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STATISTICAL THEORY FOR THE PARAMETERS WITH STUDENT's t-DISTRIBUTION

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SUMMARY

Consider the location model $\mathbf{Y} = \theta\mathbf{1}_n + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon}$ is distributed n-dimensional Student's t-distribution. We study the properties of the estimators of θ and variance σ_ϵ^2 and compare them with normal theory estimators. In addition, we present some results which help to obtain risks of preliminary test related estimators.

Keywords and phrases: Location-Scale Mode; MSE; Preliminary Test; Relative Efficiency; Risk Function; Shrinkage Estimator

AMS Classification: 62F03,62F10

1 Introduction

In the wake of increasing criticism on the inappropriate use of the normal distribution to model the errors, there is a growing trend to use, often more appropriate, Student-t model. Fisher (1956, p. 133) warned against the consequences of inappropriate use of the traditional normal model. He (1960, p. 46) also analyzed Darwin's data (cf. Box and Taio, 1992, p. 133) by using a non-normal model. Later, Fraser and Fick (1975) analyzed the same data by the Student-t model and Zellner (1976) provided both Bayesian and frequentist analyses of the multiple regression model with Student-t errors. Fraser (1979) illustrated the robustness of the Student-t model. Prucha and Kelegian (1984) proposed an estimating equation for the simultaneous equation model with the Student-t errors. Ullah and Walsh (1984) investigated the optimality of different types of tests used in econometric studies for the multivariate Student-t model. The interested readers may refer to the more recent work of Singh (1988), Lange et al. (1989), Giles (1990, 1991), Anderson (1993), Spanos (1994) and Lucus (1997) for different applications of the Student-t models. For a wide range of applications of the Student-t models refer to Lange et al. (1989).

There have been many studies in the area of the 'improved' estimation following the seminal work of Bancroft (1944) and later Han and Bancroft (1968). They developed the preliminary test estimator that uses uncertain non-sample prior information (not in the form of prior distributions), in addition to the sample information. Stein (1956) dominates the usual maximum likelihood estimators under the squared error loss function. In a series of papers Saleh and Sen (1978, 1985) explored the preliminary test approach to Stein-rule estimation. Many authors have contributed to this area, notably Sclove et al. (1972), Judge and Bock (1978), Stein (1981), Maatta and Casella (1990) and Khan and Saleh (1995, 1997). The recent book of Saleh (2006) presents a comprehensive discussion of this area.

The object of this paper is to present a review of the developments with some new materials in the estimation of the variance of the error distribution in a location model.

Consider the response vector $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ such that it satisfies the model

$$\mathbf{Y} = \theta \mathbf{1}_n + \boldsymbol{\epsilon}, \quad \mathbf{1}_n = (1, \ldots, 1)', \quad \text{a vector n-tuples of 1's,} \tag{1.1}$$

where $\boldsymbol{\epsilon}$ is distributed as n-dimensional Student's t-variable with pdf

$$f_n(\boldsymbol{\epsilon}) = \frac{\Gamma(\frac{n+\nu_0}{2})}{(\pi\nu_0)^{n/2}\Gamma(\frac{\nu_0}{2})\sigma^n|\mathbf{V}_n|^{1/2}} \left(1 + \frac{1}{\nu_0\sigma^2}\boldsymbol{\epsilon}'\mathbf{V}_n\boldsymbol{\epsilon}\right)^{-\frac{1}{2}(n+\nu_0)} \tag{1.2}$$

having the positive-definite matrix $\mathbf{V}_n$. The mean and covariance matrix of $\boldsymbol{\epsilon}$ are $\mathbf{0}$ and $\sigma_\epsilon^2\mathbf{V}_n$ where $\sigma_\epsilon^2 = \frac{\nu_0\sigma^2}{\nu_0-2}$, $\nu_0 > 2$. This distribution may be obtained as a mixture of $\mathcal{N}(0, \tau^2\mathbf{V}_n)$ and the inverse gamma distribution, $IG(\tau^2, \nu_0\sigma^2)$ given by

$$\omega(\tau^2) = \frac{1}{\Gamma(\frac{\nu_0}{2})}\left(\frac{\nu_0\sigma^2}{2\tau^2}\right)^{\frac{\nu_0}{2}} \exp\left(-\frac{\nu_0\sigma^2}{2\tau^2}\right)\tau^{-2}. \tag{1.3}$$

The n-dimensional Student's t-distribution will be denoted by $\mathcal{M}_t^{(n)}(\mathbf{0}, \sigma^2\mathbf{V}_n, \nu_0)$. It is easy to show that

$$E(\tau^2) = \frac{\nu_0\sigma^2}{\nu_0 - 2} = \sigma_\epsilon^2 \text{ (say)} \tag{1.4}$$

and

$$E(\tau^{2h}) = \left(\frac{\nu_0\sigma^2}{2}\right)^h \frac{\Gamma(\frac{\nu_0}{2} - h)}{\Gamma(\frac{\nu_0}{2})} \text{ for } h = 0, 1, 2, \ldots. \tag{1.5}$$

We discuss the estimation θ and σ_ϵ^2 and test of hypothesis problem in the next section.

2 Estimation of θ and σ_ϵ^2 and Test of Hypothesis

To obtain the unbiased estimators of θ and σ_ϵ^2 we use the LS method and minimize

$$(\mathbf{Y} - \theta\mathbf{1}_n)'\mathbf{V}_n^{-1}(\mathbf{Y} - \theta\mathbf{1}_n) \tag{2.1}$$

w.r.t. θ to obtain the unrestricted estimators

$$\tilde{\theta}_n = (\mathbf{1}_n'\mathbf{V}_n^{-1}\mathbf{1}_n)^{-1}(\mathbf{1}_n'\mathbf{V}_n^{-1}\mathbf{Y}) \tag{2.2}$$

and

$$S_u^2 = \frac{1}{m}(\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n)' \mathbf{V}_n^{-1}(\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n), \quad m = n-1 \tag{2.3}$$

as the **unbiased** estimators of θ and σ_ϵ^2 respectively. Further, the variances of $\tilde{\theta}_n$ and S_u^2 are given by

$$\mathrm{Var}(\tilde{\theta}_n) = \frac{\sigma_\epsilon^2}{K_1}; \quad K_1 = (\mathbf{1}_n' \mathbf{V}_n^{-1} \mathbf{1}_n) \tag{2.4}$$

and

$$\mathrm{Var}(S_u^2) = \frac{2\sigma_\epsilon^4 (m + \nu_0 - 2)}{m(\nu_0 - 4)}; \quad \nu_0 > 4 \tag{2.5}$$

respectively.

For the test of the null hypothesis $H_0 : \theta = \theta_0$ vs. $H_A : \theta \neq \theta_0$, we use the LR test and obtain

$$\lambda = \frac{L_0}{L_A} = a \left[\frac{S_u^2}{\tilde{\sigma}_\epsilon^2}\right]^{\frac{n}{2}} = a \left[\frac{(\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n)' \mathbf{V}_n^{-1}(\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n)}{(\mathbf{Y} - \theta_0 \mathbf{1}_n)' \mathbf{V}_n^{-1}(\mathbf{Y} - \theta_0 \mathbf{1}_n)}\right] \tag{2.6}$$

with $a = \left(\frac{n}{n-1}\right)^{\frac{n}{2}} \left(\frac{n-1+\nu_0}{n+\nu_0}\right)^{\frac{1}{2}(n+\nu_0)}$. Now, we have

$$\begin{aligned} (\mathbf{Y} - \theta_0 \mathbf{1}_n)' \mathbf{V}_n^{-1}(\mathbf{Y} - \theta_0 \mathbf{1}_n) &= (\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n)' \mathbf{V}_n^{-1}(\mathbf{Y} - \tilde{\theta}_n \mathbf{1}_n) + K_1(\tilde{\theta}_n - \theta_0)^2 \\ &= m S_u^2 + K_1(\tilde{\theta}_n - \theta_0)^2. \end{aligned} \tag{2.7}$$

Hence (2.6) reduces to

$$\lambda = a \left(\frac{1}{1 + \frac{1}{m}\mathcal{L}_n}\right) \quad \text{where} \quad \mathcal{L}_n = K_1 \frac{(\tilde{\theta}_n - \theta_0)^2}{S_u^2} \tag{2.8}$$

giving the LR-test as

$$\mathcal{L}_n = K_1 \frac{(\theta - \theta_0)^2}{S_u^2}. \tag{2.9}$$

The following theorem gives the distributions of $\tilde{\theta}_n$, S_u^2 and $\mathcal{L}_n$.

Theorem 1. *If* $\boldsymbol{\epsilon} \sim \mathcal{M}_t^{(n)}(\mathbf{0}, \sigma^2 V_n, \nu_0)$ *and* $\mathbf{Y} = \theta \mathbf{1}_n + \boldsymbol{\epsilon}$, *then the* **distribution** *of* (a) $\tilde{\theta}_n$ *is*

$$\frac{\Gamma(\frac{1+\nu_0}{2})}{(\pi\nu_0)^{\frac{1}{2}}\Gamma(\frac{\nu_0}{2})} \left(1 + \frac{K_1}{\nu_0 \sigma^2}(\tilde{\theta}_n - \theta)^2\right)^{-1/2(1+\nu_0)}, \tag{2.10}$$

Hence, $E(\tilde{\theta}_n) = \theta$ *and* $Var(\tilde{\theta}_n) = \frac{\sigma_\epsilon^2}{K_1}$*. Similarly, the distribution of* (b) S_u^2 *is*

$$\frac{(S_u^2)^{\frac{1}{2}m-1}}{B\left(\frac{\nu_0}{2}, \frac{m}{2}\right)(\nu_0\sigma^2)^{\frac{m}{2}}}\left(1 + \frac{S_u^2}{\nu_0\sigma^2}\right)^{\frac{1}{2}(m+\nu_0)} \tag{2.11}$$

and of (c) $\mathcal{L}_n$*, is given by the pdf*

$$g_{1,m}(\mathcal{L}_n; \Delta^2, \nu_0) = \sum_{r\geq 0} K_r^{(0)}(\Delta^2)\frac{(\frac{1}{m})^{\frac{1}{2}(1+2r)}\mathcal{L}_n^{r-\frac{1}{2}}}{B(\frac{1+2r}{2}, \frac{m}{2})(1+\frac{1}{m}\mathcal{L}_n)^{\frac{1}{2}(1+2r+\nu_0)}} \tag{2.12}$$

where

$$K_r^{(0)}(\Delta^2) = \frac{\Gamma\left(\frac{\nu_0+2r}{2}\right)}{\Gamma(r+1)\Gamma\left(\frac{\nu_0}{2}\right)} \frac{\left(\frac{\Delta^2}{\nu_0-2}\right)^r}{\left(1+\frac{\Delta^2}{\nu_0-2}\right)^{\frac{1}{2}(1+2r+\nu_0)}} \tag{2.13}$$

and

$$\Delta^2 = \frac{K_1(\theta-\theta_0)^2}{\sigma_\epsilon^2}. \tag{2.14}$$

The cdf of $\mathcal{L}_n$ *is then given by*

$$G_{1,m}(c_\alpha; \Delta^2) = \sum_{r\geq 0} K_r^{(0)}(\Delta^2) I_x\left(\frac{1}{2}(1+2r); \frac{m}{2}\right), \quad x = \frac{c_\alpha}{m+c_\alpha}. \tag{2.15}$$

Proof. For (a) we first find the distribution of $\tilde{\theta}_n$ assuming $\boldsymbol{\epsilon}$ to have $\mathcal{N}_n(0, \tau^2\mathbf{V}_n)$ so that $(\tilde{\theta}_n - \theta)$ follows the distribution $\mathcal{N}(0, \frac{\tau^2}{K_1})$. Taking expectation w.r.t. the IG$(\tau^2, \nu_0\sigma^2)$, we obtain the pdf given at (2.10).

For (b) again we find the distribution of $\frac{mS_u^2}{\tau^2}$ under $\mathcal{N}(0, \tau^2)$ which follows the chi-square distribution with m d.f. Taking expectation w.r.t. IG$(\tau^2, \nu_0\sigma^2)$ we obtain (2.11).

For (c), we note that under normal theory as in (b) above,

$$\mathcal{L}_n = \frac{K_1(\tilde{\theta}_n - \theta_0)^2}{S_u^2} \tag{2.16}$$

follows the non-central F-distribution with (1,m) d.f. and non-centrality parameter $\Delta_{\tau^2}^2 = \frac{K_1(\theta-\theta_0)^2}{\tau^2}$. Integrating τ^2 w.r.t. IG$(\tau^2, \nu_0\sigma^2)$ we obtain (2.12). Similarly we obtain (2.15) for the c.d.f. of $\mathcal{L}_n$.

In addition to $\tilde{\theta}_n$ and S_u^2, we present a few more estimators of θ and σ_ϵ^2. First, we consider the case when it is apriori suspected that θ **may be** equal to θ_0. In this case, following Bancroft and Han (1968) and Saleh (2006), we define the estimators given below:
(i) restricted estimator (RE) of θ is $\tilde{\theta}_n^{RE} = \tilde{\theta}_n - k_0(\tilde{\theta}_n - \theta_0)$, $0 < k_0 < 1$.

(ii) preliminary test estimator (PTE) of θ is given by

$$\tilde{\theta}_n^{PT} = \tilde{\theta}_n - k_0(\tilde{\theta}_n - \theta_0)I(\mathcal{L}_n < c_\alpha). \tag{2.17}$$

(iii) shrinkage type estimator (SE) of θ is given by

$$\tilde{\theta}_n^{S} = \tilde{\theta}_n - \frac{c_0 k_0(\tilde{\theta}_n - \theta_0)S_u}{\sqrt{K_1}|\tilde{\theta}_n - \theta_0|}, \quad c_0 > 0,\ 0 < k_0 < 1 \tag{2.18}$$

where $I(A)$ is the indicator function of the set A and c_α is the α-level critical value of the F-distribution with (1,m) d.f.

For the estimation of σ_ϵ^2, we consider the following:
(i) the unrestricted estimator of σ_ϵ^2 is S_u^2
(ii) restricted estimator of σ_ϵ^2 is defined by $(m+1)S_R^2 = mS_u^2 + K_1(\tilde{\theta}_n - \theta_0)^2$. Further, the best invariant estimators of σ_ϵ^2 are given by

$$\text{(iii)} \quad \tilde{\sigma}_\epsilon^2 = \frac{mS_u^2}{m+2} \qquad \text{(iv)} \quad \hat{\sigma}_\epsilon^2 = \frac{(m+1)S_R^2}{n+3}. \tag{2.19}$$

Let c_α be the α-level critical value of the F-distribution with (1,m) d.f. then we define three more **preliminary test** estimators of σ_ϵ^2

$$\text{(v)} \qquad S_{PT[1]}^2 = \Psi_1(\mathcal{L}_n)m\, S_u^2 \tag{2.20}$$

$$\text{(vi)} \qquad S_{PT[2]}^2 = \Psi_2(\mathcal{L}_n)mS_u^2 \tag{2.21}$$

and

$$\text{(vii)} \qquad S_{[s]}^2 = \Psi_s(\mathcal{L}_n)mS_u^2 \tag{2.22}$$

where

$$\Psi_1(\mathcal{L}_n) = \frac{1}{m}I(\mathcal{L}_n \geq c_\alpha) + \frac{(1+\frac{1}{m}\mathcal{L}_n)}{m+1}I(\mathcal{L}_n < c_\alpha), \tag{2.23}$$

$$\Psi_2(\mathcal{L}_n) = \frac{1}{m+2}I(\mathcal{L}_n \geq c_\alpha) + \frac{(1+\frac{1}{m}\mathcal{L}_n)}{m+3}I(\mathcal{L}_n < c_\alpha), \tag{2.24}$$

and

$$\Psi_s(\mathcal{L}_n) = \frac{1}{m+2}I\Big(\mathcal{L}_n \geq \frac{m}{m+2}\Big) + \frac{(1+\frac{1}{m}\mathcal{L}_n)}{m+3}I\Big(\mathcal{L}_n < \frac{m}{m+2}\Big), \tag{2.25}$$

respectively. □

In the next section we obtain some theorems needed to obtain the bias and MSE expressions of these estimators.

3 Some Theorems for Bias and MSE Calculations.

This section contains some lemmas and theorems for the calculation of bias and MSE expressions for the estimators. We begin with the following theorem:

Theorem 2. *If Z follows $M_t^{(1)}(\theta, \sigma^2, \nu_0)$ and ϕ is a measurable function of Z^2, then*
(i) *the distribution of Z^2 is given by*

$$\begin{aligned} h_1(\chi^2(\Delta^2)) &= \sum_{r\geq 1} K_r^{(0)}(\Delta^2) h_{1+2r}(\chi^2; 0), \\ \text{and} & \\ H(x;\Delta^2) &= \sum_{r\geq 1} K_r^{(0)}(\Delta^2) H_{1+2r}(x; 0), \quad c \geq 0 \end{aligned} \tag{3.1}$$

where $h_\nu(\chi^2; 0)$ and $H_\nu(x; 0)$ are the pdf and cdf of a central chi-square distribution with ν d.f.

(ii) $$E[\phi(Z^2)] = \sum_{r\geq 0} K_r^{(0)}(\Delta^2) E_N[\phi(\chi^2_{1+2r}(0))] = E^{(0)}[\phi(\chi_1^2(\Delta))]$$

(iii) $$E[Z\phi(Z^2)] = \theta E^{(0)}[\phi(\chi_3^2(\Delta^2))]$$

(iv) $$E[Z^2\phi(Z^2)] = \sigma_\epsilon^2 E^{(1)}[\phi(\chi_3^2(\Delta^2))] + \theta^2 E^{(0)}[(\chi_5^2(\Delta^2))]$$

where

$$E^{(h)}[\phi(\chi_\nu^2(\Delta^2))] = \sum_{r\geq 0} K_r^{(h)}(\Delta^2) E_N[\phi(\chi^2_{\nu+2r}(0))], \tag{3.2}$$

$$K_r^{(h)}(\Delta^2) = \left(\frac{\nu_0 - 2}{2}\right)^h \frac{\Gamma\left(\frac{\nu_0}{2} + r - h\right)}{\Gamma(r+1)\Gamma\left(\frac{\nu_0}{2}\right)} \frac{\left(\frac{\Delta^2}{\nu_0 - 2}\right)^r}{\left(1 + \frac{\Delta^2}{\nu_0-2}\right)^{\frac{\nu_0}{2}+r-h}}, \quad \text{with } \Delta^2 = \frac{\theta^2}{\sigma_\epsilon^2}, \tag{3.3}$$

and

$$E[\tau^{2h}\phi(\chi_\nu^2(\Delta^2_{\tau^2}))] = \sigma_\epsilon^{2h} E^{(h)}[\phi(\chi_\nu^2(\Delta^2))] \tag{3.4}$$

for integer values of h.

Proof. Under $N(\theta, \tau^2)$, Z^2 is distributed as $\chi_1^2(\Delta^2_{\tau^2})$ with pdf $\sum_{r\geq 0} \frac{e^{-\frac{\Delta^2_{\tau^2}}{2}}}{r!}(\frac{\Delta^2_{\tau^2}}{2})^r h_{1+2r}(\chi^2; 0)$. Integrating w.r.t. $IG(\tau^2, \nu_0\sigma^2)$ we have the result given by $h_1(\chi_1^2 \Delta^2)$. (ii) is already given.

(iii) $$E[Z\phi(Z^2)] = \theta E_{\tau^2} E_N[\phi(\chi_3^2(\Delta^2_{\tau^2}))] = \theta E^{(0)}[\phi(\chi_3^2(\Delta^2))], \tag{3.5}$$

and

$$\text{(iv)} \quad E[Z^2\phi(Z^2)] = E_{\tau^2}\left\{\tau^2 E_N[\phi(\chi_3^2(\Delta_{\tau^2}^2))] + \theta^2 E[\phi(\chi_5^2(\Delta_{\tau^2}^2))]\right\}.$$

Using the formula (3.2)-(3.4) we have the R.H.S. equal to

$$= \sigma_\epsilon^2 E^{(1)}[\phi(\chi_3^2(\Delta^2))] + \theta^2 E^{(0)}[\phi(\chi_5^2(\Delta^2))]. \tag{3.6}$$

□

Theorem 3. *If $Z \sim M_t^{(1)}(\theta, \sigma^2, \nu_0)$ and U is an independently distributed central chi-square variable with m d.f., then the distribution of $F = (mZ^2)U^{-1}$ is given by the pdf/cdf*

$$g_{1,m}^{(0)}(F(\Delta^2)) = \sum_{r\geq 0} K_r^{(0)}(\Delta^2) g_{1+2r,m}(F;0) \tag{3.7}$$

and

$$G_{1,m}^{(0)}(x;\Delta^2) = \sum_{r\geq 0} K_r^{(0)}(\Delta^2) G_{1+2r,m}(x;0), \quad \nu_0 > 2, \tag{3.8}$$

respectively where $g_{\nu_1,\nu_2}(\cdot)$ and $G_{\nu_1,\nu_2}(\cdot)$ are pdf and cdf of central F-distribution with (ν_1,ν_2) d.f.

Thus, one may obtain the formulae

$$\text{(i)} \qquad E\left[\phi\left(\frac{mZ^2}{U}\right)\right] = E^{(0)}[\phi(3F_{3,m}(\Delta^2))] \tag{3.9}$$

$$\text{(ii)} \qquad E\left[Z\phi\left(\frac{mZ^2}{U}\right)\right] = \theta E^{(0)}[\phi(3F_{3,m}(\Delta^2))]$$

$$\text{(iii)} \qquad E\left[Z^2\phi\left(\frac{mZ^2}{U}\right)\right] = \sigma_\epsilon^2 E^{(1)}[\phi(3F_{3,m}(\Delta^2))] + \theta^2 E^{(0)}[\phi(5F_{5,m}(\Delta^2))]$$

where

$$E^{(h)}[\phi(F_{\nu_1,\nu_2}(\Delta^2))] = \sum_{r\geq 0} K_r^{(h)}(\Delta^2) E_N\left[\phi\left(\frac{\nu_1+2r}{\nu_1} F_{\nu_1+2r,\nu_2}(0)\right)\right] \tag{3.10}$$

with $K_r^{(h)}(\Delta^2)$ defined by (3.3).

If $G_{q,m}(c_\alpha;\Delta_{\tau^2}^2)$ denotes the non-central F-distribution with (q,m) d.f. with noncentrality parameter $\Delta_{\tau^2}^2$, then

$$E_{\tau^2}[(\tau^2)^h G_{q,m}(c_\alpha;\Delta_{\tau^2}^2)] = \sigma_\epsilon^{2h} G_{q,m}^{(h)}(\ell_\alpha;\Delta^2)$$

where

$$G_{q,m}^{(h)}(\ell_\alpha;\Delta^2) = \sum_{r\geq 0} K_r^{(h)}(\Delta^2) I_{\ell_\alpha}\left(\frac{1}{2}(q+2r);\frac{m}{2}\right) \tag{3.11}$$

with $\ell_\alpha = \frac{qc_\alpha}{m+qc_\alpha}$ and $\Delta^2 = \frac{\theta^2}{\sigma_\epsilon^2}$.

4 Bias and MSE Expressions of the Estimators of Locations

First we present the bias expressions of the estimators of θ:

$$\begin{aligned}
&\text{(i)} \quad b_1(\bar{\theta}_n) = 0 && (4.1)\\
&\text{(ii)} \quad b_2(\theta_0) = -k_0(\theta-\theta_0) = -k_0\sigma_\epsilon\Delta, \quad \Delta = (\theta-\theta_0)\sigma_\epsilon^{-1}\\
&\text{(iii)} \quad b_3(\hat{\theta}_n^{PT}) = -k_0\sigma_\epsilon\Delta G_{3,m}^{(0)}(\ell_\alpha;\Delta^2), \quad \ell_\alpha = \frac{c_\alpha}{m+c_\alpha}\\
&\text{(iv)} \quad b_4(\hat{\theta}_n^S) = -\frac{c_0k_0c_n\sigma_\epsilon}{\sqrt{K_1}}E_{\tau^2}[2\Phi(\Delta_{\tau^2})-1], \quad \Delta_{\tau^2} = \frac{(\theta-\theta_0)}{\tau}
\end{aligned}$$

where

$$c_n = \sqrt{\frac{2}{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \quad \text{and} \quad E_N[\frac{Z}{|Z|}] = 1-2\Phi(-\Delta_{\tau^2}). \tag{4.2}$$

The related MSE expressions for these estimators are given by

$$\begin{aligned}
\text{(i)} \quad M_1(\tilde{\theta}_n) &= \frac{\sigma_\epsilon^2}{K_1} && (4.3)\\
\text{(ii)} \quad M_2(\hat{\theta}_n) &= \frac{\sigma_\epsilon^2}{K_1}\{(1-k_0)^2 + k_0^2\Delta^2\}\\
\text{(iii)} \quad M_3(\hat{\theta}_n^{PT}) &= \frac{\sigma_\epsilon^2}{K_1}\Big\{1-k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha;\Delta^2)\\
&\quad + k_0\Delta^2[2G_{3,m}^{(0)}(\ell_\alpha;\Delta^2) - (2-k_0)G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)]\Big\}\\
\text{(iv)} \quad M_4(\hat{\theta}_n^S) &= -\frac{\sigma_\epsilon^2}{K_1}\left\{1-\frac{2}{\pi}c_n^2\left[\frac{2}{(1+\frac{\Delta^2}{\nu_0-2})^{\nu_0/2}}-1\right]\right\}.
\end{aligned}$$

Proof. (i) Since $\tilde{\theta}_n$ is distributed as $M_t^{(1)}(\theta, \frac{\sigma^2}{K_1}, \nu_0)$, and $E(\tilde{\theta}_n) = \theta$, hence $M_1(\tilde{\theta}_n) = \frac{\sigma_\epsilon^2}{K_1}$.

$$\begin{aligned}
\text{(ii)} \ M_2(\hat{\theta}_n) &= E(\hat{\theta}_n-\theta)^2 = E[(1-k_0)(\tilde{\theta}_n-\theta) - k_0(\theta-\theta_0)]^2\\
&= E[(1-k_0)^2(\tilde{\theta}_n-\theta)^2 + k_0^2(\theta-\theta_0)^2 - 2k_0(1-k_0)(\tilde{\theta}_n-\theta)(\theta-\theta_0)]\\
&= (1-k_0)^2\frac{\sigma_\epsilon^2}{K_1} + k_0^2\sigma_\epsilon^2\Delta^2 = \frac{\sigma_\epsilon^2}{K_1}[(1-k_0)^2+k_0^2\Delta^2] && (4.4)
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \ M_3(\hat{\theta}_n^{PT}) &= E[(\tilde{\theta}_n-\theta) - k_0(\tilde{\theta}_n-\theta_0)I(\mathcal{L}_n < c_\alpha)]^2\\
&= E[(\tilde{\theta}_n-\theta)^2 - 2k_0(\tilde{\theta}-\theta)(\tilde{\theta}_n-\theta_0)I(\mathcal{L}_n < c_\alpha + k_0^2(\tilde{\theta}_n-\theta_0)^2I(\mathcal{L}_n < c_\alpha)]\\
&= \frac{\sigma_\epsilon^2}{K_1}\Big\{1-k_0(2-k_0)G_{3,m}^{(1)}(\frac{1}{3}c_\alpha;\Delta^2)\\
&\quad + k_0\Delta^2\left[2G_{3,m}^{(0)}(\frac{1}{3}c_\alpha;\Delta^2) - G_{5,m}^{(0)}(\frac{1}{5}c_\alpha;\Delta^2)\right]\Big\} && (4.5)
\end{aligned}$$

using (3.10) - (3.13).

$$
\begin{aligned}
\text{(iv)} \qquad M_4(\hat{\theta}_n^S) &= E[\hat{\theta}_n^S - \theta]^2 \\
&= E\left[(\tilde{\theta}_n - \theta) - \frac{c_0 k_0 S_u(\tilde{\theta}_n - \theta_0)}{\sqrt{K_1}|\tilde{\theta} - \theta_0|}\right]^2 \\
&= E\left[(\tilde{\theta} - \theta)^2 + \frac{c_0^2 k_0^2 S_u^2}{K_1} - \frac{2c_0 k_0 S_u(\tilde{\theta}_n - \theta)(\tilde{\theta}_n - \theta_0)}{\sqrt{K_1}|\tilde{\theta}_n - \theta_0|}\right] \\
&= \frac{\sigma_\epsilon^2}{K_1} + \frac{c_0^2 k_0^2 \sigma_\epsilon^2}{K_1} - \frac{2c_0 k_0 c_n \sigma_\epsilon^2}{K_1}\left[E_{\tau^2}\left\{\sqrt{\frac{2}{\pi}} e^{-\frac{\Delta^2_{\tau^2}}{2}}\right\}\right]. \qquad (4.6)
\end{aligned}
$$

Choosing $k_0 c_0$ as $k_0 c_0^*$ to minimize $M_4(\hat{\theta}_n^S)$ given by

$$k_0 c_0^* = c_n \sqrt{\frac{2}{\pi}} \frac{1}{(1 + \frac{\Delta^2}{\nu_0 - 2})^{\nu_0/2}}.$$

The optimum value of $M_4(\hat{\theta}_n^S)$ reduces to

$$M_4(\hat{\theta}_n^S) = \frac{\sigma_\epsilon^2}{K_1}\left\{1 - \frac{2}{\pi} c_n^2 \left[\frac{2}{(1 + \frac{\Delta^2}{\nu_0 - 2})^{\nu_0/2}} - 1\right]\right\} \qquad (4.7)$$

by choosing $k_0 c_0^* = c_n \sqrt{\frac{2}{\pi}}$ to make $k_0 c_0^*$ independent of Δ^2. □

5 Bias and MSE Expressions of the Estimators of σ_ϵ^2

In this section we obtain the bias and MSE expressions for the estimators of the variance, σ_ϵ^2. They may be classified as follows

(a) Unrestricted estimators (i) S_u^2 and (ii) $\tilde{\sigma}_\epsilon^2$

(b) Restricted estimators (i) S_R^2 and (ii) $\hat{\sigma}_\epsilon^2$

(c) Preliminary test estimators when θ is suspected to be θ_0:

$$(i) S^2_{PT[1]}, \ \ (ii) S^2_{PT[2]} \ \text{and} \ \ (iii) S^2_{[s]}$$

.

The bias and MSE expressions are given in the following theorems

Theorem 4. *If* $\mathbf{Y} \sim M_t^{(n)}(\theta, \sigma^2\mathbf{V}_n, \nu_0)$*, then the bias expressions of* S_U^2*,* S_R^2*,* $\tilde{\sigma}_\epsilon^2$*,* $\hat{\sigma}_\epsilon^2$*,* $S_{PT[1]}^2$*,* $S_{PT[2]}^2$ *and* $S_{[s]}^2$ *are given by*

$$\text{(i)} \quad b_1(S_U^2) = 0, \qquad \text{(ii)} \quad b_2(\tilde{\sigma}_\epsilon^2) = -\frac{2\sigma_\epsilon^2}{m+2} \tag{5.1}$$

$$\text{(iii)} \quad b_3(S_R^2) = \frac{\sigma_\epsilon^2 \Delta^2}{m+1}, \qquad \text{(iv)} \quad b_4(\hat{\sigma}_\epsilon^2) = \frac{\sigma_\epsilon^2}{m+3}(\Delta^2 - 2)$$

$$\text{(v)} \quad b_5(S_{PT[1]}^2) = -\frac{\sigma_\epsilon^2}{m+1}\left\{G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) - G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\right\}$$

$$\text{(vi)} \quad b_6(S_{PT[2]}^2) = -\ \frac{\sigma_\epsilon^2}{m+2} - \frac{\sigma_\epsilon^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) + \ G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G^{(0)}(\ell_\alpha;\Delta^2)\Big] \ \text{ and}$$

$$\text{(vii)} \quad b_7(S_{[s]}^2) = -\ \frac{\sigma_\epsilon^2}{m+2} - \frac{\sigma_\epsilon^2}{(m+2)(m+3)}\Big[m(m+2)G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) + \ G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\Big].$$

Proof. (i) - (iv) are simple. For (v) we consider

$$\begin{aligned} b_5(S_{PT[1]}^2 &= e[S_{PT[1]}^2 - \sigma_\epsilon^2] \\ &= S_u^2 - (S_u^2 - S_R^2)I(\mathcal{L}_n < c_\alpha) - \sigma_\epsilon^2 \\ &= -E[(S_U^2 - S_R^2)I(\mathcal{L}_n < c_\alpha). \end{aligned} \tag{5.2}$$

Now

$$(m+1)S_R^2 = mS_U^2 + K_1(\tilde{\theta}_n - \theta_0)^2.$$

Then

$$S_U^2 - \frac{m}{m+1}S_U^2 - \frac{K_1(\tilde{\theta}_n - \theta_0)^2}{m+1} = \frac{1}{m+1}(S_U^2 - K_1(\tilde{\theta}_n - \theta_0)^2) \tag{5.3}$$

so that

$$E(S_U^2 I(\mathcal{L}_n < c_\alpha)] = \sigma_\epsilon^2 G_{1,m+2}^{(1)}(\mathcal{L}_n;\Delta^2), \tag{5.4}$$

and

$$E[K_1(\tilde{\theta}_n - \theta_0)^2 I(\mathcal{L}_n < c_\alpha)] = \sigma_\epsilon^2 G_{3,m}^{(1)}(\ell_n;\Delta^2) + K_1(\theta - \theta_0)^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2), \tag{5.5}$$

so that

$$b_5(S_{PT[1]}^2) = -\frac{\sigma_\epsilon^2}{m+1}\left\{G_{1,m+2}^{(1)}(\ell_\alpha;\Delta^2) - G_{3,m}^{(1)}(\ell_n;\Delta^2) - \Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\right\} \tag{5.6}$$

by (3.2)–(3.3) and $\ell_\alpha = \frac{c_\alpha}{m+c_\alpha}$.

For (vi) we have

$$
\begin{aligned}
b_6(S^2_{PT[2]}) &= E(S^2_{PT[2]} - \sigma^2_\epsilon) \\
&= E\left[\frac{mS^2_U}{m+2} - \left(\frac{mS^2_U}{m+2} - \frac{S^2_R}{m+3}\right) I(\mathcal{L}_n < c_\alpha) - \sigma^2_\epsilon\right] \\
&= -\frac{\sigma^2_\epsilon}{m+2} - E\left(\frac{mS^2_U}{m+2} - \frac{mS^2_U}{(m+2)(m+3)} - \frac{K_1(\tilde{\theta}_n - \theta_0)^2}{(m+2)(m+3)}\right) I(\mathcal{L}_n < c_\alpha) \\
&= -\frac{\sigma^2_\epsilon}{m+2} - \frac{\sigma^2_\epsilon}{m+3} \\
&\quad \times \left[mG^{(1)}_{1,m+2}(\ell_\alpha; \Delta^2) - \frac{1}{m+2}\left\{G^{(1)}_{3,m}(\ell_\alpha; \Delta^2) + \Delta^2 G^{(0)}_{5,m}(\ell_\alpha; \Delta^2)\right\}\right] \\
&= -\frac{\sigma_\epsilon}{m+2} - \frac{\sigma^2_\epsilon}{(m+2)(m+3)} \\
&\quad \times \left[m(m+2)G^{(1)}_{1,m+2}(\ell_\alpha; \Delta^2) + G^{(1)}_{3,m}(\ell_\alpha; \Delta^2) - \ \Delta^2 G^{(0)}_{5,m}(\ell_\alpha; \Delta^)\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
b_7(S^2_{[s]}) &= -\frac{\sigma^2_\epsilon}{m+2} - \frac{\sigma^2_\epsilon}{m+3} \\
&\quad \times \left[mG^{(1)}_{1,m+2}(\ell^*_\alpha; \Delta^2) - \frac{1}{m+2}\left\{G^{(1)}_{3,m}(\ell^*_\alpha; \Delta^2) + \Delta^2 G^{(0)}_{5,m}(\ell^*_\alpha; \Delta^2)\right\}\right] \\
&= -\frac{\sigma^2_\epsilon}{m+2} - \frac{\sigma^2_\epsilon}{(m+2)(m+3)} \\
&\quad \times \left[m(m+2)G^{(1)}_{1,m+2}(\ell^*_\alpha; \Delta^2) + G^{(1)}(\ell^*_\alpha; \Delta^2) - \Delta^2 G^{(0)}_{5,m}(\ell^*_\alpha; \Delta^2)\right]
\end{aligned}
$$

with $\ell^*_\alpha = \frac{m}{m+3}$. □

The next theorem gives the MSE expressions of the estimators.

Theorem 5. *If* $\mathbf{Y} \sim M^{(n)}_t(\theta, \sigma^2 V_n, \nu_0)$, *then the MSE expressions of* S^2_U, S^2_R, $\tilde{\sigma}^2_\epsilon$, $\hat{\sigma}^2_\epsilon$,

$S^2_{PT[1]}$, $S^2_{PT[2]}$ *and* $S^2_{PT[5]}$ *are given by*

$$\text{(i)}\qquad M_1(S_U^2) = \frac{2(m+\nu_0-2)}{m(\nu_0-4)}\sigma_\epsilon^4, \quad \nu_0 > 4;$$

$$\text{(ii)}\qquad M_2(S_R^2) = \frac{2\sigma_\epsilon^4}{(m+1)}\frac{(m+\nu_0-1)}{(\nu_0-4)} + \frac{\Delta^2(\Delta^2+4)}{(m+1)^2}$$

$$\text{(iii)}\qquad M_3(\tilde{\sigma}_\epsilon^2) = \frac{2\sigma_\epsilon^4(m+\nu_0-4)}{(m+2)(\nu_0-4)}, \quad \nu_0 > 4$$

$$\text{(iv)}\qquad M_4(\hat{\sigma}_\epsilon^2) = \frac{2\sigma_\epsilon^4}{(m+3)}\frac{(m-\nu_0+5)}{(\nu_0-4)} + \frac{\sigma_\epsilon^4\Delta^2(\Delta^2+4)}{(m+3)^2}$$

$$\begin{aligned}\text{(v)}\qquad M_5(S_{PT[1]}) &= M_1(S_U^2) - \frac{\sigma_\epsilon^4(m+2)(2m+1)}{(m+1)^2}G^{(2)}_{1,m+4}(\ell_\alpha;\Delta^2) + \frac{\sigma_\epsilon^4}{(m+1)^2} \\ &\times\Bigg[\Big\{3G^{(2)}_{5,m}(\ell_\alpha;\Delta^2) + m(G^{(2)}_{3,m+2}(\ell_\alpha;\Delta^2) + 2(m+1)\Big[G^{(1)}_{1,m+2}(\ell_\alpha;\Delta^2) - G^{(1)}_{3,m}(\ell_\alpha;\Delta^2)\Big]\Big\} \\ &+\Delta^2\Big\{6G^{(1)}_{7,m}(\ell_\alpha;\Delta^2) + mG^{(1)}_{5,m+2}(\ell_\alpha;\Delta^2) - 2(m+1)G^{(0)}_{5,m}(\ell_\alpha;\Delta^2)\Big\} + \Delta^2 G^{(0)}_{9,m}(\ell_\alpha;\Delta^2)\Bigg].\end{aligned}$$

$$\begin{aligned}M_6\Big(S^2_{PT[2]}\Big) \quad = \quad & M_3(\tilde{\sigma}_\epsilon^2) - \frac{m(2m+5)}{(m+2)(m+3)}\sigma_\epsilon^4 G^{2)}_{1,m+4}(\ell_\alpha;\Delta^2) + \frac{3\sigma_\epsilon^4}{m^2(m+3)^2}G^{(2)}_{5,m}(\ell_\alpha;\Delta^2) \\ & + \frac{2\sigma_\epsilon^4}{(m+3)^2}\Big\{G^{(2)}_{3,m+2}(\ell_\alpha;\Delta^2) - (m+3)G^{(2)}_{3,m}(\ell_\alpha;\Delta^2)\Big\} \\ & + \frac{\Delta^2\sigma_\epsilon^4}{(m+2)(m+3)^2}\Big\{2G^{(2)}_{5,m+2}(\ell_\alpha;\Delta^2) + 6(m+2)G^{(1)}_{7,m}(\ell_\alpha;\Delta^2) \\ & + 2(m+3)G^{(1)}_{5,m}(\ell_\alpha;\Delta^2)\Big\} + \frac{\Delta^4\sigma_\epsilon^4}{m^2(m+3)^2}G^{(0)}_{9,m}(\ell_\alpha;\Delta^2)\end{aligned}$$

$$\begin{aligned}M_7(S^2_{[s]}) \quad = \quad & M_3(\tilde{\sigma}_\epsilon^2) - \frac{m(2m+5)}{(m+2)(m+3)}\sigma_\epsilon^4 G^{2)}_{1,m+4}(\ell^*_\alpha;\Delta^2) + \frac{3\sigma_\epsilon^4}{m^2(m+3)^2}G^{(2)}_{5,m}(\ell^*_\alpha;\Delta^2) \\ & + \frac{2\sigma_\epsilon^4}{(m+3)^2}\Big[G^{(2)}_{3,m+2}(\ell^*_\alpha;\Delta^2) - (m+3)G^{(2)}_{3,m}\Big] \\ & + \frac{\Delta^2\sigma_\epsilon^4}{(m+2)(m+3)^2}\Big\{2G^{(2)}_{5,m+2}(\ell^*_\alpha;\Delta^2) + 6(m+2)G^{(1)}_{7,m}(\ell^*_\alpha;\Delta^2) \\ & + 2(m+3)G^{(1)}_{5,m}(\ell^*_\alpha;\Delta^2)\Big\} + \frac{\Delta^4\sigma_\epsilon^4}{m^2(m+3)^2}G^{(0)}_{9,m}(\ell^*_\alpha;\Delta^2)\end{aligned}$$

with $\ell^*_\alpha = \frac{m}{m+3}$.

Proof.

(i)
$$\begin{aligned} M_1(S_U^2) &= E(S_U^2-\sigma_\epsilon^2)^2 = \mathrm{Var}(S_U^2) = E_{\tau^2}\mathrm{Var}_N(S_U^2|\tau^2) + \mathrm{Var}_{\tau^2}E_N(S_U^2|\tau^2) \\ &= E_{\tau^2}\Big(\frac{2\tau^4}{m}\Big) + \mathrm{Var}_{\tau^2}(\tau^2) \\ &= \frac{2}{m}\sigma_\epsilon^4\Big(\frac{\nu_0-2}{\nu_0-4}\Big) + \frac{\sigma_\epsilon^4(\nu_0-2)}{(\nu_0-4)} - \sigma_\epsilon^4 \\ &= \sigma_\epsilon^4\frac{\nu_0-2}{\nu_0-4}\Big(\frac{m+2}{m}\Big) - \sigma_\epsilon^4 \\ &= \frac{2\sigma_\epsilon^4(m+\nu_0-2)}{m(\nu_0-4)}, \quad \nu_0>4. \end{aligned}$$

(ii) $\mathrm{Var}(S_R^2)$
$$\begin{aligned} &= \mathrm{Var}_{\tau^2}\{E_N[S_R^2|\tau^2]\} + E_{\tau^2}\{\mathrm{Var}_N[S_R^2|\tau^2]\} \\ &= \mathrm{Var}_{\tau^2}\Big\{\frac{\tau^2}{(m+1)}E_N[\chi^2_{m+1}(\Delta^2_{\tau^2})]\Big\} + E_{\tau^2}\Big\{\frac{\tau^4}{(m+1)^2}\mathrm{Var}_N[\chi^2_{m+1}(\Delta^2_{\tau^2})]\Big\} \\ &= \frac{1}{(m+1)^2}\mathrm{Var}_{\tau^2}[(m+1)\tau^2 + K_1(\theta-\theta_0)^2] + E_{\tau^2}\Big[\frac{\tau^4}{(m+1)^2}2((m+1)+2\Delta^2_{\tau^2})\Big] \\ &= \frac{1}{(m+1)^2}\{(m+1)^2\mathrm{Var}(\tau^2)\} + \frac{1}{(m+1)}E_{\tau^2}(\tau^4) + \frac{4K_1(\theta-\theta_0)^2}{(m+1)^2}E_{\tau^2}(\tau^2) \\ &= E_{\tau^2}(\tau^4) - \sigma_\epsilon^2 + \frac{2}{(m+1)}E_{\tau^2}(\tau^4) + 4\sigma_\epsilon^4\frac{\Delta^2}{(m+1)^2} \\ &= \frac{m+3}{m+1}\sigma_\epsilon^4\Big(\frac{\nu_0-2}{\nu_0-4}\Big) - \sigma_\epsilon^4\Big(1-\frac{4\Delta^2}{(m+1)^2}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} M_2(S_R^2) &= \frac{m+3}{m+1}\sigma_\epsilon^4\Big(\frac{\nu_0-2}{\nu_0-4}\Big) - \sigma_\epsilon^4 + \frac{\Delta^2(4+\Delta^2)\sigma_\epsilon^4}{(m+1)^2} \\ &= \frac{2\sigma_\epsilon^4(m+\nu_0-1)}{(m+1)(\nu_0-4)} + \frac{\Delta^2(4+\Delta^2)\sigma_\epsilon^4}{(m+1)^2}. \end{aligned}$$

(iii)
$$\begin{aligned} \mathrm{Var}[\tilde\sigma_\epsilon^2] &= \mathrm{Var}_{\tau^2}\Big\{E_N\Big(\frac{mS_U^2}{m+2}\Big|\tau^2\Big)\Big\} + E_{\tau^2}\Big\{\mathrm{Var}_N\Big[\frac{mS_U^2}{m+2}\Big|\tau^2\Big]\Big\} \\ &= \mathrm{Var}_{\tau^2}\Big(\frac{m\tau^2}{m+2}\Big) + E_{\tau^2}\Big\{\Big(\frac{1}{m+2}\Big)^2\tau^4(2m)\Big\} \\ &= \frac{m^2}{(m+2)^2}\Big\{E_{\tau^2}(\tau^4) - \sigma_\epsilon^4\Big\} + \frac{2m}{(m+2)^2}E_{\tau^2}(\tau^4) \\ &= \frac{m}{m+2}\sigma_\epsilon^4\Big(\frac{\nu_0-2}{\nu_0-4}\Big) - \frac{m^2\sigma_\epsilon^4}{(m+2)^2}. \end{aligned}$$

$$
\begin{aligned}
M_3(\tilde{\sigma}_\epsilon^2) &= \frac{m\sigma_\epsilon^4}{m+2}\Big(\frac{\nu_0-2}{\nu_0-4}\Big) - \frac{m^2}{(m+2)^2}\sigma_\epsilon^4 + \frac{4\sigma_\epsilon^4}{(m+2)^2} \\
&= \frac{m\sigma_\epsilon^4}{(m+2)}\Big(\frac{\nu_0-2}{\nu_0-4}\Big) - \frac{(m-2)\sigma_\epsilon^4}{(m+2)} \\
&= \frac{2\sigma_\epsilon^4(m+\nu_0-4)}{(m+2)(\nu_0-4)}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\mathrm{Var}\Big[\frac{m+1}{m+3}S_R^2\Big] &= \Big(\frac{m+1}{m+3}\Big)^2 \mathrm{Var}(S_R^2) \\
&= \frac{m+1}{m+3}\Big(\frac{\nu_0-2}{\nu_0-4}\Big)\sigma_\epsilon^4 - \sigma_\epsilon^4\bigg(1-\frac{4\Delta^2}{(m+3)^2}\bigg).
\end{aligned}
$$

$$
\begin{aligned}
M_4(\hat{\sigma}_\epsilon^2) &= \frac{m+1}{m+3}\Big(\frac{\nu_0-2}{\nu_0-4}\Big)\sigma_\epsilon^4 - \sigma_\epsilon^4\bigg(1-\frac{4\Delta^2}{(m+3)^2}\bigg) + \frac{\sigma_\epsilon^4(\Delta^2-2)^2}{(m+3)^2} \\
&= \Big(\frac{m+1}{(m+3)}\Big)\Big(\frac{\nu_0-2}{\nu_0-4}\Big)\sigma_\epsilon^4 - \sigma_\epsilon^4 + \frac{\sigma_\epsilon^4(\Delta^4-4\Delta^2+4)}{(m+3)^2} \\
&= \frac{2\sigma_\epsilon^4(m+\nu_0-5)}{(m+3)(\nu_0-4)} + \frac{\Delta^2(4+\Delta^2)\sigma_\epsilon^4}{(m+3)^2}.
\end{aligned}
$$

(v)

$$
\begin{aligned}
& M_5(S^*_{PT[1]}) \\
&= E[S^2_{PT[1]} - \sigma_\epsilon^2]^2 \\
&= E[S_U^2 - \sigma_\epsilon^2]^2 - \frac{2}{(m+1)}E\Big[(S_U^2-\sigma_\epsilon^2)\{S_U^2 - K_1(\tilde{\theta}_n-\theta_0)^2\}I(\mathcal{L}_n < c_\alpha)\}\Big] \\
&\quad + \frac{1}{(m+1)^2}E[(S_U^2 - K_1(\tilde{\theta}_n-\theta_o)^2)I(\mathcal{L}_n<c_\alpha)]^2 \\
&= M_1(S_U^2) - \frac{(2m+1)}{m+1)^2}E[(S_U^4 I(\mathcal{L}_n<c_\alpha)] \\
&\quad + \frac{1}{(m+1)^2}E[K_1^2(\tilde{\theta}_n-\theta_0)^4 I(\mathcal{L}_n<c_\alpha)] \\
&\quad + \frac{m}{(m+1)^2}E[S_U 2K_1(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n<c_\alpha)] \\
&\quad + \frac{2\sigma_\epsilon^2}{m+1}E[S_U^2 I(\mathcal{L}_n<c_\alpha)] - \frac{2\sigma_\epsilon^2}{m+1}E[K_1(\tilde{\theta}-\theta_0)^2 I(\mathcal{L}_n<c_\alpha)].
\end{aligned}
$$

Now

$$
\begin{aligned}
E[S_U^2 I(\mathcal{L}_n<c_\alpha)] &= E_{\tau^2}\left\{\frac{\tau^2}{m}E_N\left[\chi_m^2 I\left(\frac{\chi_1^2(\Delta_{\tau^2}^2)}{\chi_m^2} < \frac{1}{m}c_\alpha\right)\right]\right\} \\
&= E_{\tau^2}[\tau^2 G_{1,m+2}(\ell_\alpha;\Delta^2_{\tau^2})] = \sigma_\epsilon^2 G^{(1)}_{1,m+2}(\ell_\alpha;\Delta^2)
\end{aligned}
$$

by (3.13) where $\ell_\alpha = \frac{c_\alpha}{m+c_\alpha}$.

$$
\begin{aligned}
E[S_U^4 I(\mathcal{L}_n < c_\alpha)] &= E_{\tau^2}\left[\frac{\tau^2}{m^2} E_N\left\{\chi_m^4 I\left(\frac{\chi_1^2(\Delta_{\tau^2}^2)}{\chi_m^2} < \frac{1}{m}c_\alpha\right)\right]\right\} \\
&= E_{\tau^2}\left[\frac{\tau^4}{m}\{m(m+2)G_{1,m+4}(\ell_\alpha;\Delta_{\tau^2}^2)\right] \\
&= \frac{m+2}{m}\sigma_\epsilon^4 G_{1,m+4}^{(2)}(\ell_\alpha;\Delta^2) \quad \text{by (3.13).}
\end{aligned}
$$

Next we have,

$$
\begin{aligned}
E\big[K_1(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n < c_\alpha)\big] &= E_{\tau^2}\left\{\tau^2 E\left[\chi_1^2(\Delta_{\tau^2}^2) I\left(\frac{\chi_1^2(\Delta_{\tau^2}^2)}{\chi_m^2} < \frac{1}{m}C_\alpha\right)\right]\right\} \\
&= E_{\tau^2}\big\{\tau^2\big\{G_{3,m}(\ell_\alpha;\Delta_{\tau^2}^2) + \Delta_{\tau^2}^2 G_{5,m}(\ell_\alpha;\Delta^2)\big\}\big] \\
&= \sigma_\epsilon^2 G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) + \sigma_\epsilon^2\Delta^2 G_{5,m}^{(0)}(\ell_\alpha;\Delta^2).
\end{aligned}
$$

and

$$
\begin{aligned}
&E\big[K_1^2(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n < c_\alpha)\big] \\
&= E_{\tau^2}\left\{\tau^4 E_N\left[\big(\chi_1^2(\Delta_{\tau^2}^2)\big]^2 I\left(\frac{\chi_1^2(\Delta_{\tau^2}^2)}{\chi_m^2} < \frac{1}{m}c_\alpha\right)\right]\right\} \\
&= E_{\tau^2}\left\{\tau^4\left[3G_{5,m}(\ell_\alpha;\Delta_{\tau^2}^2) + 6\Delta_{\tau^2}^2 G_{7,m}^{(1)}(\ell_\alpha;\Delta_{\tau^2}^2) + \Delta_{9,m}^4(\ell_\alpha;\Delta_{\tau^2}^2)\right]\right\} \\
&= 3\sigma_\epsilon^4 G_{5,m}^{(2)}(\ell_\alpha;\Delta^2) + 6\sigma_\epsilon^4\Delta^2 G_{7,m}^{(1)}(\ell_\alpha;\Delta^2) + \sigma_\epsilon^4\Delta^4 G_{9,m}^{(0)}(\ell_\alpha;\Delta^2).
\end{aligned}
$$

Further,

$$
\begin{aligned}
E\big[S_U^2 K_1^2(\tilde{\theta}_n-\theta_0)^2 I(\mathcal{L}_n < c_\alpha)\big] &= E_{\tau^2}\left\{\frac{\tau^4}{m} E\left[\chi_m^2\chi_1^2(\Delta_{\tau^2}^2) I\left(\frac{\chi_1^2(\Delta_{\tau^2}^2)}{\chi_m^2} < \frac{1}{m}d_\alpha\right)\right]\right\} \\
&= E_{\tau^2}\left[\tau^4\left\{G_{3,m+2}(\ell_\alpha;\Delta_{\tau^2}^2 + \Delta_{\tau^2}^2 G_{5,m+2}(\ell_\alpha;\Delta_{\tau^2}^2)\right\}\right] \\
&= \sigma_\epsilon^2 G_{3,m+2}^{(2)}(\ell_\alpha;\Delta^2) + \sigma_\epsilon^4\Delta^2 G_{5,m+2}^{(1)}(\ell_\alpha;\Delta^2)
\end{aligned}
$$

$$
\begin{aligned}
&M_7(S^*_{PT[2]})\\
&= E[(S^2_{PT[2]}] - 2\sigma^2_\epsilon E[S^2_{PT[2]}] + \sigma^4_\epsilon\\
&= E_{\tau^2}\left\{\frac{\tau^4}{(m+2)^2}E_N\Big(\frac{mS^2_U}{\tau^2}\Big)^2|\tau^2\right\}\\
&\quad + \frac{1}{m^2(m+3)^2}E_{\tau^2}\left\{\tau^4 E_N\left[\Big(\frac{m}{m+2}-\mathcal{L}_n\Big)^2\Big(\frac{mS^2_U}{\tau^2}\Big)^2 I(\mathcal{L}_n<c_\alpha)|\tau^2\right]\right\}\\
&\quad - \frac{2}{m(m+2)(m+3)}E_{\tau^2}\left\{\tau^4 E_N\left[\Big(\frac{mS^2_U}{\tau^2}\Big)^2\Big(\frac{m}{m+2}-\mathcal{L}_n\Big)I(\mathcal{L}_n<c_\alpha)|\tau^2\right]\right\}\\
&\quad + \frac{2}{m(m+3)^2}E_{\tau^2}\left\{\tau^4 E_N\left[\Big(\frac{mS^2_U}{\tau^2}\Big)\Big(\frac{m}{m+2}-\mathcal{L}_n)I(\mathcal{L}_n<c_\alpha)|\tau^2\right]\right\} - \sigma^4_\epsilon\\
&= \frac{m}{(m+2)}E_{\tau^2}(\tau^4) + \frac{1}{m^2(m+3)^2}E_{\tau^2}\Big\{\tau^4 E_N\left[\Big(\frac{m}{m+2}\Big)^2 - \frac{2m}{m+2}\mathcal{L}_n + \mathcal{L}^2_n\right]\\
&\quad \times\Big(\frac{mS^2_U}{\tau^2}\Big)^2 I(\mathcal{L}_n<c_\alpha)|\tau^2\Big\} - \frac{2}{m(m+2)(m+3)}E_{\tau^2}\\
&\quad \times\left\{\tau^4 E_N\Big[\Big(\frac{m}{m+2}\Big)\Big(\frac{mS^2_U}{\tau^2}\Big)^2 I(\mathcal{L}_n<c_\alpha) - \Big(\frac{mS^2_U}{\tau^2}\Big)^2\mathcal{L}_n I(\mathcal{L}_n<c_\alpha)|\tau^2\right\}\\
&\quad + \frac{2}{m(m+3)^2}E_{\tau^2}\left\{\tau^4 E_N\left[\frac{m}{m+2}\frac{mS^2_U}{\tau^2}I(\mathcal{L}_n<c_\alpha) - \Big(\frac{mS^2_U}{\tau^2}\Big)\mathcal{L}_n I(\mathcal{L}_n<c_\alpha)\right]\right\} - \sigma^4_\epsilon\\
&= \frac{m}{m+2}E_{\tau^2}(\tau^4) + \frac{E_{\tau^2}}{m^2(m+3)^2}\left[\tau^4 E_N\left\{\Big(\frac{m}{m+2}\Big)^2\chi^4_m I(F_{1,m}(\Delta^2_{\tau^2})<c_\alpha)\right\}\right]\\
&\quad - \frac{2}{(m+2)^2(m+3)^2}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^4_m F_{1,m}(\Delta^2_{\tau^2})I(F_{1,m}(\Delta^2_{\tau^2})<c_\alpha)\right\}\right]\\
&\quad + \frac{1}{m^2(m+3)^2}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^4_m(F_{1,m}(\Delta^2_{\tau^2})^2 I(F_{1,m}(\Delta^2_{\tau^2})<c_\alpha)\right\}\right]\\
&\quad - \frac{2}{(m+2)^2(m+3)}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^4_m I(F_{1,m}(\Delta^2_{\tau^2})<c_\alpha)\right\}\right]\\
&\quad - \frac{2}{(m+2)(m+3)}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^4_m(F_{1,m}(\Delta^2_{\tau^2})I(F_{1,m}(\Delta^2_{\tau^2}<c_\alpha)\right\}\right]\\
&\quad + \frac{2}{(m+2)(m+3)^2}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^4_m I(F_{1,m}<c_\alpha)\right\}\right]\\
&\quad - \frac{2}{(m+2)(m+3)^2}E_{\tau^2}\left[\tau^4 E_N\left\{\chi^2_m F_{1,m}(\Delta^2_{\tau^2})I(F_{1,m}(\Delta^2_{\tau^2})<c_\alpha)\right\}\right] - \sigma^4_\epsilon.
\end{aligned}
$$

Simplification leads to $M_7(S^2_{PT[2]})$. Similarly, $M_8(S^2_{[s]})$ may be obtained by replacing c_α by $\frac{m}{m+2}$. □

6 Analysis of the Estimators

In this section, we provide the analysis of the various estimators. In section 6.1 we consider the estimators of the location parameter, θ, and section 6.2 contains the analysis of MSE expressions for the estimators of the variance, σ_ϵ^2.

6.1 Location Parameter

We considered four estimators of θ, namely,

(a) Unrestricted estimator, $\tilde{\theta}_n$

(b) Restricted estimator, $\hat{\theta}_n^{\mathrm{RE}}$

(c) Preliminary Test estimator, $\hat{\theta}_n^{\mathrm{PT}}$

(d) Shrinkage type estimator, $\hat{\theta}_n^{\mathrm{SE}}$.

The bias and mse expressions are given by (i)-(iv) of equations (4.1) and by (i)-(iv) of equation (4.3) respectively.

Comparison of $\hat{\theta}_n^{RE}$ and $\tilde{\theta}_n$

The bias of $\tilde{\theta}_n$ is zero and the bias of $\hat{\theta}_n^{RE}$ is $-k_0\sigma_\epsilon$. At $\Delta = 0$, both are unbiased but as Δ moves away from the origin, bias $\hat{\theta}_n^{RE}$ is unbounded. As regards the mse of the estimators, we have the mse-difference given by

$$M_1(\tilde{\theta}_n) - M_2(\hat{\theta}_n^{RE}) = \frac{\sigma_\epsilon^2}{K_1}\left\{1-(1-k_0)^2 - k_0^2\Delta^2\right\} \gtreqless 0 \tag{6.1}$$

whenever

$$\Delta^2 \gtreqless (2k_0^{-1}-1), \quad 0 < k_0 \leq 1.$$

Thus, if $\Delta^2 \leq (2k_0^{-1}-1)$, then $\hat{\theta}_n^{RE}$ is better than $\tilde{\theta}_n$ and if $\Delta^2 > (2k_0^{-1}-1)$, then $\tilde{\theta}_n$ dominates $\hat{\theta}_n$. The relative efficiency of the estimator $\hat{\theta}_n$ is

$$RE(\hat{\theta}_n : \tilde{\theta}_n) = [(1-k_0)^2 + k_0^2\Delta^2]^{-1}. \tag{6.2}$$

Comparison of $\hat{\theta}_n^{PT}$ and $\tilde{\theta}_n$

Here the bias of $\hat{\theta}_n^{PT}$ is $-k_0\sigma_\epsilon\Delta\, G_{3,m}(\ell_\alpha;\Delta^2)$. If $\Delta^2 = 0$, then both $\tilde{\theta}_n$ and $\hat{\theta}_n^{PT}$ are unbiased. Otherwise $|b_3(\hat{\theta}_n^{PT})| > 0 \;\forall\; \Delta^2 > 0$. As regards mse for the estimators, we have

$$\begin{aligned} &M_1(\tilde{\theta}_n) - M_3(\hat{\theta}_n^{PT}) \\ &= \frac{\sigma_\epsilon^2}{K_1}\left\{k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) - k_0\Delta^2\left[2G_{3,m}^{(0)}(\ell_\alpha;\Delta^2) - (2-k_0)G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\right]\right\}. \end{aligned}$$

Thus, mse-difference is $\gtreqless 0$ whenever

$$\Delta^2 \lesseqgtr \frac{(2k_0^{-1}-1)G_{3,m}^{(1)}(\ell_\alpha;\Delta^2)}{[2G_{3,m}^{(0)}(\ell_\alpha;\Delta^2) - (2-k_0)G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)]} \tag{6.3}$$

The relative efficiency of $\hat{\theta}_n^{PT}$ w.r.t. $\tilde{\theta}_n$ is given by

$$E(\alpha,\Delta^2) = RE(\hat{\theta}_n^{PT};\tilde{\theta}_n) = \Big[1 - k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha;\Delta^2) + k_0\Delta^2\{2G^{(0)}(\ell_\alpha;\Delta^2) - (2-k_0)G_{5,m}^{(0)}(\ell_\alpha;\Delta^2)\}\Big]^{-1}. \quad (6.4)$$

Note that

(i) If $\Delta^2 = 0$, then it reduces to $[1 - k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha;0)]^{-1} \geq 1$.

(ii) If $\Delta^2 \to \infty$, then, $\mathrm{RE}(\hat{\theta}_n^{PT};\tilde{\theta}_n) \to 1$.

(iii) The $\mathrm{RE}(\hat{\theta}_n^{PT};\tilde{\theta}_n)$ crosses the 1-line in the interval $\left(1 - \frac{1}{2}k_0, \frac{1}{k_0} - \frac{1}{2}\right)$.

(iv) $\mathrm{Re}(\hat{\theta}_n^{PT};\tilde{\theta}_n)$ equals $[1 - k_0(2-k_0)G_{3,m}^{(1)}(\ell_\alpha;0)]^{-1}$ at $\Delta^2 = 0$, then drops monotonically crossing the 1-line in the interval $(1-\frac{1}{2}k_0,\ \frac{1}{k_0}-\frac{1}{2})$ keeping to a minimum, then increases towards the 1-line. Thus, an optimum α-level mse is obtained by solving the equation for $\alpha \in \mathcal{A} = \{\alpha | RE(\alpha,\Delta^2) \geq E_0\}$

$$\min_{\Delta^2} RE(\alpha,\Delta^2) = E(\alpha,\Delta_0^2(\alpha)) = E_0 \quad (6.5)$$

where E_0 is a prefixed or guaranteed relative efficiency.

Comparison of $\tilde{\theta}_n$ and $\hat{\theta}_n^S$

The bias expression is given by $b_4(\hat{\theta}_n^S) = \left\{-\frac{c_0k_0c_n\sigma_\epsilon}{\sqrt{K_1}}\right\} E_{\tau^2}[2\Phi(\Delta_{\tau^2}) - 1]$. As $\Delta_\tau^2 \to 0$, $|b_4(\hat{\theta}_n^S)| \to 0$ and as $\Delta_{\tau^2}^1 \to \infty$, $|b_4(\hat{\theta}_n^S)| = \frac{c_0k_0c_n\sigma_\epsilon}{\sqrt{K_1}}$. The absolute bias is a non-decreasing function of Δ_{τ^2}. Thus, near the origin the bias is smallest and becomes largest when $\Delta_{\tau^2} \to \infty$.

As regards mse comparison, the relative efficiency $\mathrm{RE}(\hat{\theta}_n^S;\tilde{\theta}_n)$ is given by

$$\left\{1 - \frac{2}{\pi}c_n^2\left[\frac{2}{\left(1+\frac{\Delta^2}{\nu_0-2}\right)^{\nu_0/2}} - 1\right]\right\}^{-1}. \quad (6.6)$$

Under $\Delta^2 = 0$,

$$RE(\hat{\theta}_n^S;\tilde{\theta}_n) = \left(1 - \frac{2}{\pi}c_n^2\right)^{-1} \geq 1$$

and (6.1.6) decreases to $(1 + \frac{2}{\pi}c_n^2)^{-1}(\leq 1)$ as $\Delta^2 \to \infty$. The relative loss of efficiency of $\hat{\theta}_n^S$ relative to $\tilde{\theta}_n$ is $1-(1+\frac{2}{\pi}c_n^2)^{-1}$, while the gain in efficiency is $(1-\frac{2}{\pi}c_n^2)^{-1}$. The efficiency is 1 when $\Delta^2 = \nu_0 - 2$. If $\Delta^2 < \nu_0 - 2$, $\hat{\theta}_n^S$ performs better than $\tilde{\theta}_n$, otherwise $\tilde{\theta}_n$ is better. Note that $\hat{\theta}_n^S$ does not depend on the level of significance while $\hat{\theta}_n^{PT}$ does. As Δ^2 the efficiency of PTE w.r.t. $\tilde{\theta}_n$ tends to 1 while that of $\hat{\theta}_n^S$ w.r.t. $\tilde{\theta}_n$ tends to $(1+\frac{2}{\pi}c_n^2)^{-1} < 1$. Thus, $\hat{\theta}_n^S$ is better near the null hypothesis than that of $\hat{\theta}_n^{PT}$.

6.2 Analysis of the Estimators of Scale Parameter

In section 2, we have defined seven estimators of σ_ϵ^2. The bias and mse expressions of these estimators are given in section 4. In this section, we present the analysis of the mse expressions.

First we note that the mse expression for S_U^2 is constant while the restricted estimate, S_R^2 depends on the departure parameter, Δ^2. Under H_0, i.e. for $\Delta^2 = 0$, $M_2(S_R^2) = \frac{2\sigma_\epsilon^4(m+\nu_0-1)}{(m+1)(\nu_0-4)}$ so that $M_2(S_R^2) < M_1(S_U^2)$. The mse's are equal when Δ^2 equals

$$\Delta_*^2 = -2 + 2\sqrt{1 + \frac{(m+1)(\nu_0-2)}{2m(\nu_0-4)}}, \quad \nu_0 > 4. \tag{6.1}$$

Hence, the range of Δ^2 for which S_R^2 dominates S_U^2 is given by $[0, \Delta_*^2]$ otherwise S_U^2 dominate S_R^2. Note that the mse of S_R^2 is unbounded as $\Delta^2 \to \infty$.

Similarly, under H_0, $M_4(\hat\sigma_\epsilon^2) = \frac{2\sigma_\epsilon^4(m-\nu_0+5)}{(m+3)(\nu_0-4)}$ so that $M_4(\hat\sigma_\epsilon^2) < M_3(\tilde\sigma_\epsilon^2)$. Hence, the range of Δ^2 for which $\hat\sigma_\epsilon^2$ dominate $\tilde\sigma_\epsilon^2$ is given by $[0, \Delta_{**}^2]$ where Δ_{**}^2 is defined by the solution of the equation

$$\Delta^2(\Delta^2 + 4) = \frac{2(2m+\nu_0-2)(m+3)}{(m+2)(\nu_0-4)} \tag{6.2}$$

i.e.

$$\Delta_{**}^2 = -2 + 2\sqrt{1 + \frac{(m+3)(2m+\nu_0-2)}{2(m+2)(\nu_0-4)}} \tag{6.3}$$

otherwise, $\tilde\sigma_\epsilon^2$ dominates $\hat\sigma_\epsilon^2$.

Now, we show the uniform dominance of $S_{[s]}^2$ over $\tilde\sigma_\epsilon^2$ under the quadratic loss function $\frac{1}{\sigma_\epsilon^4}(\sigma_x^2 - \sigma_\epsilon^2)^2$. For this, we consider the risk of $S_{[s]}^2$ with respect to the quadratic loss-function.

Then, we have

$$\begin{aligned}
&\frac{1}{\sigma_\epsilon^4} E[mS_U^2\psi_s(\mathcal{L}_n) - \sigma_\epsilon^2]^2 \\
&= E_{\mathcal{L}_n}\Bigg\{\psi_s^2(\mathcal{L}_n) E\left[\left(\frac{\tau^2}{\sigma_\epsilon^2}\right)^2 \left(\frac{mS_U^2}{\tau^2}\right)^2 \Bigg| \mathcal{L}_n\right] \\
&\quad -2\psi_s(\mathcal{L}_n) E\left[\left(\frac{\tau^2}{\sigma - \epsilon^2}\right)\left(\frac{mS_U^2}{\tau^2}\right)\Bigg|\mathcal{L}_n\right] + 2\Bigg\}.
\end{aligned} \tag{6.4}$$

Now, consider the term inside the curly bracket of (6.2.4). For fixed Δ^2 and for each $\mathcal{L}_n$, this is a quadratic form in $\psi_S(\mathcal{L}_n)$ with the minimum at

$$\psi_S^*(\mathcal{L}_n) = \frac{E\left[\left(\frac{\tau^2}{\sigma_\epsilon^2}\right)\left(\frac{mS_U^2}{\tau^2}\right)\Big|\mathcal{L}_n\right]}{E\left[\left(\frac{\tau^2}{\sigma_\epsilon^2}\right)^2\left(\frac{mS_U^2}{\tau^2}\right)^2\Big|\mathcal{L}_n\right]} \tag{6.5}$$

which is a function of $\mathcal{L}_n$ and Δ^2.

The optimum $\psi_0(\mathcal{L}_n)$ is given by

$$\psi_0(\mathcal{L}_n) = \max_{\Delta^2} \psi_S^*(\mathcal{L}_n) = \frac{(1+\frac{1}{m}\mathcal{L}_n)(\nu_0 - 4)}{(m+3)(\nu_0 - 2)}. \tag{6.6}$$

If $\mathcal{L}_n < \frac{m}{m+2}$, then $\frac{1+\frac{1}{m}\mathcal{L}_n}{m+3} < \frac{1}{m+2}$ which implies also that

$$\psi_S^*(\mathcal{L}_n) < \psi_0(\mathcal{L}_n) \leq \frac{1}{m+2}$$

for all Δ^2, that is $\psi_0(\mathcal{L}_n)$ is closer to the minimizing value than $\frac{1}{m+2}$. So it is obvious that for each Δ^2 and $\mathcal{L}_n$

$$\frac{1}{\sigma_\epsilon^4} E\Big\{[\psi_S(\mathcal{L}_n) m S_U^2 - \sigma_\epsilon^2]^2 \Big| \mathcal{L}_n\Big\} \leq \frac{1}{\sigma_\epsilon^4} E\left\{\left[\frac{mS_U^2}{m+2} - \sigma_\epsilon^2\right]^2 \middle| \mathcal{L}_n\right\} \tag{6.7}$$

so that $m\psi_S(\mathcal{L}_n)S_U^2$ dominates $\frac{mS_U^2}{m+2} = \tilde{\sigma}_\epsilon^2$ uniformly in $\Delta^2 \in (0,\infty)$.

Similarly, we consider the $S_{PT[1]}^2$ with the mean square error $M_5(S_{PT[1]}^2)$ which is optimum at the critical value 1 for all $(1,m)$ under H_0. Then,

$$S_{PT[1]}^2 = S_U^2 I\ (\mathcal{L}_n \geq 1) + S_R^2 I\ (\mathcal{L}_n < 1).$$

Using Stein's method, we have optimum *psi*-function as

$$\psi_{10}(\mathcal{L}_n) = \frac{1+\frac{1}{m}\mathcal{L}_n}{m+1} < \frac{1}{m} \quad \text{for } \mathcal{L}_n \leq 1$$

for all Δ^2. This means that $\psi_{10}(\mathcal{L}_n)$ is closer to the minimum value than $1/m$. Hence,

$$\begin{aligned} E\Big[\{\psi_1(\mathcal{L}_n)\chi_m^2 - 1\}^2 \Big| \mathcal{L}_n\Big] &\leq E\Big[\Big(\frac{\chi_m^2}{m+2} - 1\Big)^2 \Big| \mathcal{L}_n\Big] \\ &\leq \frac{1}{(m+2)^2} E[\{\psi_1(\mathcal{L}_n)\chi_m^2 - (m+2)\}^2 \Big| \mathcal{L}_n] \\ &\leq \frac{1}{m^2} E[\{\psi_1(\mathcal{L}_n)\chi_m^2 - m\}^2 \Big| \mathcal{L}_n]. \end{aligned} \tag{6.8}$$

Thus the estimator $m\psi_S(\mathcal{L}_n)S_U^2$ dominates the PTE(1) of σ_ϵ^2 with critical value 1.

Further, $m\psi_S(\mathcal{L}_n)S_U^2 \leq m\psi_2(\mathcal{L}_n)S_U^2$ and equality holds when the critical value is $(m/m+2)$. Thus, Stein type estimator $m\psi_S(\mathcal{L}_n)S_U^2$ is superior to S_U^2 as well as PTE(1) and PTE(2) uniformly in Δ^2.

7 Conclusion

We have studied the properties of four estimators of location and seven estimators of the scale-parameter of the t-distribution. In the case of location parameter estimators, the biased estimators do better than the unbiased estimators. Also, the shrinkage estimators do better than the PTE under H_0 or near it. In the case of variance estimation, the Stein-type estimator which is a PT-type estimator with given critical value, does better than any other though the improvement may not be significant.

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