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E-OPTIMAL INCOMPLETE BLOCK DESIGNS FOR CORRELATED ERRORS

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SUMMARY

Sufficient conditions are derived for an incomplete block design to be E-optimal in a subclass of connected designs when within-block observations are correlated and generalized least squares estimation method is used. Constructions of the proposed E-optimal designs are given.

Keywords and phrases: E-optimal design; Incomplete block design; Information matrix; Universal optimality.

1 Introduction

Let d be a block design for v treatments having b blocks each with k experimental units or plots and $D(v, b, k)$ be the class of all competing designs for a given v, b, and k. The optimality problem considered in this paper is addressed under the following fixed effects model:

$$Y_d = 1_{bk}\mu + Z\beta + X_d\tau + \epsilon, \qquad \mathrm{cov}(\epsilon) = \Sigma. \tag{1.1}$$

Here Y_d is the $bk \times 1$ vector of observed responses obtained from a design d written in block major order, 1_{bk} is the $bk \times 1$ column vector of ones, τ is the $v \times 1$ vector of treatment effects; X_d is a $bk \times v$ plot-treatment design matrix and β is the vector of parameters for fixed block effects. The plot-block incidence matrix Z is equal to $(I_b \otimes 1_k)$. The error covariance matrix Σ is assumed to be specified as

$$\Sigma = I(b) \otimes \frac{1}{1-\rho^2} \begin{pmatrix} 1 & \rho & \dots & \rho^{k-1} \\ \rho & 1 & \dots & \rho^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{k-1} & \rho^{k-2} & \dots & 1 \end{pmatrix}$$

which says that the observations within each block are correlated according to autoregressive process of order one and that observations from different blocks are uncorrelated. In this paper, we shall only consider nonnegative correlations since negative correlations due to competition between neighboring plots are unlikely to occur in practice. Generalized least squares information matrix C_d for the estimation of contrasts involving treatment parameters in the above model can be written as

$$C_d = X_d'\Sigma^{-1}X_d - X_d'\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-}Z'\Sigma^{-1}X_d. \tag{1.2}$$

The information matrix C_d, for any design d, is non-negative definite and has rank $v-1$ whenever d is a connected design. For a connected design d, we shall let

$$0 < \mu_{d1} \leq \mu_{d2} \leq \ldots, \leq \mu_{d(v-1)}$$

denote the $v-1$ nonzero eigenvalues of C_d. Henceforth, we assume that $D(v,b,k)$ denotes the class of all connected designs for a given v, b and k. Proposition 1 of Kiefer (1975) gives two sufficient conditions for a design d^* to be universally optimal in $D(v,b,k)$. These are

(i) $\mathrm{tr}(C_{d^*}) \geq \mathrm{tr}(C_d)$ for all $d \in D(v,b,k)$, and

(ii) C_{d^*} is completely symmetric.

The universal optimality includes some other widely used optimality criteria including E-, A- and D-optimality. The strongly equineighbored designs constructed by Martin and Eccleston (1991) satisfy the conditions (i) and (ii) within some subclasses of $D(v,b,k)$ for a general dependence structure and hence solves the universal optimality problem fully for all error dependence structures. However, these designs require that the number of blocks b be a multiple of $v(v-1)/2$ if v is odd and multiple of $v(v-1)$ if v is even. Kunert (1985) and Morgan and Chakravarti (1988) noted that universally optimal designs under some specified known dependence structures such as the $AR(1)$ process considered in this paper require the same number of blocks. For other sets of design parameters, literature is very slow in addressing the optimality and construction problems even for specified known error dependence structures.

However, several authors have concentrated on designs that are optimal or highly efficient under generalized least squares method for specified dependence structures, see Gill and Shukla (1985), Kunert (1987), Russel and Eccleston (1987a, 1987b), Uddin (2007, 2008a, 2008b, 2008c, 2008d), for example. We see that this problem can be simplified to some extent by restricting the search to some specific optimality criteria such as E- and MV-optimality within some restricted classes of competing designs. The four recent papers by Uddin (2008a, 2008b, 2008c, 2008d) deal with E- and MV-optimal designs for three treatments in blocks of size three whereas Uddin (2007) deals with E-optimal incomplete block designs in blocks of size $k=3$ for all $v \geq 4$. The present paper is devoted to the investigation of E-optimal incomplete block designs in blocks of size $k \geq 3$ under model (1.1) within the class of designs in which each treatment is replicated either r or $r+1$ times where

$$r = \mathrm{int}(bk/v) = \text{the largest integer not exceeding } bk/v.$$

We shall use $D_r(v,b,k)$ to denote this subclass of designs. A design $d^* \in D_r(v,b,k)$ is said to be E-optimal if it maximizes the smallest nonzero eigenvalue μ_{d1} over all $d \in D_r(v,b,k)$. This is equivalent to saying that the design d^* is E-optimal if $\mu_{d^*1} \geq \mu_{d1}$ for all $d \in D_r(v,b,k)$. Following some notations in section 2, we establish lower and upper bounds for μ_{d1} and identify E-optimal designs within the subclass $D_r(v,b,k)$. These bounds are then utilized in section 3 to construct infinite series of the proposed E-optimal designs. However, we restrict all discussions to $v \geq 3$ and $k \geq 3$ since $v = 2$ and $k \geq 3$ is covered by the work of Mathews (1987) and since, for $k = 2$, the generalized least squares estimation problem reduces to that of ordinary least squares problem in our model.

2 Bounds on the Smallest Nonzero Eigenvalue of C_d and E-Optimal Designs

In this section, first we establish bounds on the smallest nonzero eigenvalue of C_d for an arbitrary design $d \in D_r(v,b,k)$. For the purpose, we utilize the following result from Jacroux (1986).

Lemma 2.1. *Let $C = (c_{ij})_{v\times v}$ be an $v \times v$ matrix which satisfies the following conditions:*

(a) *C is positive semi-definite symmetric matrix with $C_d 1_v = 0$, $1_v' C_d = 0$,*

(b) *$c_{ii} \geq 0$, for all $i = 1, 2, \ldots, v$, and $c_{ij} \leq 0$ for $i, j = 1, 2, \ldots, v, i \neq j$.*

(c) *If $\mu_0 = 0 \leq \mu_1 \leq \mu_2 \cdots \leq \mu_{v-1}$ are the eigenvalues of C, then μ_1 must satisfy the following inequality:*

$$\mu_1 \leq \frac{v}{v-1}\min(c_{11}, c_{22}, \ldots, c_{vv}).$$

For any design $d \in D(v,b,k)$, the information matrix C_d satisfies the conditions (a) and (b) of Lemma 1. Thus the inequality (c) of Lemma 1 holds for the smallest nonzero eigenvalue of C_d for all $d \in D_r(v,b,k)$. We take advantage of this inequality in our search for E-optimal designs by maximizing $\min(c_{d11}, c_{d22}, \ldots, c_{dvv})$ with respect to all $d \in D_r(v,b,k)$. For this purpose, we first introduce the following notations.

e_{diu} = the number of times treatment i appears in the two end plots of the u-th block.

f_{diu} = the number of times treatment i appears in the $k-2$ interior plots of the u-th block.

$n_{dii'}$ = the number of times treatments i and i' occur as neighbors in all b blocks with the convention that this neighbor count is doubled when $i = i'$.

and

$$e_{di} = \textstyle\sum_{u=1}^{b} e_{diu}, \quad f_{di} = \sum_{u=1}^{b} f_{diu}, \quad \lambda_{dEii'} = \sum_{u=1}^{b}(e_{diu}e_{di'u})$$
$$\lambda_{dIii'} = \textstyle\sum_{u=1}^{b}(f_{diu}f_{di'u}), \quad \lambda_{dIEii'} = \sum_{u=1}^{b}(e_{diu}f_{di'u} + e_{di'u}f_{diu}).$$

With these notations, the diagonal elements ($c_{dii}, i = 1, 2, \ldots, v$) and off-diagonal elements ($c_{dii'}, i, i' = 1, 2, \ldots, v; i \neq i'$) of the information matrix C_d can be expressed as follows:

$$\begin{aligned}
c_{dii} &= r_{di} + \rho^2\, f_{di} - \rho\, n_{dii} - \frac{1-\rho}{2+(k-2)(1-\rho)} \sum_{u=1}^{b} (e_{diu} + (1-\rho) f_{diu})^2 \\
c_{dii'} &= -\rho\, n_{dii'} - \frac{1-\rho}{2+(k-2)(1-\rho)} \sum_{u=1}^{b} (e_{diu} + (1-\rho) f_{diu})(e_{di'u} + (1-\rho) f_{di'u}) \\
&= -\rho n_{dii'} - \frac{1-\rho}{2+(k-2)(1-\rho)} (\lambda_{dEii'} + (1-\rho)^2 \lambda_{dIii'} + (1-\rho)\lambda_{dIEii'}).
\end{aligned}$$

Note here that the trace$(C_{d^*}) = \sum_{i=1}^{v} c_{d^*ii}$ is maximum with respect to all designs in $D_r(v,b,k)$ if d^* is binary and if, for the binary design d^*, the quantities $n_{d^*ii'}$, e_{d^*iu}, f_{d^*iu}, $\lambda_{d^*Eii'}$, $\lambda_{d^*Iii'}$, $\lambda_{d^*IEii'}$ are all constants, then d^* is universally optimal. We shall utilize this fact in our construction of E-optimal designs for other design parameters in section 3.

With $r = \text{int}(\frac{bk}{v})$, we now assume $bk = rv + m, 0 \leq m \leq v-1$. Note that $r_{di} \geq r$ for all $d \in D_r(v,b,k)$. For an arbitrary design $d \in D_r(v,b,k)$, we shall use M to denote the set of all treatments i for which $r_{di} = r$, and M^c to denote the set of the remaining treatments. Note that M is a nonempty set with $v - m$ treatment symbols and M^c is an empty set whenever $m = 0$. With

$$\begin{aligned}
\theta_1 &= 1 - \frac{1-\rho}{2+(k-2)(1-\rho)} \quad \text{and} \\
\theta_2 &= (1+\rho^2)(1 - \frac{1-\rho}{2+(k-2)(1-\rho)}) + \frac{2\rho(1-\rho)}{2+(k-2)(1-\rho)},
\end{aligned}$$

we state our first result in the following Theorem.

Theorem 1. *Consider the class $D_r(v,b,k)$ with $bk = rv + m, 0 \leq m \leq v-1$. If $m \geq 1$, we further assume that the parameters of a design $d \in D_r(v,b,k)$ and the correlation coefficient ρ satisfy the condition*

$$(i) \qquad \left[int\left(\frac{b(k-2)}{v}\right) - int\left(\frac{b(k-2) - (v-m)(int(\frac{b(k-2)}{v}) + 1)}{m}\right)\right](\theta_2 - \theta_1) - \theta_1 \geq 0.$$

Then for all $d \in D_r(v,b,k)$, $\mu_{d1} \leq \frac{v}{v-1}(r\theta_1 + int(\frac{b(k-2)}{v})(\theta_2 - \theta_1))$ with equality if d is binary with at least one treatment $i \in M$ having $f_{di} \leq int(b(k-2)/v)$.

Proof. For an arbitrary design $d \in D_r(v,b,k)$ and for all $i = 1, 2, \ldots, v$, we have

$$\begin{aligned} c_{dii} &= r_{di} + \rho^2 f_{di} - \rho\, n_{dii} - \frac{1-\rho}{2+(k-2)(1-\rho)} \sum_{u=1}^{b} [e_{diu} + (1-\rho) f_{diu}]^2 \\ &\leq r_{di} + \rho^2 f_{di} - \frac{1-\rho}{2+(k-2)(1-\rho)} (e_{di} + (1-\rho)^2 f_{di}) \text{ with equality if } d \text{ is binary.} \\ &= r_{di}\theta_1 + f_{di}(\theta_2 - \theta_1) \end{aligned}$$

To establish the upper bound for μ_{d1}, we now consider the following two cases $m = 0$ and $m \geq 1$ separately.

Case I. $m = 0$. In this case, all competing designs are equireplicate and M^c is an empty set. Thus for any design $d \in D_r(v,b,k)$, we have $r_{di} = r$ for all $i \in M$ and $f_{di} \leq \mathrm{int}(\frac{b(k-2)}{v})$ for at least one $i' \in M$. Hence $c_{di'i'} \leq r\theta_1 + \mathrm{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1)$ with equality if d is binary.

Case II. $m \geq 1$. In this case, the set M^c is a nonempty set. For any design $d \in D_r(v,b,k)$, we have $r_{di} = r$ for all $i \in M$ and $f_{di'} \leq \mathrm{int}(\frac{b(k-2)}{v})$ for at least one $i' \in M \cup M^c$. If $i' \in M$, then $c_{di'i'} \leq r\theta_1 + \mathrm{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1)$. If the binary design d assigns treatments satisfying $f_{di} \geq \mathrm{int}(\frac{b(k-2)}{v}) + 1$ for all $i \in M$, then there must be at least one treatment $i^* \in M^c$ for which $f_{di^*} \leq \mathrm{int}(\frac{b(k-2)-(v-m)(\mathrm{int}(\frac{b(k-2)}{v})+1)}{m})$. For this treatment i^*, we have

$$\begin{aligned} c_{di^*i^*} &= (r+1)\theta_1 + \mathrm{int}\left(\frac{b(k-2)-(v-m)(\mathrm{int}(\frac{b(k-2)}{v})+1)}{m}\right)(\theta_2 - \theta_1) \\ &\leq r\theta_1 + \mathrm{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1) \ \text{ by condition}(i). \end{aligned}$$

Thus under both cases I and II, $\min(c_{d11}, c_{d22}, \ldots, c_{dvv}) \leq r\theta_1 + \mathrm{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1)$. The proof is now completed under both cases using Lemma 1. □

Remark 1. The upper limit established for μ_{d1} in Theorem 1 above is independent of the choice of any design $d \in D_r(v,b,k)$. This means that if there exists a design $d^* \in D_r(v,b,k)$ for which $\mu_{d^*1} = \frac{v}{v-1}(r\theta_1 + \mathrm{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1))$ then $\mu_{d^*1} \geq \mu_{d1}$ for all $d \in D_r(v,b,k)$ and d^* is optimal in $D_r(v,b,k)$.

The search for such a design d^* is simplified by establishing a lower bound for μ_{d1} in the following theorem.

Theorem 2. *Suppose $d \in D(v,b,k)$. If w_d is the smallest off-diagonal element of $-C_d$, then $\mu_{d1} \geq w_d v$ for all $d \in D(v,b,k)$.*

Proof. For an arbitrary design $d \in D_r(v,b,k)$, define

$$T_{dx} = C_d - \frac{xv}{v-1}(I_v - \frac{1}{v}1_v1_v')$$

where x is any real number. Then the eigenvalues of T_{dx} are $0, \mu_{di} - \frac{xv}{v-1}, i = 1, 2, \ldots, v-1$. Let $t_{dxii'}$ be the (i,i')-th element of T_{dx}. Then $t_{dxii} = c_{dii} - x$ and $t_{dxii'} = c_{dii'} + \frac{x}{v-1}$. With $x = (v-1)w$, we have $t_{dxii} = c_{dii} - (v-1)w \geq c_{dii} + \sum_{i'=1(i' \neq i)}^{v}(c_{dii'}) = 0$ for all $i = 1, 2, \ldots, v$. Thus with $x = (v-1)w$, we have $t_{dxii} \geq 0$ for all $i = 1, 2, \ldots, v$. Furthermore, with $x = (v-1)w$, $t_{dxii'} = c_{dii'} + w \leq 0$ for all $i \neq i'$. This implies that the matrix T_{dx} with $x = (v-1)w$ is a positive-semidefinite matrix and hence $\mu_{d1} - \frac{xv}{v-1} = \mu_{d1} - vw \geq 0$, completing the proof. □

The lower bound obtained for μ_{d1} in Theorem 2 above depends on the design d. That is to say that this lower bound changes as the design $d \in D_r(v,b,k)$ changes. The trick here is to find a design d for which the lower bound for μ_{d1} is equal to its upper bound given by Theorem 1. This is summarized in the following theorem.

Theorem 3. *Consider the class $D_r(v,b,k)$ with $bk = rv + m$ for some positive $m \leq v-1$. Assume that the parameters satisfy condition (i) of Theorem 1. If there exists a design $d^* \in D_r(v,b,k)$ for which $w_{d^*} = \frac{1}{v-1}(r\theta_1 + int(\frac{b(k-2)}{v})(\theta_2 - \theta_1))$ then $\mu_{d^*1} = vw_{d^*}$ and d^* is E-optimal in $D_r(v,b,k)$.*

The proof of Theorem 3 follows from the inequalities established in Theorems 1 and 2, and Remark 1 stated above following Theorem 1. Since the lower bound of μ_{d1} depends on the largest off-diagonal element of C_d, one may take advantage of this fact in constructing the proposed E-optimal designs.

Example 2.1. The smallest nontrivial example of an E-optimal design of Theorem 3 is a complete block design for $v = 3, b = 2, k = 3$ for which the two blocks may be chosen as $(3,1,2)$ and $(1,2,3)$. The C-matrix for this design is

$$C = \frac{1}{3-\rho}\begin{pmatrix} 4+2\rho & -2-2\rho & -2 \\ -2-2\rho & 4+2\rho & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

with eigenvalues $\mu_0 = 0$, $\mu_1 = 6/(3-\rho)$, and $\mu_2 = (6+4\rho)/(3-\rho)$. Note that, for this design, $r_{di} = 2, i = 1,2,3,\ f_{d1} = 1, f_{d2} = 1, f_{d3} = 0,$. Thus $c_{d33} \leq r\theta_1 = 4/(3-\rho)$ and $w_d = 2/(3-\rho) = \frac{1}{v-1}(r\theta_1 + \text{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1))$ satisfying the conditions of Theorem 3.

3 Construction of E-Optimal Designs

Under model (1.1) with uncorrelated errors (i.e. when $\Sigma = I$), E-optimal designs are constructed in the literature using an augmentation process in which blocks of certain types are added to the blocks of previously known optimal designs, see Jacroux (1980) and Constantine (1981), for example. For our correlated error model, we offer a similar construction technique that augments a class of neighbor balanced incomplete block designs by additional blocks to give E-optimal designs in $D_r(v,b,k)$. We only consider constructions of designs for $v \geq 4$, $k \geq 3$ since optimal designs for $v = 3$, $k = 3$ are covered by the work of Uddin (2008a, 2008b, 2008c, 2008d).

For unequally replicated (bk/v is not an integer) E-optimal incomplete block designs, our construction technique utilizes the method of differences (see Raghavarao, 1971). For the purpose of our construction, we shall let $F_v(x)$ denote the finite field of order v with primitive root x and denote the v treatments by the v elements of $F_v(x) = \{0, x^0, x^1, \ldots, x^{v-1}\}$. We now state a series of unequally replicated E-optimal designs in the following Theorem.

Theorem 4. *Let $v = 2t+1$ be an odd prime or prime power number. Let $x_1, x_2, \ldots, x_k$ be k distinct elements of $F_v(x)$. Let d_1 be the design obtained by developing t initial blocks $(x_1, x_2, \ldots, x_k)x^{(i-1)}, i = 1, 2, \ldots, t,$ over $F_v(x)$. Let d^* be a design obtained by augmenting d_1 by b_2 disjoint binary blocks of size k each such that*

$$(i') \ \ int\left(\frac{2b_2+v}{b_2 k}\right) - \delta \geq \left\{\rho^2 + \frac{2\rho(1-\rho)(1+\rho^2)}{2+(k-3)(1-\rho)}\right\}^{-1}$$

where

$$\delta = \begin{cases} 2 & \textit{if } \left(2b_2 + v/b_2 k\right) \textit{ is an integer} \\ 1 & \textit{otherwise,} \end{cases}$$

then the design d^* is E-optimal in $D_r(v, b = b_1 + b_2, k)$.

Proof. We first note that the design d_1 is universally optimal over the class $D(v,b,k)$ (see Martin and Eccleston, 1991) with the following properties: $b_1 = v(v-1)/2$, $r_{d_1 i} = k(v-1)/2$, $f_{d_1 i} = (k-2)(v-1)/2$, $n_{d_1 ii} = 0$, for all i, $n_{d_1 ii'} = k-1$, $\lambda_{d_1 Eii'} = 1$, $\lambda_{d_1 Iii'} = (k-2)(k-3)/2$, $\lambda_{d_1 IEii'} = 2(k-2)$ for all $i \neq i'$.

By construction, the design d^* is binary and it has at least one treatment $s \in F_v(x)$ for which replications and concurrence numbers are same as those under the design d_1. That is, for this treatment s, the following properties hold:

$r_{d^* s} = k(v-1)/2 \leq r_{d^* i}$, $f_{d^* s} = (k-2)(v-1)/2 \leq f_{d^* i}$, $n_{d^* ss} = 0 \leq n_{d^* ii}$, for all i, and $n_{d^* si'} = k-1 \leq n_{d^* ii'}$, $\lambda_{d^* Esi'} = 1 \leq \lambda_{d^* Eii'}$, $\lambda_{d^* Isi'} = (k-2)(k-3)/2 \leq \lambda_{d^* Iii'}$, $\lambda_{d^* IEsi'} = 2(k-2) \leq \lambda_{d^* IEii'}$ for some $i' \neq s$, and for all $i' \neq i$.

This implies that the smallest off-diagonal element of $-C_{d^{**}}$ is

$$w_{d^*} \quad = \quad \frac{(1-\rho+\rho^2-\rho^3)k^2 - (1-5\rho+5\rho^2-5\rho^3)k - 2\rho(2-3\rho+3\rho^2)}{2(2+(k-2)(1-\rho))}.$$

With $r = k(v-1)/2, \text{int}(b(k-2)/v) = (k-2)(v-1)/2$, and θ_1 and θ_2 as defined above, it can be shown that the expression $\frac{1}{v-1}(r\theta_1 + \text{int}(\frac{b(k-2)}{v})(\theta_2 - \theta_1))$ simplifies to w_{d^*}. The E-optimality of d^* now follows from Theorem 3 by noting that $\text{int}(\frac{b(k-2)}{v}) - \text{int}(\frac{b(k-2)-(v-m)(\text{int}(\frac{b(k-2)}{v})+1)}{m})$ for the design d^* simplifies to $\text{int}(\frac{2b_2+v}{b_2 k})$ - δ and $\theta_1/(\theta_2 - \theta_1) = \{\rho^2 + \frac{2\rho(1-\rho)(1+\rho^2)}{2+(k-3)(1-\rho)}\}^{-1}$. $\square$

One may take $n \geq 1$ copies of b_1 blocks of the design d_1 and augment it by the above b_2 blocks to obtain an E-optimal design in $D_r(v, b = nb_1 + b_2, k)$. The E-optimal designs constructed above in Theorem 4 require that the correlation coefficient ρ and design parameters v, b_2, and k satisfy the condition (i'). We see that this condition is generally satisfied for large positive ρ whenever $m = b_2k$ is small compared to v, see Table 1 where we have evaluated the condition (i') for all odd prime and prime power number v, $9 \leq v \leq 29$. For given v, all possible b_2 and k for which $\rho \in (0, 1)$ are listed in Table 1.

Finally, we like to note that this paper addresses the E-optimality problem within the class of those designs for which $r \in \{\text{int}(bk/v), \text{int}(bk/v) + 1\}$. If bk/v is an integer, the competing designs are restricted to equireplicate designs which are often used in practice. In addition, complete block designs are often preferred in practice whenever $k = v$. However, with no restriction on the class of competing designs, E-optimal designs may be nonbinary and may lie outside the class of equireplicate designs especially when ρ is large, see Uddin (2008a, 2008b).

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Table 1: Correlation coefficient ρ satisfying the inequality (i') for some v, b_2, and k.

v	b_2	k	ρ	v	b_2	k	ρ
9	1	3	$[.45, 1)$	25	1	5	$[.33, 1)$
11	1	3	$[.32, 1)$	25	1	6	$[.43, 1)$
11	1	4	$[.50, 1)$	25	1	7	$[.58, 1)$
13	1	3	$[.32, 1)$	25	1	8	$[.59, 1)$
13	1	4	$[.50, 1)$	27	1	3	$[.13, 1)$
17	1	3	$[.20, 1)$	27	2	3	$[.24, 1)$
17	2	3	$[.45, 1)$	27	3	3	$[.45, 1)$
17	1	4	$[.37, 1)$	27	1	4	$[.21, 1)$
17	1	5	$[.54, 1)$	27	2	4	$[.50, 1)$
17	1	6	$[.56, 1)$	27	1	5	$[.33, 1)$
19	1	3	$[.20, 1)$	27	2	5	$[.54, 1)$
19	2	3	$[.45, 1)$	27	1	6	$[.43, 1)$
19	1	4	$[.30, 1)$	27	1	7	$[.45, 1)$
19	1	5	$[.40, 1)$	27	1	8	$[.59, 1)$
19	1	6	$[.56, 1)$	27	1	9	$[.60, 1)$
23	1	3	$[.15, 1)$	29	1	3	$[.11, 1)$
23	2	3	$[.32, 1)$	29	2	3	$[.24, 1)$
23	3	3	$[.45, 1)$	29	3	3	$[.45, 1)$
23	1	4	$[.25, 1)$	29	4	3	$[.45, 1)$
23	2	4	$[.50, 1)$	29	1	4	$[.21, 1)$
23	1	5	$[.40, 1)$	29	2	4	$[.37, 1)$
23	1	6	$[.43, 1)$	29	1	5	$[.28, 1)$
23	1	7	$[.58, 1)$	29	2	5	$[.54, 1)$
23	1	8	$[.59, 1)$	29	1	6	$[.35, 1)$
25	1	3	$[.15, 1)$	29	1	7	$[.45, 1)$
25	2	3	$[.32, 1)$	29	1	8	$[.59, 1)$
25	3	3	$[.45, 1)$	29	1	9	$[.60, 1)$
25	1	4	$[.25, 1)$	29	1	10	$[.61, 1)$
25	2	4	$[.50, 1)$				