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EXTENDED CONFLUENT HYPERGEOMETRIC SERIES DISTRIBUTION AND SOME OF ITS PROPERTIES

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SUMMARY

Here we introduce a new family of distributions namely the extended confluent hypergeometric series (ECHS) distribution as a generalization of confluent hypergeometric series distributions, Crow and Bardwell family of distributions, displaced Poisson distributions and generalized Hermite distributions. Some important aspects of the ECHS distributions such as probability mass function, mean, variance and recursion formulae for probabilities, moments and factorial moments are obtained.

Keywords and phrases: Crow and Bardwell family of distributions; Displaced Poisson distribution; Hyper-Poisson distribution; Probability generating function

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1 Introduction

Hall (1956) has obtained some hypergeometric series distributions occurring in the study of birth and death processes and named it as the confluent hypergeometric series (CHS) distributions. Bhattacharya (1966) has discussed some of its properties and defined it as follows:

A random variable X is said to have CHS distribution with parameters ν, λ and η if

$$f(x) = P(X = x) = \frac{\Gamma(\nu + x)\Gamma(\lambda)}{\Gamma(\lambda + x)\Gamma(\nu)} \times \frac{\eta^x}{{}_1F_1(\nu; \lambda; \eta)}, \tag{1.1}$$

where $\nu > 0$, $\lambda > 0$, $\eta > 0$; $x = 0, 1, 2, \ldots$ and ${}_1F_1(\nu; \lambda; \eta)$ is the confluent hypergeometric function (for details see Mathai and Saxena, 1973 or Slater, 1960). Several well-known discrete distributions such as Poisson distribution, displaced Poisson distribution of Staff (1964) and Hyper-Poisson distribution of Bardwell and Crow (1964) are special cases of the CHS distribution. Inference for this family has been attempted by Gurland and Tripathi (1975) and Tripathi and Gurland (1977, 1979). The probability generating function (p.g.f.) of the CHS distribution is the following, in which $\delta = [{}_1F_1(\nu; \lambda; \eta)]^{-1}$.

$$H(z) = \delta\ {}_1F_1(\nu; \lambda; \eta z) \tag{1.2}$$

In this article, we obtain an extended version of the CHS distribution and call it as 'the extended confluent hypergeometric series distributions' or in short 'the ECHS distributions'. In section 2 we establish that the ECHS distributions possess a random sum structure and have listed its important special cases. In section 3, we present some of its properties by deriving expressions for its probability mass function, mean and variance. There we also obtain certain recursion formulae for probabilities, moments and factorial moments. It is important to note that since the ECHS distributions possess a random sum structure, it is applicable wherever such a structure arises. The random sum structure arises in several areas of scientific research such as ecology, biology, genetics, physics, operations research etc. For details see Johnson et al. (1992). The ECHS distributions can also have applications in the areas of accident statistics, epidemiological studies and analysis of linguistic data. The results concerning statistical inference in connection with the ECHS distributions are given in the sequel.

2 Genesis and Special Cases

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables, where X_1 has the following p.g.f., in which m is a positive integer.

$$P(z) = \alpha z + (1-\alpha)z^m,$$

where $\alpha = \eta_1/\eta$, $\eta = \eta_1 + \eta_2$, $\eta_1 > 0$ and $\eta_2 \geq 0$. Consider a non-negative integer valued random variable Y having CHS distribution with p.g.f. (1.2). Assume that $Y, X_1, X_2, \ldots$ are independent and let $S_0 = 0$. Define $S_Y = \sum_{n=0}^{Y} X_n$. Then the p.g.f. of S_Y is

$$\begin{aligned} Q(z) = E(z^{S_Y}) = E_Y\{E[z^{S_Y}|Y]\} &= H(P(z)) \\ &= \delta \quad {}_1F_1(\upsilon; \lambda; \eta_1 z + \eta_2 z^m). \end{aligned} \tag{2.1}$$

We define a distribution with p.g.f. (2.1), as the 'the extended confluent hypergeometric series distribution' or in short 'the ECHS distribution'. Clearly, the ECHS distribution with $\eta_2 = 0$ and/or $m = 1$ is the CHS distribution. This family of distributions include several well-known discrete distributions as shown below.

1. When $\upsilon = \lambda$ and $m = 2$, from (2.1) we get the p.g.f. of the Hermite distribution, with parameters $\eta_1 > 0$ and $\eta_2 > 0$, of Kemp and Kemp (1965).

2. When $\upsilon = \lambda$, from (2.1) we get the p.g.f. of the generalized Hermite distribution, with parameters $m > 0$, $\eta_1 > 0$ and $\eta_2 > 0$, of Gupta and Jain (1974).

3. When $\upsilon = 1$, $m = 2$ and λ is a positive integer, from (2.1) we get the p.g.f. of the extended displaced Poisson distribution of type I, with parameters λ, $\eta_1 > 0$ and $\eta_2 > 0$.

4. When $\upsilon = 1$ and λ is a positive integer, from (2.1) we get the p.g.f. of the extended displaced Poisson distribution of type II, with parameters λ, m, $\eta_1 > 0$ and $\eta_2 > 0$.

5. When $\upsilon = 1$ and $m = 2$, from (2.1) we get the p.g.f. of the extended Crow and Bardwell family of distribution of type I, with parameters $\lambda > 0$, $\eta_1 > 0$ and $\eta_2 > 0$.

6. When $\upsilon = 1$, from (2.1) we get the p.g.f. of the extended Crow and Bardwell family of distribution of type II, with parameters $m > 0$, $\lambda > 0$, $\eta_1 > 0$ and $\eta_2 > 0$.

7. When $m = 2$, from (2.1) we get the p.g.f. of the extended Crow and Bardwell family of distributions of type III, with parameters $\upsilon > 0$, $\lambda > 0$, $\eta_1 > 0$ and $\eta_2 > 0$.

3 Properties

Let V be a random variable distributed as the ECHS distributions with p.g.f.

$$\begin{aligned} Q(z) &= \delta \;\; {}_1F_1(\upsilon; \lambda; \eta_1 z + \eta_2 z^m) \\ &= \sum_{r=0}^{\infty} g_r(\upsilon, \lambda) z^r, \end{aligned} \tag{3.1}$$

in which $g_r(\upsilon, \lambda) = P(V = r)$, $r = 0, 1, 2, ...$ and $\delta = [{}_1F_1(\nu; \lambda; \eta_1 + \eta_2)]^{-1}$. On expanding (3.1) and equating the coefficients of z^r, we obtain the following proposition.

Proposition 3.1. *The probability mass function (p.m.f.) $g_r(\upsilon, \lambda)$ of the ECHS distribution with p.g.f. (3.1) is the following, for $r = 0, 1, 2, \ldots$.*

$$g_r(\upsilon, \lambda) = \delta \sum_{n=0}^{[r/m]} \frac{(\upsilon)_{r-(m-1)n}}{(\lambda)_{r-(m-1)n}} \frac{\eta_1^{r-mn} \eta_2^n}{(r - mn)! n!}, \tag{3.2}$$

where $(a)_0 = 1$, $(a)_n = a(a+1)...(a+n-1)$, for $n \geq 1$, and $[r/m]$ denote the integer part of (r/m).

Further we have the following propositions.

Proposition 3.2. *The mean and variance of the ECHS distribution with p.g.f. (3.1) are as follows:*

$$Mean = \frac{\upsilon}{\lambda} \Lambda_1 (\eta_1 + m\eta_2), \tag{3.3}$$

$$Variance = \frac{\upsilon}{\lambda} \left(\frac{\upsilon + 1}{\lambda + 1} \Lambda_2 - \frac{\upsilon}{\lambda} \Lambda_1^2 \right) (\eta_1 + m\eta_2)^2 + \frac{\upsilon}{\lambda} \Lambda_1 (\eta_1 + m^2 \eta_2), \tag{3.4}$$

where

$$\Lambda_j = \delta \;\; {}_1F_1(\upsilon + j; \lambda + j; \eta_1 + \eta_2), j = 1, 2. \tag{3.5}$$

The proof is simple and hence omitted.

Proposition 3.3. *The following is a simple recursion formula for the probabilities $g_r(\upsilon,\lambda)$ of the ECHS distribution with p.g.f. (3.1), for $r \geq 1$.*

$$g_{r+1}(\upsilon,\lambda) = \frac{\upsilon\Lambda_1}{\lambda(r+1)}\{\eta_1 g_r(\upsilon+1,\lambda+1) + m\eta_2 g_{r-m+1}(\upsilon+1,\lambda+1)\}, \tag{3.6}$$

where Λ_1 is as defined in (3.5).

Proof. On differentiating (3.1) with respect to z, we have

$$\sum_{r=0}^{\infty}(r+1)g_{r+1}(\upsilon,\lambda)z^r = \frac{\upsilon}{\lambda}\delta(\eta_1 + m\eta_2 z^{m-1})\ {}_1F_1(\upsilon+1;\lambda+1;\eta_1 z + \eta_2 z^m). \tag{3.7}$$

By replacing υ by $\upsilon+1$ and λ by $\lambda+1$ in (3.1) we get the following.

$$\delta^*\ {}_1F_1(\upsilon+1;\lambda+1;\eta_1 z + \eta_2 z^m) = \sum_{r=0}^{\infty} g_r(\upsilon+1,\lambda+1)z^r, \tag{3.8}$$

where $\delta^* = [{}_1F_1(\upsilon+1;\lambda+1;\eta_1+\eta_2)]^{-1}$. Relations (3.7) and (3.8) together lead to the following relationships:

$$\sum_{r=0}^{\infty}(r+1)g_{r+1}(\upsilon,\lambda)z^r = \frac{\upsilon\Lambda_1}{\lambda}\sum_{r=0}^{\infty}\left\{\eta_1 g_r(\upsilon+1,\lambda+1)z^r + m\eta_2 g_r(\upsilon+1,\lambda+1)z^{r+m-1}\right\} \tag{3.9}$$

On equating coefficients of z^r on both sides of (3.9) we get (3.6). □

Proposition 3.4. *The following is a recursion formula for the factorial moments $\mu_{[n]}(\upsilon,\lambda)$ of the ECHS distribution for $n \geq 1$, in which $\mu_{[0]}(\upsilon,\lambda) = 1$.*

$$\mu_{[n+1]}(\upsilon,\lambda) = \frac{\upsilon\Lambda_1}{\lambda}\left[\eta_1\mu_{[n]}(\upsilon+1,\lambda+1) + m\eta_2\sum_{r=0}^{m-1}\binom{m-1}{r}n^{(r)}\mu_{[n-r]}(\upsilon+1,\lambda+1)\right], \tag{3.10}$$

where $n^{(r)} = n(n-1)(n-2)\cdots(n-r+1)$, for any positive integer r and $n^{(0)} = 1$.

Proof. The factorial moment generating function $F_V(t)$ of the ECHS distribution with p.g.f. (3.1) has the following series representation.

$$F_V(t) = Q(1+t) = \delta\ {}_1F_1[\upsilon;\lambda;\eta_1(1+t) + \eta_2(1+t)^m] = \sum_{n=0}^{\infty}\mu_{[n]}(\upsilon,\lambda)\frac{t^n}{n!} \tag{3.11}$$

Differentiate (3.11) with respect to t to obtain

$$\frac{\upsilon}{\lambda}\delta[\eta_1 + m\eta_2(1+t)^{m-1}]{}_1F_1[\upsilon+1;\lambda+1;\eta_1(1+t)+\eta_2(1+t)^m] = \sum_{n=1}^{\infty}\mu_{[n]}(\upsilon,\lambda)\frac{t^{n-1}}{(n-1)!}. \tag{3.12}$$

By using (3.11) with υ, λ replaced by $\upsilon+1, \lambda+1$ respectively, we obtain the following from (3.12).

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_{[n+1]}(\upsilon,\lambda)\frac{t^n}{n!} &= \frac{\upsilon}{\lambda}\Lambda_1[\eta_1+m\eta_2(1+t)^{m-1}]\sum_{n=0}^{\infty}\mu_{[n]}(\upsilon+1,\lambda+1)\frac{t^n}{n!}\\ &= \frac{\upsilon\Lambda_1}{\lambda}\sum_{n=0}^{\infty}\Bigg[\eta_1\mu_{[n]}(\upsilon+1,\lambda+1)\frac{t^n}{n!}\\ &\quad +m\eta_2\sum_{r=0}^{m-1}\binom{m-1}{r}\mu_{[n]}(\upsilon+1,\lambda+1)\frac{t^{n+r}}{n!}\Bigg] \qquad (3.13)\end{aligned}$$

Now, on equating coefficients of $\frac{t^n}{n!}$ on both sides of (3.13) we get (3.10). □

Proposition 3.5. *The following is a recursion formula for the raw moments $\mu_n(\upsilon,\lambda)$ of the ECHS distribution, for $n \geq 0$.*

$$\mu_{n+1}(\upsilon,\lambda) = \frac{\upsilon}{\lambda}\Lambda_1\sum_{r=0}^{n}\binom{n}{r}(\eta_1+m^{r+1}\eta_2)\mu_{n-r}(\upsilon+1,\lambda+1). \qquad (3.14)$$

Proof. The characteristic function $\phi_V(t)$ of the ECHS distribution with p.g.f. (3.1) has the following series representation. For $t\epsilon R$,

$$\phi_V(t) = Q(e^{it}) = \delta\ {}_1F_1(\upsilon;\lambda;\eta_1e^{it}+\eta_2e^{mit}) = \sum_{n=0}^{\infty}\mu_n(\upsilon,\lambda)\frac{(it)^n}{n!}. \qquad (3.15)$$

On differentiating (3.15) with respect to t, we obtain

$$\frac{\upsilon}{\lambda}\delta(\eta_1e^{it}+m\eta_2e^{mit})_1F_1(\upsilon+1;\lambda+1;\eta_1e^{it}+\eta_2e^{mit}) = \sum_{n=1}^{\infty}\mu_n(\upsilon,\lambda)\frac{(it)^{n-1}}{(n-1)!}. \qquad (3.16)$$

By using (3.15) with υ, λ replaced by $\upsilon+1$, $\lambda+1$ respectively, we get the following from (3.16).

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_{n+1}(\upsilon,\lambda)\frac{(it)^n}{n!} &= \frac{\upsilon}{\lambda}\Lambda_1(\eta_1e^{it}+m\eta_2e^{mit})\sum_{n=0}^{\infty}\mu_n(\upsilon+1,\lambda+1)\frac{(it)^n}{n!}\\ &= \frac{\upsilon\Lambda_1}{\lambda}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}\left\{\frac{\eta_1(it)^k}{k!}+\frac{m\eta_2(mit)^k}{k!}\right\}\right]\frac{\mu_n(\upsilon+1,\lambda+1)(it)^n}{n!}.\end{aligned}$$

On equating coefficients of $\frac{(it)^n}{n!}$ on both sides we get (3.14). □

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References

[1] Bardwell, G.E. and Crow, E.L. (1964). A two parameter family of hyper-Poisson distributions. *Journal of American Statistical Association*, **59**, 133–141.

[2] Bhattacharya, S.K. (1966). Confluent hypergeometric distributions of discrete and continuous type with applications to accident proneness. *Bulletin of the Calcutta Statistical Association*, **15**, 20–31.

[3] Gupta, R.P. and Jain, G.C. (1974). A generalized Hermite distribution and its properties. *SIAM Journal of Applied Mathematics*, **27**, 359–363.

[4] Gurland, J. and Tripathi, R.C. (1975). Estimation of parameters on some extensions of the Katz family of discrete distributions involving hypergeometric functions. *Statistical distributions in scientific work, 1: Models and Structures*. G.P. Patil, S. Kotz and J.K. Ord (editors) 59–82 Dordrecht,Reidel.

[5] Hall, W.J. (1956). Some hypergeometric series distributions occurring in birth-and-death processes at equilibrium (abstract). *Annals of Mathematical Statistics*, **27**, 221.

[6] Johnson, N.L., Kotz, S., and Kemp, A.W. (1992). *Univariate Discrete Distributions*, 2^{nd} edition. Wiley, New York.

[7] Kemp, C.D. and Kemp, A.W. (1965). Some properties of the Hermite distribution. *Biometrika*, **52**, 381–394.

[8] Mathai, A.M. and Saxena, R.K. (1973). *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lecture Notes No. 348. Springer-Verlag, Heidelberg.

[9] Slater, L.J. (1960). *Confluent Hypergeometric functions*. Cambridge University Press, Cambridge.

[10] Staff, P.J. (1964). The displaced Poisson distribution. *Australian Journal of Statistics*, **6**, 12–20.

[11] Tripathi, R.C. and Gurland, J. (1977). A general family of discrete distributions with hypergeometric probabilities. *Journal of the Royal Statistical Society, Series B*, **39**, 349–356.

[12] Tripathi, R.C. and Gurland, J. (1979). Some aspects of the Kemp families of distributions. *Communications in Statistics-Theory and Methods*, **8**, 855–869.