigma^2}$ for nonnegative $\mathcal{L}^{-1}(f(s))$. Then we have

$$\begin{aligned} R(\hat{\beta};\beta) - R(\hat{\beta}_{k_1,k_2};\beta) &= E_{EC}[(\hat{\beta}-\beta)'\boldsymbol{W}(\hat{\beta}-\beta) - (\hat{\beta}_{k_1,k_2}-\beta)'\boldsymbol{W}(\hat{\beta}_{k_1,k_2}-\beta)] \\ &= \int_0^\infty w_*(t)E_{N^*}[(\hat{\beta}-\beta)'\boldsymbol{W}(\hat{\beta}-\beta) \\ &\quad -(\hat{\beta}_{k_1,k_2}-\beta)'\boldsymbol{W}(\hat{\beta}_{k_1,k_2}-\beta)], \end{aligned} \tag{3.3}$$

where E_{EC} and E_{N^*} denote taking expectation w.r.t elliptical and normal assumptions respectively.

Then in order to prove $R(\hat{\beta};\beta) - R(\hat{\beta}_{k_1,k_2};\beta) > 0$, it is enough to show that

$$D_{N^*} \quad = \quad E_{N^*}[(\hat{\beta}-\beta)'\boldsymbol{W}(\hat{\beta}-\beta)] - E_{N^*}[(\hat{\beta}_{k_1,k_2}-\beta)'\boldsymbol{W}(\hat{\beta}_{k_1,k_2}-\beta)] > 0$$

Now let

$$z^* = \sigma^{-1}(tG_1)^{\frac{1}{2}}\hat{\beta}, \quad \Delta = \sigma^{-1}(tG_1)^{\frac{1}{2}}\beta$$
$$v = \frac{\hat{\epsilon}'\boldsymbol{V}^{-1}\hat{\epsilon}}{\sigma^2 t^{-1}}, \; u = \frac{v}{z^{*\prime}z^* + (1-k_2)v}.$$

In this case we obtain

$$\begin{aligned} \hat{\beta}_{k_1,k_2} - \beta &= \hat{\beta} - \beta - \frac{k_1\hat{\epsilon}'\boldsymbol{V}^{-1}\hat{\epsilon}}{y'\boldsymbol{V}^{-1}y - k_2\hat{\epsilon}'\hat{\epsilon}} \\ &= \sigma(tG_1)^{-\frac{1}{2}}[(z^*-\Delta) - k_1 u z^*] \end{aligned}$$

Similar to Srivastava and Chaturvedi (1985), one can get

$$\begin{aligned} D_{N^*} &= \sigma^2 t^{-1}\Big\{2k_1 E_{N^*}[u(Z^*-\Delta)'G_1^{\frac{1}{2}}\boldsymbol{W}G_1^{\frac{1}{2}}z^*] \\ &\quad -k_1^2 E_{N^*}[u^2 z^{*\prime}G_1^{\frac{1}{2}}\boldsymbol{W}G_1^{\frac{1}{2}}z^*]\Big\} \end{aligned} \tag{3.4}$$

Then using Stein lemma, we obtain

$$\begin{aligned} E_{N^*}[u(z^*-\Delta)' G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*] &= E_{N^*}\left[\frac{\partial(uG_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*)}{\partial z^{*\prime}}\right] \\ &= E_{N^*}[utr(G_1\boldsymbol{W})] \\ &\quad -E_{N^*}[2u^2 v^{-1} z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*] \end{aligned} \tag{3.5}$$

Also by direct computations similar to Singh (1991), we get

$$\begin{aligned} E_{N^*}[u^2 z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*] &= (m+2)E_{N^*}\left[\frac{v z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*}{[z^{*\prime} z^* + (1-k_2)v]^2}\right] \\ &\quad -4(1-k_2)E_{N^*}\left[\frac{u^2 z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*}{z^{*\prime} z^* + (1-k_2)v}\right] \\ &< (m+2)E_{N^*}\left[\frac{u^2}{v z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^*}\right] \end{aligned} \tag{3.6}$$

Then, using (3.4), (3.5) and (3.6) we have

$$\begin{aligned} & E_{N^*}[(\hat{\beta}-\beta)' \boldsymbol{W} (\hat{\beta}-\beta) - (\hat{\beta}_{k_1,k_2}-\beta)' \boldsymbol{W} (\hat{\beta}_{k_1,k_2}-\beta)] \\ \geq\ & t^{-1}\{2k_1 tr(G_1\boldsymbol{W}) - [4k_1 + (m+2)k_1^2]\lambda_p(G_1\boldsymbol{W})\}E_{N^*}(u) > 0, \end{aligned}$$

provided $0 < k_1 \leq \frac{2(\mathcal{F}-2)}{m+2}$, since

$$uv^{-1} z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^* \leq (z^{*\prime} z^*)^{-1} z^{*\prime} G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}} z^* \leq \lambda_1(G_1^{\frac{1}{2}} \boldsymbol{W} G_1^{\frac{1}{2}}).$$

Singh (1991) proposed the same theorem changing $\mathcal{F}$ with $d = \frac{tr[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{W}]}{\lambda_p[G_1\boldsymbol{W}]}$ under multivariate Student's t model. As it can be obtained, we will be in reach of the same result as in Theorem 1 under normal assumptions. Because the density generator $\boldsymbol{g}$ plays no role in obtaining the dominance conditions using Lemma 1.1.Therefore we can conclude the dominance conditions given in Theorem 1 are robust under normal theory.

Corollary 3.1. Taking $k_1 = n^{-1}\psi'(0)$ and $k_2 = 1 - k_1$ in (2.2), $\hat{\beta}_{k_1,k_2} \succ \hat{\beta}$ provided

$$\mathcal{F} \geq \frac{(m+2)\psi'(0)}{2n} + 2.$$

For the proof consider that from Theorem 1, $\hat{\beta}_{k_1,k_2} \succ \hat{\beta}$ for all $k_2 \leq 1$ and for all $0 < k_1 \leq \frac{\mathcal{F}_0 - 2}{m+2}$, where $\mathcal{F}_0 = \lambda_i^{-1} \sum \lambda_i > 2$ and the result follows directly.

Corollary 3.2. Ignoring the term $O\left(\delta_*^{-3}\right)$ for $\delta_* = \beta' G_1^{-1}\beta$, we have

$$R(\hat{\beta};\beta) - R(\hat{\beta}_{k_1,k_2};\beta) = \mathcal{D}, \tag{3.7}$$

where

$$\begin{aligned}\mathcal{D} &= \frac{mk_1\wp_2}{\delta_*}\left[2tr(G_1\boldsymbol{W}) - \frac{[(m+2)k_1+4]\beta'\boldsymbol{W}\beta}{\delta_*}\right] \\ &+\frac{mk_1\wp_3}{\delta_*}\left[\frac{2tr(G_1\boldsymbol{W})[(2k_2-k_1)(m+2)-2n]}{\delta_*}\right. \\ &\left.+\frac{8[(m+2)k_1+4]tr(\beta\boldsymbol{W}\beta')}{\delta_*^2}\right],\end{aligned}$$

where

$$\wp_i = \int_0^\infty t^{-i}w_*(t)dt \ \text{ and } \ w_*(t) = (2\pi)^{n/2}\sigma^n|V|^{1/2}t^{-n/2}\mathcal{L}^{-1}(f(s)). \tag{3.8}$$

Proof. Using equations (8) and (14) and the proof of Theorem 3 from Singh (1991), by some direct computations we can obtain

$$\begin{aligned}R(\hat{\beta};\beta) - R(\hat{\beta}_{k_1,k_2};\beta) &= \int_0^\infty \frac{mk_1t^{-2}w_*(t)}{\delta_*}\left\{tr(G_1\boldsymbol{W})\right. \\ &\times\left(1+\frac{t^{-1}[(2k_2-k_1)(m+2)-2n]}{\delta_*}\right) \\ &\left.-\left[\frac{\beta'\boldsymbol{W}\beta}{\delta_*} - \frac{8t^{-1}tr(\beta\boldsymbol{W}\beta')}{\delta_*^2}\right]\times[(m+2)k_1+4]\right\} \\ &= \frac{mk_1\wp_2}{\delta_*}\left[2tr(G_1\boldsymbol{W}) - \frac{[(m+2)k_1+4]\beta'\boldsymbol{W}\beta}{\delta_*}\right] \\ &+\frac{mk_1\wp_3}{\delta_*}\left[\frac{2tr(G_1\boldsymbol{W})[(2k_2-k_1)(m+2)-2n]}{\delta_*}\right. \\ &\left.+\frac{8[(m+2)k_1+4]tr(\beta\boldsymbol{W}\beta')}{\delta_*^2}\right]\end{aligned}$$

4 General Remarks

In this paper, we obtained the dominance conditions under which DKC estimators perform better than the UMVUE of regression coefficient vector. We assumed that the error term has elliptically contoured distribution. We used the inverse Laplace transform to obtain the main results. But it is important to consider the following notes

1. Based on Lemma 1.1, we derived the results for the case $\epsilon \sim EC_n(0, \sigma^2\boldsymbol{V}, \boldsymbol{g})$. Those are the same for the case $\epsilon \sim EC_n(0, \sigma^2\boldsymbol{V}, \boldsymbol{g}_n)$.

2. Involving Lemma 1.1, the inverse Laplace transform of $h(.)$ exists if the following conditions are satisfied.

   (a) $h(t)$ is differentiable when t is sufficiently large.

   (b) $h(t) = o(t^{-m})$ as $t \to \infty$, $m > 1$.

However, it is rather difficult to calculate the inverse Laplace transform of some functions, we can handle it for many density generators of elliptical densities. See Debnath and Bhatta (2007) for more details.

3. Theorem 1 is significantly true even for some elliptical distributions that inverse Laplace transform do not exist. Because the density generator $\boldsymbol{g}$ ($\boldsymbol{g}_n$) plays no role in obtaining the sufficient conditions of the Theorem 1 and those are robust under normal assumptions.

4. Involving Corollary 3.2, there exist four method to determine the $\wp_i$, $i = 1, 2, 3$ as follows.

   (a) Partial fraction decomposition.

   (b) Convolution theorem.

   (c) Contour integral of the Laplace inverse integral.

   (d) Heaviside's expansion theorem.

   See Debnath and Bhatta (2007) for more details.

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