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INFERENCE ON RELIABILITY $P(Y < X)$ IN TRUNCATED ARCSINE DISTRIBUTIONS

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SUMMARY

We consider inference on reliability $R = P(Y < X)$, where the random variables X and Y are independent truncated arcsine distributions with parameters β_1 and β_2, respectively. We obtain the maximum likelihood estimator of R in terms of $\rho = \beta_1/\beta_2$ and observe that inference on R is equivalent to inference on ρ, where R is a monotone function of ρ. We obtain an exact and an approximate confidence interval for ρ and also provide a test for ρ.

*Keywords and phrases:*Maximum likelihood estimator (MLE); Method of moment estimator (MME); Mean square error (MSE).

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1 Introduction

The problem of estimating the probability that a random variable Y is less than an independent random variable X, arises in reliability studies. When the random variable Y represents a stress that a device is subjected to in service and the random variable X represents the strength that varies from item to item in the population of devices, then the reliability R, i.e., the probability that a randomly selected device functions successfully, is equal to $P(Y < X)$. The same problem also arises in the context of statistical tolerance where Y represents the diameter of a shaft and X denotes the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then $P(Y < X)$. In biometry, Y represents a patient's remaining years of life if treated with drug A and X represents a patient's remaining years when treated with drug B. If the choice of drug is left to the patient, then the person's deliberation will center on whether $P(Y < X)$ is less than or greater than 1/2.

Estimation of $R = P(Y < X)$ has been considered over the years for many probability distributions, e.g., exponential, normal, gamma, Pareto etc., with unknown parameters. Maximum likelihood, minimum variance unbiased estimator (MVUE), and Bayes estimators of R were found by various authors. See for example Ali and Woo (2005a), (2005b), Ali, Pal and Woo (2005), Ali, Woo and Pal (2004), Basu (1964), Beg (1980), Church and Harris (1970), Downtown (1973), Ivshin (1996), Iwase (1987), Kelley, Kelley, and Suchany (1976), McCool (1991), Pal, Ali, and Woo (2005), Tong (1977, 1974), and Sathe and Verde (1969).

A truncated distribution arises in the statistical theory of communications. Norton (1978) and Arnold and Groenveld (1980) derived the characterization of the arcsine distribution. These characterizations are potentially useful for inference whether data arises from a random walk. Woo and Ali (1997) considered an unbiased estimation of the standard deviation of the arcsine distribution. Ali, Woo, and Lee (1999) studied a right-tail probability estimation in a truncated arcsine distribution. In this paper, we consider inference on reliability $R = P(Y < X)$ in two independent truncated arcsine distributions.

2 Parameter Estimation

Let X be a truncated arcsine random variable with the pdf

$$f(x) = \frac{2}{\pi} \cdot \frac{1}{\sqrt{\beta^2 - x^2}}, \quad 0 < x < \beta, \tag{2.1}$$

(see Woo and Ali, 1997).

Let $X_1, X_2, \ldots, X_n$ be a random sample from (2.1) and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics.

Lemma 2.1. *Let $X = \beta \cos\theta$. Then X has a truncated arcsine pdf (2.1) iff θ follows a uniform distribution over (0, $\frac{\pi}{2}$) (see, Woo and Ali, 1997).*

From formula 3.14 in Oberhettinger and Badii (1973), we obtain the moment generating function of the truncated arcsine random variable X as

$$M_X(t) = I_0(-\beta t) - L_0(-\beta t)$$

where $I_0(x)$ and $L_0(x)$ are modified Bessel function and modified Struve function of order zero, respectively.

From formulas 9.6.10 and 12.2.1 in Abramowitz and Stegun (1970), $I_0(-\beta t)$ and $L_0(-\beta t)$ can be represented by the following infinite sums

$$I_0(-\beta t) = \sum_{k=0}^{\infty} \frac{(\beta T)^{2k}}{(2^k k!)^2} \quad \text{and} \quad L_0(-\beta t) = \sum_{k=0}^{\infty} \frac{(-\frac{\beta}{2}t)^{2k+1}}{\Gamma^2(k + \frac{3}{2})}.$$

From formulas 3.248(2) and 3.248(3) in Gradshteyn and Ryzhik (1965), the nth moment of X can be guaranteed by the existence of its mgf and is given by

$$E(X^n) = \begin{cases} \frac{(2k-1)!!}{(2k)!!}\beta^n, & \text{if} \quad n = 2k, \qquad k = 1, 2, \ldots \\ \frac{(2k-1)!!}{(2k)!!}\frac{2}{\pi}\beta^n, & \text{if} \quad n = 2k+1, \quad k = 0, 1, 2, \ldots \end{cases}$$

where $(2k)!! \equiv 2 \cdot 4 \cdots (2k)$ and $(2k+1)!! \equiv 1 \cdot 3 \cdot 5 \cdots (2k+1)$, and $0!! = 1$.

From the nth moment of X, mean and variance of a truncated arcsine random variable X are given by

$$E(X) = \frac{2}{\pi}\beta \text{ and } Var(X) = (\frac{1}{2} - \frac{4}{\pi^2})\beta^2, \text{ (see Woo and Ali (1997)).}$$

The method of moment estimator (MME) and the maximum likelihood estimator (MLE) of β are

$$\tilde{\beta} = \frac{\pi}{2n}\sum_{i=1}^{n} X_i, \text{ and } \hat{\beta} = X_{(n)}, \text{ respectively.}$$

From the nth moment of X, we obtain the following:

$$E(\tilde{\beta}) = \beta, \text{ and } Var(\tilde{\beta}) = \frac{1}{n}(\frac{\pi^2}{8} - 1)\beta^2.$$

Since θ follows a uniform distribution over $(0, \pi/2)$, the corresponding first order statistic $\theta_{(1)}$ converges to zero in probability and hence from Lemma 2.1 and variance of MME of β, we have the following.

Proposition 2.1. *(a) The MLE is a consistent estimator of β. (b) The MME is a consistent estimator of β.*

From Lemma 2.1, formulas 2.631(4), 2.631(5), 2.633(1), and 2.633(2) in Gradshteyn and Ryzhik (1965), we can obtain the rth moment of the MLE $\hat{\beta} = X_{(n)}$ as follows.

Fact 1:

(a) If $r = 2p$, then

$$\begin{aligned} E(X_{(n)}^r) \quad = \quad & \beta^r \Bigg[\begin{pmatrix} 2p \\ p \end{pmatrix} 2^{-2p} + \frac{(-1)^p}{2^{2p-1}} \sum_{k=0}^{p-1} ((-1)^k \begin{pmatrix} 2p \\ k \end{pmatrix} \cdot \sum_{j=0}^{n-1} \frac{n!}{(n-j-1)!} \\ & \frac{\pi^{-j-1}}{(p-k)^{j+1}} \sin((2p-2k+j)\frac{\pi}{2}) - \frac{n!}{(p-k)^n \pi^n} \sin((n-1)\pi))) \Bigg] \end{aligned}$$

(b) If $r = 2p+1$, then

$$\begin{aligned} E(X_{(n)}^r) \quad = \quad & \beta^r \Bigg[\frac{(-1)^p}{2^{2p}} \sum_{k=0}^{p} ((-1)^k \begin{pmatrix} 2p \\ k \end{pmatrix} \cdot \frac{n!}{(p-k+1/2)^n \pi^n} \cos((n-1)\frac{\pi}{2} - \\ & \sum_{j=0}^{n-1} \frac{n!}{(n-j-1)!(p-k+1/2)^{j+1}\pi^{j+1}} \cos((2p-2k+j+1)\frac{\pi}{2}))) \Bigg]. \end{aligned}$$

From Lemma 2.1, formulas 2.643(1), 2.643(3) and 2.643(7) in Gradshteyn and Ryzhik (1965), we obtain the rth moment of reciprocal of the MLE $\hat{\beta} = X_{(n)}$ as given below.

Fact 2:

(a) If $r = 1$, then

$$E\left(\frac{1}{X_{(n)}}\right) = \frac{1}{\beta}\left[\frac{n}{n-1}\frac{2}{\pi} + 2n\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{k-1}-1)(\pi/2)^{2k-1}}{(2k)!(n-1+2k)} \cdot B_{2k}\right].$$

(b) If $r = 2$, then

$$E\left(\frac{1}{X_{(n)}^2}\right) = \frac{1}{\beta^2}\left[\frac{n(n-1)}{n-2}(\frac{2}{\pi})^2 + n(n-1)\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}(\pi/2)^{2k-2}}{(2k)!(n-2+2k)} \cdot B_{2k}\right].$$

(c) (Recursion formula) If $n > r+1 > 3$, then

$$\begin{aligned} E\left(\frac{1}{X_{(n)}^r}\right) &= \frac{1}{\beta^r}\left[-\frac{n(n-1)}{(r-1)(r-2)}(\frac{2}{\pi})^2 + n(\frac{2}{\pi})^n \cdot \frac{r-2}{r-1} \cdot I(n-1, r-2)\right. \\ &\quad \left. + \frac{n(n-1)(n-2)}{(r-1)(r-2)}(\frac{2}{\pi})^n \cdot I(n-3, r-2)\right], \end{aligned}$$

where $I(n-1, r) \equiv \int_0^{\pi/2} \frac{x^{n-1}}{\sin^r x} dx$ and B_n is a Bernoulli number.

Remark 2. The Bernoulli numbers B_n are given as follows:

$$B_0 = 1, B_1 = -1/2, B_{2k+1} = 0, k = 1, 2, 3 \ldots, \text{ and } B_n = \sum_{k=0}^{n}\binom{n}{k}B_k.$$

Remark 3. To apply the recursion formula (c) in Fact 2, we use the following.

$$I(p, 1) = \frac{1}{p}(\frac{\pi}{2})^p + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2(2^{k-1}-1)}{(p+2k)(2k)!} \cdot B_{2k} \cdot (\frac{\pi}{2})^{p+2k} \text{ and}$$

$$I(p, 2) = \frac{p}{p-1}(\frac{\pi}{2})^{p-1} + p \cdot \sum_{k=1}^{\infty}(-1)^k\frac{\pi^{p+2k-1} \cdot 2^{1-p}}{(2k)!(p+2k-1)} \cdot B_{2k}, \text{ if } p > 1.$$

From Fact 1(b), we can obtain the expectation of the MLE $\hat{\beta}$ as

$$E(\hat{\beta}) = \beta \cdot [n!(\frac{2}{\pi})^n \cos((n-1)\frac{\pi}{2}) - \sum_{k=0}^{n-1}\frac{n!}{(n-k-1)!}(\frac{2}{\pi})^{k+1}\cos((k+1)\frac{\pi}{2})].$$

From Fact 1(a), we can obtain the second moment of the MLE $\hat{\beta}$ as:

$$E(\hat{\beta}^2) = \beta^2[\frac{1}{2} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{n!}{(n-j-1)!}\pi^{-j-1}\sin(\frac{\pi}{2} \cdot j) + \frac{n!}{2 \cdot \pi^n}\sin((n-1)\frac{\pi}{2})].$$

Table 1: MSE of MME and MLE (unit: β^2)

n	MME	MLE
10	0.1234	0.0114
20	0.0618	0.0057
30	0.0411	0.0019

Therefore, we can obtain the variance of the MLE $\hat{\beta}$. Table 1 shows the numerical values of MSE of the MME and MLE.

From Table 1, the MLE has less MSE than the MME when $n = 10$, 20, 30. Hence we recommend the MLE of β in parametric estimation in a truncated arcsine distribution rather than the MME of β in the sense of MSE.

3 Reliability

Let X and Y be independent truncated arcsine random variables each having parameter β_1 and β_2, respectively. Then the reliability can be obtained as follows.

$$R = P(Y < X) = \begin{cases} 1 - \frac{4}{\pi^2}\int_0^1 \cos^{-1}\frac{\beta_1}{\beta_2}y \cdot \frac{1}{\sqrt{1-y^2}}dy, & \text{if } \frac{\beta_1}{\beta_2} < 1 \\ 1 - \frac{4}{\pi^2}\int_0^1 \cos^{-1} y \cdot \frac{1}{\sqrt{(\beta_1/\beta_2)^2 - y^2}}dy, & \text{if } \frac{\beta_1}{\beta_2} > 1. \end{cases} \tag{3.1}$$

Therefore, reliability $R = P(Y < X)$ depends on $\rho \equiv \frac{\beta_1}{\beta_2}$ only and since the reliability function is a monotonic function of ρ, inference on ρ is equivalent to inference on R (see McCool (1991)).

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent samples from the above random variables X and Y, respectively. From the result of Section 2,

$$\hat{\rho} = \frac{\hat{\beta}_1}{\hat{\beta}_2} = \frac{X_{(m)}}{Y_{(n)}}$$

is recommended as the estimator of $\rho \equiv \frac{\beta_1}{\beta_2}$ which is the MLE of ρ and which is also a consistent estimator of ρ from Proposition 2.1(a).

From Facts 1, 2(a) and 2(b), we can obtain the first and second moments of $\hat{\rho}$ to find the mean and variance of $\hat{\rho}$.

$$\begin{aligned} E(\hat{\rho}) \quad = \quad & \rho[m!(\frac{2}{\pi})^m \cos((m-1)\frac{\pi}{2}) - \sum_{k=0}^{m-1} \frac{m!}{(m-k-1)!}(\frac{2}{\pi})^{k+1}\cos((k+1)\frac{\pi}{2})] \cdot \\ & [\frac{n}{n-1}\frac{2}{\pi} + 2n\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{k-1}-1)(\pi/2)^{2k-1}}{(2k)!(n-2+2k)} \cdot B_{2k}], \text{ and} \end{aligned}$$

$$E(\hat{\rho}^2) = \rho^2[\frac{1}{2} + \frac{1}{2}\sum_{j=0}^{m-1} \frac{m!}{(m-j-1)!}\pi^{-j-1}\sin(\frac{\pi}{2}\cdot j) + \frac{m!}{2\cdot\pi^m}\sin((m-1)\frac{\pi}{2})]\cdot$$
$$[\frac{n(n-1)}{n-2}(\frac{2}{\pi})^2 + n(n-1)\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}(\pi/2)^{2k-2}}{(2k)!(n-2+2k)!}\cdot B_{2k}].$$

Next, the reliability (3.1) can be represented by infinite series as follows. From the formula 1.641(1) in Gradshteyn and Ryzhik (1965) and formula 2.20 in Oberhettinger (1974),

$$R = P(Y < X) = \frac{4}{\pi^{3/2}}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k+1}k!(2k+1)\Gamma(k+3/2)}\rho^{2k+1},\ \text{if } \rho < 1,$$

and from the formulas 2.814(1) and 4.523(3) in Gradshteyn and Ryzhik (1965),

$$R = P(Y < X) = 1 - \frac{4}{\pi^2\cdot\rho}[1 + \sum_{k=1}^{\infty}\frac{k!(2k-1)!!2^k}{(2k)!!(2k+1)!!(2k+1)\cdot\rho^k}],\ \text{if } \rho > 1.$$

Especially, if $\rho = 1$, then $R = P(Y < X) = \frac{1}{2}$.

The MLE $\hat{R}$ of reliability $R = P(Y < X)$ is obtained by replacing ρ by $\hat{\rho}$. However, by the result of McCool (1991), it is sufficient to consider inference on ρ instead of R.

Next, we find a pivot quantity to find a confidence interval for ρ. From the quotient pdf of Theorem 7 in Rohatgi (1976, p.141), formula 2.23 in Oberhettinger (1974), and formula 15.1.4 in Abramowitz and Stegun (1970), the distribution of $W \equiv \frac{1}{\rho}\cdot\frac{X_{(m)}}{Y_{(n)}}$ can be obtained as follows.

$$f_W(w) = \begin{cases} \frac{4}{\pi}\cdot\frac{1}{w}\cdot\tan^{-1}\frac{w}{\sqrt{1-w^2}}, & \text{if} \quad 0 < w < 1 \\ \frac{4}{\pi}\cdot\frac{1}{w}\cdot\tan^{-1}\frac{w}{\sqrt{w^2-1}}, & \text{if} \quad w > 1. \end{cases} \tag{3.2}$$

Hence, since distribution of W does not involve the parameter ρ, the quantity W is a pivot quantity.

From the result (3.2), for given $0 < \alpha < 1$, we can obtain a $(1-\alpha)100\%$ confidence interval for ρ as follows.

$$\left(b_{\alpha/2}\cdot\frac{x_{(m)}}{y_{(n)}},\ \ \frac{1}{b_{\alpha/2}}\cdot\frac{x_{(m)}}{y_{(n)}}\right)$$

is a $(1-\alpha)100\%$ confidence interval for ρ where $b_{\alpha/2}$ is defined by

$$\frac{\alpha}{2} = \frac{4}{\pi^2}\cdot\int_0^{b_{\alpha/2}}\frac{1}{x}\cdot\tan^{-1}\frac{x}{\sqrt{1-x^2}}dx = \frac{4}{\pi^2}\cdot\int_{\frac{1}{b_{\alpha/2}}}^{\infty}\frac{1}{x}\cdot\tan^{-1}\frac{1}{\sqrt{x^2-1}}dx.$$

We now consider the asymptotic confidence interval based on the MLE $\hat{\rho}$. From Proposition 2.1, since $\hat{\rho}$ is a consistent estimator of ρ, we can obtain an asymptotic confidence intervals given below.

$$\left(\hat{\rho} - z_{\alpha/2}\cdot\sqrt{\widehat{Var(\hat{\rho})}},\ \ \hat{\rho} + z_{\alpha/2}\cdot\sqrt{\widehat{Var(\hat{\rho})}}\right)$$

is an $(1-\alpha)100\%$ confidence interval for ρ where $1-\alpha/2 = \int_{-\infty}^{z_{\alpha/2}} \phi(t)dt$, and where $\phi(t)$ is the standard normal pdf.

To test $H_0 : \rho = 1$ against $H_1 : \rho \neq 1$, the critical region can be obtained using likelihood ratio test as:

$$\hat{\rho} \leq c_1 \text{ or } \hat{\rho} \geq c_2.$$

Therefore, from result (3.2), for a given significance level $0 < \alpha < 1$, if the null hypothesis is true, the critical region is given by

$$\frac{X_{(m)}}{Y_{(n)}} \leq b_{\alpha/2} \text{ or } \frac{X_{(m)}}{Y_{(n)}} \geq \frac{1}{b_{\alpha/2}}.$$

Also, the two outside tails of the preceding asymptotic confidence interval for ρ can be used as the critical region for the test.

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