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STEIN-RULE ESTIMATION IN ULTRASTRUCTURAL MODEL UNDER EXACT LINEAR RESTRICTIONS

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summary

The role and construction of Stein-rule estimators in multivariate ultrastructural model is discussed when some prior information about the regression coefficients is available in the form of exact linear restrictions. The additional information in the forms of covariance matrix of measurement errors and reliability matrix of explanatory variables is used for the construction of consistent estimators. Two families of Stein-rule estimators are proposed using each type of additional information which are consistent as well as satisfy the exact linear restrictions. The distribution of measurement errors is assumed to be not necessarily normally distributed. The asymptotic distribution of the proposed families of Stein-rule estimators are derived and studied. The finite sample properties of the estimators are studied through a Monte-Carlo simulation experiment.

Keywords and phrases: Measurement errors, exact linear restriction, Stein-rule estimators, ultrastructural model, reliability matrix.

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1 Introduction

A fundamental assumption in all statistical analysis is that the recorded observations on variables are free from any error. Such an assumption is often violated in many practical situations and measurement error enters into the data. The presence of measurement error in the data agitates the optimal statistical properties of the estimators and tools. In the context of linear regression model, the ordinary least squares estimator (OLSE) is the best linear unbiased estimator of regression coefficients in the absence of measurement error in the data. When measurement errors enter into the data, the same OLSE becomes biased as well as inconsistent. An important issue in a linear measurement error model is how to obtain the consistent estimators of regression coefficients. Such consistent estimators can be obtained when some additional information can be incorporated in the model. The additional information can be available in various forms, see Cheng and Van Ness (1999) and Fuller (1987) for more details. In the context of multivariate measurement error models, the additional information in the form of covariance matrix of measurement errors associated with explanatory variables or reliability matrix of explanatory variable are generally employed to obtain the consistent estimators of regression coefficients. In both cases, the arising estimators can be viewed as if OLSE is adjusted in a particular way. When covariance matrix of measurement errors associated with explanatory variables is used, then the arising consistent estimator has been termed as "adjusted least squares estimator" by Schneeweiss (1976). The interpretation of "adjusted" arises because the estimator looks like as if an OLSE is adjusted for its inconsistency by the known covariance matrix of measurement errors. The reliability matrix is a multivariate generalization of reliability ratio in a scalar case, see Gleser (1992) for more details on reliability matrix. When additional information on reliability matrix of explanatory variables is used to obtain a consistent estimator of regression coefficient, the resulting estimator again looks like as if OLSE is adjusted for its inconsistency by multiplication of inverse of reliability matrix.

Often, it is found that the biased estimators may be superior in mean squared error terms when compared to the unbiased least squares rule. One example of such biased estimators in the context of linear model arises by shrinkage estimation. A popular family of estimators arising from the shrinkage estimation is characterized by Stein-rule estimators which are more efficient than OLSE under a simple rider that the number of explanatory variables are not less than three, see Judge and Bock (1978) and Saleh (2006) for more details on Stein-rule estimation. The use of Stein-rule family of estimators in multivariate measurement error models is analyzed by Shalabh (1998, 2000).

In many situations, some prior information about the regression coefficients is available which improves upon the efficiency of OLSE. Such prior information can be available from different types of sources, e.g., from some extraneous sources, similar kind of experiments conducted in the past, long association of experimenter with the experiment, etc. For example, the Cobb-Douglas production function in economics has a linear constraint of constant returns to scale. When such prior information is expressible in the form of exact linear restrictions binding the regression coefficients, the theory of restricted least squares estimation can be employed to estimate the regression coefficients. When there are no measurement errors in data then the resulting restricted least squares estimator (RLSE) is unbiased, consistent, satisfies the given exact linear restrictions on regression coefficients and has smaller variability around mean than the OLSE, see Toutenburg (1982) and Rao et al. (2008). However, the RLSE becomes inconsistent and biased when observations are contaminated with measurement errors. The problem of finding the estimators which are consistent as well as satisfy the exact linear restrictions in presence of measurement errors in data by using the adjusted forms of OLSE and additional information in the form of covariance matrix of the measurement errors or reliability matrix associated with explanatory variables is considered by Shalabh et al. (2007). What is the role of Stein-rule estimation in obtaining the consistent estimators of regression coefficients in measurement error models when some prior information about the regression coefficients is available in the form of exact linear restrictions is addressed in this paper.

Another popular assumption in measurement error models is that the distribution of measurement errors is assumed to be normal. When this assumption is violated, the results obtained on the assumption of normality may not remain valid. The effect of non-normally distributed measurement errors has been analyzed by Srivastava and Shalabh (1997a, 1997b) and Shalabh (2003) in univariate measurement error models. What is the effect of non-normally distributed measurement errors on the properties of Stein-rule estimation under exact linear restriction is another issue explored in this article.

The plan of the paper is as follows. The multivariate ultrastructural model, exact linear restrictions on regression coefficients and various statistical assumptions are described in Section 2 . In Section 3, the construction of consistent Stein-rule families of estimators is presented which satisfy the given restrictions. The asymptotic distributions and properties of the estimators are given in Section 4. The findings about the finite sample properties of the estimators from a Monte-Carlo simulation experiment are presented in Section 5. Section 6 contains the concluding remarks followed by some useful lemmas in Appendix.

2 Model Specification

Suppose the true values of study variable are linearly related with the true values of p independent explanatory variables. The true values of study variable and explanatory variables are unobservable due to the presence of measurement errors and only measurement ridden observations are available. Such a set up can be presented by a multivariate linear ultrastructural measurement error (MLUME) model due to Dolby (1976) which is given by

$$\left.\begin{aligned} \eta &= T\beta \\ y &= \eta + \epsilon \\ X &= T + \Delta \end{aligned}\right\}, \tag{2.1}$$

where η is a $n \times 1$ vector of n true values on the study variable, $T = (\xi_{ij},\, i = 1, 2, \ldots, n;\, j = 1, 2, \ldots, p)$ is a $n \times p$ matrix of n true values on each of the p explanatory variables, and β is a $p \times 1$ vector of regression coefficients. The true values η are observed as a $n \times 1$ vector y with measurement error vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)'$ of order $n \times 1$ and T is observed with $n \times p$ matrix of measurement errors $\Delta = (\delta_{ij},\, i = 1, 2, \ldots, n;\, j = 1, 2, \ldots, p)$. For $i = 1, 2, \ldots, n;\, j = 1, 2, \ldots, p$, we assume that $\xi_{ij} = \mu_{ij} + \phi_{ij}$, where μ_{ij} are fixed constant as mean of ξ_{ij} and ϕ_{ij} are associated random errors. Thus, in matrix notations

$$T = M + \Phi, \tag{2.2}$$

where $M = (\mu_{ij};\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ and $\Phi = (\phi_{ij},\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ are $n \times p$ matrices of means μ_{ij} and random error components ϕ_{ij}, respectively.

Further we make the following assumptions about the distributions of measurement errors and random error component:

(i) ϵ_i, $(i = 1, 2, \ldots, n)$ are independent and identically distributed with mean 0, variance σ_ϵ^2, third moment $\gamma_{1\epsilon}\sigma_\epsilon^3$ and fourth moment $(\gamma_{2\epsilon}+3)\sigma_\epsilon^4$. Here, for a random variable Z, γ_{1Z} and γ_{2Z} denote the Pearson's coefficients of skewness and kurtosis of the random variable Z.

(ii) δ_{ij}, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ are independent and identically distributed with mean 0, variance σ_δ^2, third moment $\gamma_{1\delta}\sigma_\delta^3$ and fourth moment $(\gamma_{2\delta} + 3)\sigma_\delta^4$.

(iii) ϕ_{ij}, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ are independent and identically distributed with first four finite moments given by 0, σ_ϕ^2, $\gamma_{1\phi}\sigma_\phi^3$ and $(\gamma_{2\phi} + 3)\sigma_\phi^4$, respectively.

(iv) ϵ, Δ and Φ are statistically jointly independent.

(v) The nth row of matrix M converges to $\boldsymbol{\sigma}'_\mu$.

The last assumption implies that $\lim_{n\to\infty} n^{-1}M'M = \boldsymbol{\sigma}_\mu\boldsymbol{\sigma}'_\mu$ and $\lim_{n\to\infty} n^{-1}M'e_n = \boldsymbol{\sigma}_\mu$, where e_n is a $n\times 1$ vector of elements unity. Such an assumption is needed for the application of asymptotic theory to obtain the asymptotic distribution of the estimators and to avoid the presence of any trend in the observations, see Schneeweiss (1982, 1991).

The equations (2.1)-(2.2) describe the set up of an ultrastructural model. The structural and functional forms of measurement error model as well as the classical regression model can be obtained as its particular cases. When all the row vectors of M are assumed to be identical, implying that rows of X are random and independent, having some multivariate distribution, we get the specification of a structural model. When Φ is taken identically equal to a null matrix implying that $\sigma_\phi^2 = 0$ and consequently that the matrix X is fixed but is measured with error, we obtain the specification of a functional model. When both Δ and Φ are identically equal to a null matrix, implying that $\sigma_\phi^2 = \sigma_\delta^2 = 0$ and consequently that X is fixed and is measured without any measurement error, we get the classical regression model. Thus the ultrastructural model provides a general framework for the study of three interesting models in a unified manner.

We also assume that some prior information on the regression coefficient is available in terms of $J(< p)$ exact linear restrictions as

$$r = R\beta, \tag{2.3}$$

where r is a $J \times 1$ known vector and R is a $J \times p$ known matrix of full row rank.

3 Restricted Stein-Rule Estimation of Regression Coefficients

The OLSE of β in the linear regression model without measurement errors is

$$b = S^{-1}X'y \ , \tag{3.1}$$

where $S = X'X$ and it's probability in limit ($\underset{n\to\infty}{\text{plim}}$) under the MLUME model (2.1)-(2.2) is

$$\underset{n\to\infty}{\text{plim}}\, b = (I_p - \sigma_\delta^2\Sigma^{-1})\beta, \tag{3.2}$$

which is not equal to β, in general, where $\Sigma = \boldsymbol{\sigma}_\mu\boldsymbol{\sigma}'_\mu + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$, see Shalabh (2007). Thus the OLSE is inconsistent for estimating β under measurement error ridden observations. Also, $Rb \neq r$, i.e., the OLSE does not satisfy the given restrictions (2.3).

When the regression coefficients are subjected to the restrictions (2.3) and there are no measurement errors in the data, the restricted least squares estimator (RLSE) is used which is given by

$$\begin{aligned} b_R &= b + S^{-1}R'(RS^{-1}R')^{-1}(r - Rb) \\ &= b - \{I_p - f_R(S)\}(b - \beta), \end{aligned} \tag{3.3}$$

where the function $f_R : \mathbb{R}^{p\times p} \to \mathbb{R}^{p\times p}$ is defined as

$$f_R(U) = I_p - U^{-1}R'(RU^{-1}R')^{-1}R, \qquad U \in \mathbb{R}^{p\times p}, \tag{3.4}$$

and $Rb_R = r$, i.e., b_R satisfy the exact linear restrictions. Now, under the MLUME model (2.1)-(2.2), we observe that $\underset{n\to\infty}{\text{plim}}\, n^{-1}S = \Sigma$, $\underset{n\to\infty}{\text{plim}}\, f_R(S) = f_R(\Sigma)$ and $\underset{n\to\infty}{\text{plim}}\, f_R(S - n\sigma_\delta^2 I_p) = f_R(\Sigma - \sigma_\delta^2 I_p)$ which establishes that

$$\underset{n\to\infty}{\text{plim}}\, b_R = \{I_p - \sigma_\delta^2 f_R(\Sigma)\Sigma^{-1}\}\beta \ ,$$

which is not equal to β, in general. Thus the RLSE becomes inconsistent estimator of β in the presence of measurement errors in the data.

In the classical linear regression model when measurement errors are absent, the family of Stein-rule estimators of β is given by

$$\hat\beta_S = \left[1 - \frac{k}{n-p+2}\frac{(y-Xb)'(y-Xb)}{b'X'Xb}\right] b, \tag{3.5}$$

where b is the OLSE of β and k is a positive characterizing scalar, see Vinod and Srivastava (1995). Such a family (3.5) does not satisfy the restrictions, i.e., $r \neq R\hat\beta_S$. When regression coefficients are subjected to exact linear restrictions $r = R\beta$, Srivastava and Srivastava (1984) provided two improved families of Stein-rule estimators given by

$$\hat\beta_{RS} = b_R - \frac{k}{n-p+2}\frac{(y-Xb)'(y-Xb)}{b'X'Xb}\Omega X'Xb \tag{3.6}$$

and

$$\hat{\beta}_{SR} = b_R - \frac{k}{n-p+2}\frac{(y-Xb_R)'(y-Xb_R)}{b_R'X'Xb_R}\Omega X'Xb_R\ , \tag{3.7}$$

where $\Omega = (X'X)^{-1} - (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}R(X'X)^{-1}$. Both the families of estimators (3.6) and (3.7) are consistent for β as well as satisfy the restrictions in the sense that $r = R\hat{\beta}_{RS}$ and $r = R\hat{\beta}_{SR}$.

However under the MLUME model (2.1)-(2.2), $\underset{n\to\infty}{\text{plim}}\ \hat{\beta}_{RS} \neq \beta$ and $\underset{n\to\infty}{\text{plim}}\ \hat{\beta}_{SR} \neq \beta$, in general, i.e., both $\hat{\beta}_{RS}$ and $\hat{\beta}_{SR}$ becomes inconsistent. Therefore, we attempt to obtain such families of Stein-rule estimators which are consistent as well as satisfy the restrictions (2.3) under the MLUME model (2.1)-(2.2).

It is well known that some additional information is needed to obtain the consistent estimators of regression coefficients under the MLUME model (2.1)-(2.2). Here we propose to use the following forms of additional knowledge separately to obtain the consistent estimators:
(i) covariance matrix of measurement errors associated with explanatory variables and
(ii) reliability matrix of explanatory variables.

3.1 When Covariance Matrix of Measurement Errors is Known

The covariance matrix of measurement errors δ_{ij} associated with the explanatory variables is any positive definite matrix Σ_δ which is assumed to be of the form $\sigma_\delta^2 I_p$ for simplicity in exposition. When $\Sigma_\delta = \sigma_\delta^2 I_p$, or equivalently σ_δ^2 is known, a consistent estimator of β is given by

$$b_\delta^{(1)} = (I_p - n\sigma_\delta^2 S^{-1})^{-1} b. \tag{3.8}$$

Schneeweiss (1976) has termed $b_\delta^{(1)}$ as an "adjusted least squares estimator", see also Shalabh et al. (2007). Although, the estimator $b_\delta^{(1)}$ is consistent for estimating β, it does not satisfy the given linear restrictions (2.3), i.e., $Rb_\delta^{(1)} \neq r$.

Now using the philosophy proposed by Srivastava and Srivastava (1984) to construct the families of Stein-rule estimators under (2.3) and additional information $\sigma_\delta^2 I_p$, we obtain two different families of Stein-rule estimators of β which are consistent as well as satisfy the given restrictions (2.3).

First approach consists of obtaining an estimator by replacing the inconsistent OLSE by the consistent $b_\delta^{(1)}$ in the estimator $\hat{\beta}_{RS}$ given in (3.6). The resulting estimator is

$$\hat{\beta}_{RS\delta} = b_\delta^{(2)} - \frac{k}{n-p+2}\frac{(y-Xb_\delta^{(1)})'(y-Xb_\delta^{(1)})}{b_\delta^{(1)'}X'Xb_\delta^{(1)}}\Omega X'Xb_\delta^{(1)}, \tag{3.9}$$

where

$$\begin{aligned} b_\delta^{(2)} &= b_\delta^{(1)} + S^{-1}R'(RS^{-1}R')^{-1}(r - Rb_\delta^{(1)}) \\ &= b_\delta^{(1)} - \{I_p - f_R(S)\}(b_\delta^{(1)} - \beta). \end{aligned} \tag{3.10}$$

In the second approach to find estimator, we propose to replace b by $b_\delta^{(1)}$ in the estimator $\hat{\beta}_{SR}$ given in (3.7). The resulting estimator is given by

$$\hat{\beta}_{SR\delta} = b_\delta^{(2)} - \frac{k}{n-p+2} \frac{(y - Xb_\delta^{(2)})'(y - Xb_\delta^{(2)})}{b_\delta^{(2)'} X'Xb_\delta^{(2)}} \Omega X'Xb_\delta^{(2)}. \tag{3.11}$$

Since $Rf_R(S) = 0$, this implies that $Rb_\delta^{(2)} = r$. Also, since $R\Omega = 0$, therefore we have $R\hat{\beta}_{RS\delta} = r$ and $R\hat{\beta}_{SR\delta} = r$. Thus, both of the estimators $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$ satisfy the given restrictions (2.3). It can also be proved using the results $\operatorname*{plim}_{n\to\infty} n^{-1}S = \Sigma$, $\operatorname*{plim}_{n\to\infty} f_R(S) = f_R(\Sigma)$ and $\operatorname*{plim}_{n\to\infty} f_R(S - n\sigma_\delta^2 I_p) = f_R(\Sigma - \sigma_\delta^2 I_p)$ that $\operatorname*{plim}_{n\to\infty} \hat{\beta}_{RS\delta} = \beta$ and $\operatorname*{plim}_{n\to\infty} \hat{\beta}_{SR\delta} = \beta$, i.e., both $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$ are consistent for estimating β.

3.2 When Reliability Matrix is Known

The reliability matrix associated with the explanatory variables is defined as

$$K_x = \Sigma_x^{-1}\Sigma_T, \tag{3.12}$$

where $\Sigma_x = n^{-1}M'M + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$ and $\Sigma_T = n^{-1}M'M + \sigma_\phi^2 I_p$ are considered as the measures of variances of observed and true values, respectively of explanatory variables. The reliability matrix is a multivariate generalization of reliability ratios of explanatory variables in univariate case. It is more popular in psychometric literature and can be estimated through psychological tests, see Gruijter and Kamp (2008). Gleser (1992) has suggested the approaches through which the reliability matrix can be estimated. We assume that the reliability matrix of explanatory variables K_x is known. Since $\operatorname*{plim}_{n\to\infty} b = (I_p - \sigma_\delta^2\Sigma^{-1})\beta$ and $\lim_{n\to\infty} K_x = (I_p - \sigma_\delta^2\Sigma^{-1})$, a consistent estimator of β is

$$b_K^{(1)} = K_x^{-1}b, \tag{3.13}$$

see, Gleser (1992, 1993). The estimator can also be viewed as an adjustment in OLSE by the pre-multiplication of inverse of reliability matrix. It is clear that the estimator $b_K^{(1)}$ does not satisfy the restrictions (2.3), i.e., $Rb_K^{(1)} \neq r$.

Now, using K_x, we again obtain the two families of Stein-rule estimators of β which are consistent as well as satisfy the given restrictions (2.3).

First estimator is obtained by replacing the inconsistent b by the consistent $b_K^{(1)}$ in the estimator $\hat{\beta}_{RS}$ given in (3.6). The resulting estimator is

$$\hat{\beta}_{RSK} = b_K^{(2)} - \frac{k}{n-p+2} \frac{(y - Xb_K^{(1)})'(y - Xb_K^{(1)})}{b_K^{(1)'} X'Xb_K^{(1)}} \Omega X'Xb_K^{(1)}\ , \tag{3.14}$$

where

$$\begin{aligned} b_K^{(2)} &= b_K^{(1)} + S^{-1}R'(RS^{-1}R')^{-1}(r - Rb_K^{(1)}) \\ &= b_K^{(1)} - \{I_p - f_R(S)\}(b_K^{(1)} - \beta). \end{aligned} \tag{3.15}$$

In order to obtain the second estimator, we propose to replace b by $b_K^{(1)}$ in $\hat{\beta}_{SR}$ given in (3.7). The resulting estimator is given by

$$\hat{\beta}_{SRK} = b_K^{(2)} - \frac{k}{n-p+2} \frac{(y - Xb_K^{(2)})'(y - Xb_K^{(2)})}{b_K^{(2)'} X'Xb_K^{(2)}} \Omega X'Xb_K^{(2)}. \tag{3.16}$$

Since $Rf_R(S) = 0$, this implies that $Rb_K^{(2)} = r$. Also, since $R\Omega = 0$, therefore we have $R\hat{\beta}_{RSK} = r$ and $R\hat{\beta}_{SRK} = r$. Thus, both of the estimators $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$ satisfy the given restrictions (2.3). It can also be proved using the results $\operatorname*{plim}_{n\to\infty} n^{-1}S = \Sigma$, $\operatorname*{plim}_{n\to\infty} f_R(S) = f_R(\Sigma)$ and $\operatorname*{plim}_{n\to\infty} f_R(S - n\sigma_\delta^2 I_p) = f_R(\Sigma - \sigma_\delta^2 I_p)$ that $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$ are consistent for estimating β in the sense that their probability in limits are equal to β.

4 Asymptotic Properties

The exact distribution and finite sample properties of the estimators $\hat{\beta}_{SR\delta}$, $\hat{\beta}_{RS\delta}$, $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$ are difficult to derive. Even if derived, the expressions will turn out to be complicated and it may not be possible to draw any clear inference from them. Moreover, the mean of $b_\delta^{(1)}$ does not exist under the normal distribution of measurement errors, see, Cheng and Van Ness (1999, page 58) and Cheng and Kukush (2006). So we propose to employ the large sample asymptotic approximation theory to study the asymptotic distribution of the estimators.

Theorem 1. *The asymptotic distributions of* $\sqrt{n}\left(\hat{\beta}_{RS\delta} - \beta\right)$ *and* $\sqrt{n}\left(\hat{\beta}_{SR\delta} - \beta\right)$ *are p-variate normal with common mean vector* 0 *and common covariance matrix* $f_R(\Sigma)(\Sigma - \sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma - \sigma_\delta^2 I_p)^{-1} f_R'(\Sigma)$, *where*

$$\begin{aligned} \Omega_h &= (\sigma_\epsilon^2 + \sigma_\delta^2(\beta'\beta))\Sigma + \sigma_\delta^4\beta\beta' + \gamma_{1\delta}\sigma_\delta^3\{f(\boldsymbol{\sigma}_\mu e_p', \beta\beta') \\ &\quad + \big(f(\boldsymbol{\sigma}_\mu e_p', \beta\beta')\big)'\} + \gamma_{2\delta}\sigma_\delta^4 f(I_p, \beta\beta'). \end{aligned} \tag{4.1}$$

Here $f_R'(\Sigma)$ *indicates the transpose of matrix* $f_R(\Sigma)$.

Proof. From (3.9), we have

$$\sqrt{n}\left(\hat{\beta}_{RS\delta} - \beta\right) = \sqrt{n}\left(b_\delta^{(2)} - \beta\right) - \frac{k\sqrt{n}}{n-p+2} \frac{(y - Xb_\delta^{(1)})'(y - Xb_\delta^{(1)})}{b_\delta^{(1)'} X'Xb_\delta^{(1)}} \Omega X'Xb_\delta^{(1)}.$$

Since $\operatorname*{plim}_{n\to\infty} b_\delta^{(1)} = \beta$ and $\operatorname*{plim}_{n\to\infty} n^{-1}S = \Sigma$, so

$$\operatorname*{plim}_{n\to\infty} \frac{k\sqrt{n}}{n-p+2} \frac{(y - Xb_\delta^{(1)})'(y - Xb_\delta^{(1)})}{b_\delta^{(1)'} X'Xb_\delta^{(1)}} \Omega X'Xb_\delta^{(1)} = 0. \tag{4.2}$$

Using (4.2) and Lemma A.4 given in Appendix and Slutzky's Lemma (see Arnold, 1990, p. 451)), we obtain

$$\sqrt{n}\left(\hat{\beta}_{RS\delta}-\beta\right) \quad \xrightarrow{d} \quad N_p\big(0, f_R(\Sigma)(\Sigma-\sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma-\sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)\big),$$

as $n \to \infty$.

Now, from (3.11), we have

$$\sqrt{n}\left(\hat{\beta}_{SR\delta}-\beta\right) = \sqrt{n}\left(b_\delta^{(2)}-\beta\right) - \frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_\delta^{(2)})'(y-Xb_\delta^{(2)})}{b_\delta^{(2)'}X'Xb_\delta^{(2)}}\Omega X'Xb_\delta^{(2)}.$$

Since $\operatorname*{plim}_{n\to\infty} b_\delta^{(2)} = \beta$, so

$$\operatorname*{plim}_{n\to\infty}\frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_\delta^{(2)})'(y-Xb_\delta^{(2)})}{b_\delta^{(2)'}X'Xb_\delta^{(2)}}\Omega X'Xb_\delta^{(2)} = 0. \tag{4.3}$$

Using (4.3) and Lemma A.4 given in Appendix and Slutzky's Lemma (see Arnold 1990, p. 451), we obtain, as $n \to \infty$,

$$\sqrt{n}\left(\hat{\beta}_{SR\delta}-\beta\right) \quad \xrightarrow{d} \quad N_p\big(0, f_R(\Sigma)(\Sigma-\sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma-\sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)\big).$$

$\square$

From the expression of Ω_h, given in (4.1), it is clear that the asymptotic distributions of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$ depend on the skewness and kurtosis of the distributions of measurement errors δ_{ij}'s. These effects can be considered as the non-normality effects. It is also clear that the asymptotic distributions of these estimators are not affected by the non-normality effects of the distributions of ϕ_{ij}'s and ϵ_i's. Note that under the normally distributed measurement errors in explanatory variables, the coefficients of skewness and kurtosis disappear. The degree of departure from non-normality depends on the magnitude of coefficients of skewness and kurtosis.

Next theorem presents the asymptotic distributions of $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$.

Theorem 2. *The asymptotic distributions of* $\sqrt{n}\left(\hat{\beta}_{RSK}-\beta\right)$ *and* $\sqrt{n}\left(\hat{\beta}_{SRK}-\beta\right)$ *are p-variate normal with common mean vector* 0 *and common covariance matrix* $f_R(\Sigma)(\Sigma - \sigma_\delta^2 I_p)^{-1}\Omega_K(\Sigma-\sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)$, *where*

$$\Omega_K = \Omega_h + \Omega_{hH}(\sigma_\delta^2\Sigma^{-1}\beta) + \Omega_{hH}'(\sigma_\delta^2\Sigma^{-1}\beta) + \Omega_H(\sigma_\delta^2\Sigma^{-1}\beta),$$

Ω_h *is given in (4.1),*

$$\begin{aligned}\Omega_{hH}(d) \quad = \quad & -\sigma_\delta^2[\Sigma(d\beta' + (d'\beta)I_p) + \gamma_{1\delta}\sigma_\delta\{f(\boldsymbol{\sigma}_\mu e_p', \beta d') \\ & + f(I_p, \beta d'\boldsymbol{\sigma}_\mu e_p) + (f(\boldsymbol{\sigma}_\mu e_p', d\beta'))'\} + \gamma_{2\delta}\sigma_\delta^2 f(I_p, \beta d')],\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Omega_H(d) &= (\sigma_\delta^2+\sigma_\phi^2)[\Sigma\{dd'+(d'd)I_p\}+dd'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu'+(d'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu'd)I_p] \\ &\quad +(\gamma_{1\phi}\sigma_\phi^3+\gamma_{1\delta}\sigma_\delta^3)[f(\boldsymbol{\sigma}_\mu e_p', dd')+\{f(\boldsymbol{\sigma}_\mu e_p', dd')\}' \\ &\quad +2f(I_p, e_p\boldsymbol{\sigma}_\mu dd')]+(\gamma_{2\phi}\sigma_\phi^4+\gamma_{2\delta}\sigma_\delta^4)f(I_p, dd') \end{aligned} \tag{4.5}$$

for any sequence $\{d_n\}$ *of* $p\times 1$ *non-stochastic vectors such that* $\lim_{n\to\infty} d_n = d$.

Proof. From (3.14), we have

$$\sqrt{n}\left(\hat\beta_{RSK}-\beta\right)=\sqrt{n}\left(b_K^{(2)}-\beta\right)-\frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_K^{(1)})'(y-Xb_K^{(1)})}{b_K^{(1)'}X'Xb_K^{(1)}}\Omega X'Xb_K^{(1)}.$$

Since $\operatorname*{plim}_{n\to\infty} b_K^{(1)}=\beta$, so

$$\operatorname*{plim}_{n\to\infty}\frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_K^{(1)})'(y-Xb_K^{(1)})}{b_K^{(1)'}X'Xb_K^{(1)}}\Omega X'Xb_K^{(1)}=0. \tag{4.6}$$

Using (4.6) and Lemma A.6 given in Appendix and Slutzky's Lemma (see Arnold, 1990, p. 451), we obtain

$$\sqrt{n}\left(\hat\beta_{RSK}-\beta\right) \quad \xrightarrow{d} \quad N_p\big(0, f_R(\Sigma)(\Sigma-\sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma-\sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)\big),$$

as $n\to\infty$.

Now, from (3.16), we have

$$\sqrt{n}\left(\hat\beta_{SRK}-\beta\right)=\sqrt{n}\left(b_K^{(2)}-\beta\right)-\frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_K^{(2)})'(y-Xb_K^{(2)})}{b_K^{(2)'}X'Xb_K^{(2)}}\Omega X'Xb_K^{(2)}.$$

Since $\operatorname*{plim}_{n\to\infty} b_K^{(2)}=\beta$, so

$$\operatorname*{plim}_{n\to\infty}\frac{k\sqrt{n}}{n-p+2}\frac{(y-Xb_K^{(2)})'(y-Xb_K^{(2)})}{b_K^{(2)'}X'Xb_K^{(2)}}\Omega X'Xb_K^{(2)}=0. \tag{4.7}$$

Using (4.7) and Lemma A.6 given in Appendix and Slutzky's Lemma (see Arnold, 1990, p. 451), we obtain, as $n\to\infty$,

$$\sqrt{n}\left(\hat\beta_{SRK}-\beta\right) \quad \xrightarrow{d} \quad N_p\big(0, f_R(\Sigma)(\Sigma-\sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma-\sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)\big).$$

□

From the expressions of Ω_h, $\Omega_H(\cdot)$ and $\Omega_{hH}(\cdot)$, it is clear that the asymptotic distributions of the estimators $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$ are affected by the non-normality effects of the distributions of δ_{ij}'s and ϕ_{ij}'s. However, the non-normality of the distribution of ϵ_i's does not have any effect on the asymptotic properties of these estimators. Note that under the normally distributed measurement errors in explanatory variables and random error components, the corresponding coefficients of skewness and kurtosis disappear. The degree of departure from non-normality depends on the degree of such coefficients of skewness and kurtosis.

It may be noted that the asymptotic distributions of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{RSK}$ are same as that of $\hat{\beta}_{SR\delta}$ and $\hat{\beta}_{SRK}$, respectively. This is due to the large sample size. The difference in their properties may precipitate in their finite sample properties, if they exist and found. We have tried to have some insight on this issue through a Monte-Carlo simulation experiment whose findings are reported in the next section.

5 Simulation Study

For simulation study, we adapted the following values and parameters:

$$n \in \{22, 48, 100, 500, 1000\},\ p = 5,\ \beta \in \left\{ \begin{pmatrix} 2.2 \\ 1.1 \\ 3 \\ 4.2 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 12 \\ 23 \\ 5 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ -12 \\ 5 \\ -9 \end{pmatrix} \right\},$$

$$R = \begin{pmatrix} -6 & -2.5 & 1 & -3 & -3.7 \\ 4.2 & -1.8 & 2.4 & -3.5 & -1.7 \\ 6.8 & 0.1 & -1.5 & -3.6 & 1.4 \\ 15.8 & -9.4 & -5.2 & 1.2 & -5 \end{pmatrix},$$

$$\begin{aligned} (\sigma_\epsilon^2, \sigma_\phi^2, \sigma_\delta^2) \in & \big\{(0.5, 0.5, 0.5), (0.5, 0.5, 1.25), (1.25, 0.5, 0.5), (0.5, 1.25, 0.5) \\ & (1.25, 1.25, 0.5), (1.25, 0.5, 1.25), (0.5, 1.25, 1.25), (1.25, 1.25, 1.25)\big\}. \end{aligned}$$

The chosen matrix of means M is kept fixed for $50,000$ repetitions of the simulation experiments. We adapt the following distributions to study the effect of departure from normality of the distribution of measurement errors:

(i) normal distributions which is symmetric and mesokurtic and

(ii) gamma distribution which has nonzero coefficients of skewness and kurtosis.

The generated random variables from different distributions are suitably scaled to have mean zero and same variances. So the difference in the results may be viewed as contribution due to departure from non-normality. In order to study the properties of the estimators $\hat{\beta}_{RS\delta}$, $\hat{\beta}_{SR\delta}$, $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$, we have obtained the bias vectors and mean squared error matrices empirically.

Cheng and Kukush (2006) proved that the moments of the estimator $b_{\delta}^{(1)}$ does not exist under the normality of measurement errors. The estimators $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$ also depend on $b_{\delta}^{(1)}$. Therefore, the existence of the moments of these estimators is also doubtful. The outcomes of simulation study also support this doubt. Therefore, in order to study the properties of these estimators, we adapted the criteria of median bias vector and median squared error matrices. In order to save space, we only present a few outcomes of the simulation in the Tables 1-4. The conclusions of the simulation study are as follows.

First we study the bias vectors of the estimators around their median (in case of $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$) and mean (in case of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$). It is observed that the absolute bias of all the estimators reduces as sample size increases under normal as well as gamma distributions of measurement errors under all parametric settings. This also supports the theoretical finding about the means of the asymptotic distributions of all estimators obtained in Section 3. No clear pattern of dominance in the absolute median bias (or absolute bias) of the estimators over each other can be seen. Their magnitudes are small.

Now we analyze the empirical median squared error matrices (MdSEM) (in case of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$) and mean squared error matrices (MSEM) (in case of $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$) of the estimators. The variability of all the estimators reduces as sample size increases for all different combinations of variances $(\sigma_{\epsilon}^2, \sigma_{\phi}^2, \sigma_{\delta}^2)$ and different distributions of measurement errors. This verifies the result that the estimators are consistent. The difference in the variabilities of the estimators under each type of used additional information is small in magnitude but no clear dominance is seen. We also investigated the dominance under a weaker criterion which is the trace of MdSEM but no clear dominance between the estimators $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$ as well as $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$ over each other is observed. This is due to the fact that the dominance depends on the parametric values. We observe that the variability of all the estimators is more affected by σ_{δ}^2 then other variances. As σ_{δ}^2 increases, the variability of all the estimators increases. On the other hand, there is no significant effect of σ_{ϵ}^2 and σ_{ϕ}^2 on the variability of the estimators. Moreover, the variability of the estimators is higher under gamma distribution of measurement errors than under normal distribution. This also confirms the effect of non-normality of the distributions of measurement errors on the efficiency properties of the estimators.

6 Conclusion

We have developed the families of Stein-rule estimators of regression coefficients in a multivariate linear ultrastructural measurement error model, where regression coefficients are subjected to some exact linear restrictions. Using the philosophy of Stein-rule estimators in restricted regression model without measurement errors, obtained by Srivastava and Srivastava (1984), we obtained two Stein-rule estimators in each case by using the covariance

Table 1: Absolute median bias of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$

when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have normal distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
MdB'($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	0.001	0.008	-0.002	0.000	-0.008	0.005	0.028	-0.007	0.001	-0.029
MdB'($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	0.005	0.026	-0.006	0.001	-0.028	0.006	0.034	-0.008	0.002	-0.036
MdB'($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	-0.013	-0.075	0.018	-0.003	0.080	-0.002	-0.011	0.003	0.000	0.011
MdB'($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	-0.022	-0.124	0.030	-0.006	0.132	-0.001	-0.006	0.002	0.000	0.007
when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have gamma distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
MdB'($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	-0.004	-0.024	0.006	-0.001	0.026	0.004	0.024	-0.006	0.001	-0.025
MdB'($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	-0.001	-0.007	0.002	0.000	0.008	0.005	0.029	-0.007	0.001	-0.031
MdB'($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	-0.029	-0.166	0.040	-0.008	0.177	-0.009	-0.053	0.013	-0.002	0.056
MdB'($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	-0.039	-0.221	0.054	-0.010	0.235	-0.008	-0.048	0.012	-0.002	0.051

Table 2: Absolute bias of $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$

when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have normal distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
B'($\hat{\beta}_{RSK}$)	(0.5, 0.5, 0.5)	0.025	0.141	-0.034	0.007	-0.150	0.014	0.081	-0.020	0.004	-0.086
B'($\hat{\beta}_{SRK}$)	(0.5, 0.5, 0.5)	0.030	0.172	-0.042	0.008	-0.183	0.015	0.088	-0.021	0.004	-0.094
B'($\hat{\beta}_{RSK}$)	(1.25, 1.25, 1.25)	0.051	0.290	-0.070	0.013	-0.308	0.025	0.144	-0.035	0.007	-0.153
B'($\hat{\beta}_{SRK}$)	(1.25, 1.25, 1.25)	0.060	0.339	-0.082	0.016	-0.361	0.028	0.157	-0.038	0.007	-0.167
when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have gamma distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
B'($\hat{\beta}_{RSK}$)	(0.5, 0.5, 0.5)	0.019	0.110	-0.027	0.005	-0.117	0.012	0.066	-0.016	0.003	-0.070
B'($\hat{\beta}_{SRK}$)	(0.5, 0.5, 0.5)	0.025	0.143	-0.035	0.007	-0.152	0.013	0.074	-0.018	0.003	-0.079
B'($\hat{\beta}_{RSK}$)	(1.25, 1.25, 1.25)	0.043	0.248	-0.060	0.011	-0.263	0.024	0.138	-0.033	0.006	-0.147
B'($\hat{\beta}_{SRK}$)	(1.25, 1.25, 1.25)	0.054	0.307	-0.074	0.014	-0.326	0.027	0.152	-0.037	0.007	-0.161

matrix of measurement error and reliability matrix as additional information under the MLUME model. The proposed estimators are consistent as well as satisfy the exact linear restrictions. We established the asymptotic normality of the obtained estimators in terms of coefficients of skewness and kurtosis of the distribution of measurement errors and random term in the model. The skewness and kurtosis can be considered as the effect of non-normality of the distribution. We observed that the asymptotic distributions of the two families of Stein-rule estimators under each type of additional information are same which is due to large sample size. Simulation study shows minor differences in their efficiency properties but does not give any clear dominance of any estimator over the other. However, the non-normality effects of the distribution of measurement errors is clear from the simulation study.

Table 3: Median squared error matrices of $\hat{\beta}_{RS\delta}$ and $\hat{\beta}_{SR\delta}$

when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have normal distribution											
	$(\sigma^2_\delta, \sigma^2_\phi, \sigma^2_\epsilon)$	$n = 22$					$n = 48$				
		0.026	0.146	-0.035	0.007	-0.155	0.011	0.064	-0.015	0.003	-0.068
		0.146	0.833	-0.202	0.039	-0.886	0.064	0.365	-0.088	0.017	-0.388
MdSEM($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	-0.035	-0.202	0.049	-0.009	0.215	-0.015	-0.088	0.021	-0.004	0.094
		0.007	0.039	-0.009	0.002	-0.041	0.003	0.017	-0.004	0.001	-0.018
		-0.155	-0.886	0.215	-0.041	0.942	-0.068	-0.388	0.094	-0.018	0.412
		0.026	0.149	-0.036	0.007	-0.159	0.011	0.064	-0.016	0.003	-0.068
		0.149	0.852	-0.206	0.040	-0.906	0.064	0.366	-0.089	0.017	-0.389
MdSEM($\hat{\beta}_{SR\delta}$)	(0.5, 0.5, 0.5)	-0.036	-0.206	0.050	-0.010	0.219	-0.016	-0.089	0.021	-0.004	0.094
		0.007	0.040	-0.010	0.002	-0.042	0.003	0.017	-0.004	0.001	-0.018
		-0.159	-0.906	0.219	-0.042	0.963	-0.068	-0.389	0.094	-0.018	0.413
		0.079	0.453	-0.110	0.021	-0.481	0.029	0.163	-0.039	0.008	-0.173
		0.453	2.582	-0.625	0.120	-2.745	0.163	0.931	-0.225	0.043	-0.989
MdSEM($\hat{\beta}_{RS\delta}$)	(1.25, 1.25, 1.25)	-0.110	-0.625	0.151	-0.029	0.665	-0.039	-0.225	0.055	-0.010	0.239
		0.021	0.120	-0.029	0.006	-0.127	0.008	0.043	-0.010	0.002	-0.046
		-0.481	-2.745	0.665	-0.127	2.918	-0.173	-0.989	0.239	-0.046	1.052
		0.083	0.474	-0.115	0.022	-0.504	0.029	0.164	-0.040	0.008	-0.175
		0.474	2.705	-0.655	0.126	-2.875	0.164	0.938	-0.227	0.044	-0.997
MdSEM($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	-0.115	-0.655	0.159	-0.030	0.696	-0.040	-0.227	0.055	-0.011	0.241
		0.022	0.126	-0.030	0.006	-0.133	0.008	0.044	-0.011	0.002	-0.046
		-0.504	-2.875	0.696	-0.133	3.056	-0.175	-0.997	0.241	-0.046	1.060
when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have gamma distribution											
	$(\sigma^2_\delta, \sigma^2_\phi, \sigma^2_\epsilon)$	$n = 22$					$n = 48$				
		0.028	0.161	-0.039	0.007	-0.171	0.012	0.069	-0.017	0.003	-0.073
		0.161	0.918	-0.222	0.043	-0.976	0.069	0.391	-0.095	0.018	-0.416
MdSEM($\hat{\beta}_{RS\delta}$)	(0.5, 0.5, 0.5)	-0.039	-0.222	0.054	-0.010	0.236	-0.017	-0.095	0.023	-0.004	0.101
		0.007	0.043	-0.010	0.002	-0.045	0.003	0.018	-0.004	0.001	-0.019
		-0.171	-0.976	0.236	-0.045	1.038	-0.073	-0.416	0.101	-0.019	0.442
		0.029	0.164	-0.040	0.008	-0.174	0.012	0.069	-0.017	0.003	-0.073
		0.164	0.935	-0.226	0.043	-0.994	0.069	0.392	-0.095	0.018	-0.417
MdSEM($\hat{\beta}_{SR\delta}$)	(0.5, 0.5, 0.5)	-0.040	-0.226	0.055	-0.011	0.241	-0.017	-0.095	0.023	-0.004	0.101
		0.008	0.043	-0.011	0.002	-0.046	0.003	0.018	-0.004	0.001	-0.019
		-0.174	-0.994	0.241	-0.046	1.057	-0.073	-0.417	0.101	-0.019	0.443
		0.090	0.513	-0.124	0.024	-0.545	0.031	0.179	-0.043	0.008	-0.190
		0.513	2.926	-0.708	0.136	-3.110	0.179	1.020	-0.247	0.047	-1.085
MdSEM($\hat{\beta}_{RS\delta}$)	(1.25, 1.25, 1.25)	-0.124	-0.708	0.172	-0.033	0.753	-0.043	-0.247	0.060	-0.011	0.263
		0.024	0.136	-0.033	0.006	-0.144	0.008	0.047	-0.011	0.002	-0.050
		-0.545	-3.110	0.753	-0.144	3.306	-0.190	-1.085	0.263	-0.050	1.153
		0.094	0.536	-0.130	0.025	-0.569	0.032	0.180	-0.044	0.008	-0.191
		0.536	3.055	-0.740	0.142	-3.247	0.180	1.025	-0.248	0.048	-1.090
MdSEM($\hat{\beta}_{SR\delta}$)	(1.25, 1.25, 1.25)	-0.130	-0.740	0.179	-0.034	0.786	-0.044	-0.248	0.060	-0.012	0.264
		0.025	0.142	-0.034	0.007	-0.151	0.008	0.048	-0.012	0.002	-0.051
		-0.569	-3.247	0.786	-0.151	3.452	-0.191	-1.090	0.264	-0.051	1.159

Table 4: Mean squared error matrices of $\hat{\beta}_{RSK}$ and $\hat{\beta}_{SRK}$

when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have normal distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
		0.016	0.090	-0.022	0.004	-0.096	0.007	0.041	-0.010	0.002	-0.044
		0.090	0.516	-0.125	0.024	-0.549	0.041	0.235	-0.057	0.011	-0.250
MSEM($\hat{\beta}_{RSK}$)	(0.5, 0.5, 0.5)	-0.022	-0.125	0.030	-0.006	0.133	-0.010	-0.057	0.014	-0.003	0.060
		0.004	0.024	-0.006	0.001	-0.025	0.002	0.011	-0.003	0.001	-0.012
		-0.096	-0.549	0.133	-0.025	0.583	-0.044	-0.250	0.060	-0.012	0.265
		0.016	0.092	-0.022	0.004	-0.098	0.007	0.041	-0.010	0.002	-0.044
		0.092	0.525	-0.127	0.024	-0.559	0.041	0.236	-0.057	0.011	-0.251
MSEM($\hat{\beta}_{SRK}$)	(0.5, 0.5, 0.5)	-0.022	-0.127	0.031	-0.006	0.135	-0.010	-0.057	0.014	-0.003	0.061
		0.004	0.024	-0.006	0.001	-0.026	0.002	0.011	-0.003	0.001	-0.012
		-0.098	-0.559	0.135	-0.026	0.594	-0.044	-0.251	0.061	-0.012	0.267
		0.033	0.188	-0.046	0.009	-0.200	0.015	0.083	-0.020	0.004	-0.088
		0.188	1.072	-0.260	0.050	-1.140	0.083	0.474	-0.115	0.022	-0.503
MSEM($\hat{\beta}_{RSK}$)	(1.25, 1.25, 1.25)	-0.046	-0.260	0.063	-0.012	0.276	-0.020	-0.115	0.028	-0.005	0.122
		0.009	0.050	-0.012	0.002	-0.053	0.004	0.022	-0.005	0.001	-0.023
		-0.200	-1.140	0.276	-0.053	1.212	-0.088	-0.503	0.122	-0.023	0.535
		0.034	0.193	-0.047	0.009	-0.205	0.015	0.084	-0.020	0.004	-0.089
		0.193	1.100	-0.266	0.051	-1.169	0.084	0.477	-0.115	0.022	-0.507
MSEM($\hat{\beta}_{SRK}$)	(1.25, 1.25, 1.25)	-0.047	-0.266	0.064	-0.012	0.283	-0.020	-0.115	0.028	-0.005	0.123
		0.009	0.051	-0.012	0.002	-0.054	0.004	0.022	-0.005	0.001	-0.024
		-0.205	-1.169	0.283	-0.054	1.243	-0.089	-0.507	0.123	-0.024	0.539
when $(\delta_{ij}, \phi_{ij}, \epsilon_i)$ have gamma distribution											
	$(\sigma_\delta^2, \sigma_\phi^2, \sigma_\epsilon^2)$	$n = 22$					$n = 48$				
		0.017	0.098	-0.024	0.005	-0.104	0.008	0.045	-0.011	0.002	-0.048
		0.098	0.559	-0.135	0.026	-0.594	0.045	0.257	-0.062	0.012	-0.273
MSEM($\hat{\beta}_{RSK}$)	(0.5, 0.5, 0.5)	-0.024	-0.135	0.033	-0.006	0.144	-0.011	-0.062	0.015	-0.003	0.066
		0.005	0.026	-0.006	0.001	-0.028	0.002	0.012	-0.003	0.001	-0.013
		-0.104	-0.594	0.144	-0.028	0.631	-0.048	-0.273	0.066	-0.013	0.291
		0.017	0.099	-0.024	0.005	-0.106	0.008	0.045	-0.011	0.002	-0.048
		0.099	0.566	-0.137	0.026	-0.602	0.045	0.258	-0.062	0.012	-0.274
MSEM($\hat{\beta}_{SRK}$)	(0.5, 0.5, 0.5)	-0.024	-0.137	0.033	-0.006	0.146	-0.011	-0.062	0.015	-0.003	0.066
		0.005	0.026	-0.006	0.001	-0.028	0.002	0.012	-0.003	0.001	-0.013
		-0.106	-0.602	0.146	-0.028	0.640	-0.048	-0.274	0.066	-0.013	0.291
		0.037	0.208	-0.050	0.010	-0.221	0.016	0.093	-0.022	0.004	-0.099
		0.208	1.188	-0.288	0.055	-1.263	0.093	0.529	-0.128	0.025	-0.562
MSEM($\hat{\beta}_{RSK}$)	(1.25, 1.25, 1.25)	-0.050	-0.288	0.070	-0.013	0.306	-0.022	-0.128	0.031	-0.006	0.136
		0.010	0.055	-0.013	0.003	-0.059	0.004	0.025	-0.006	0.001	-0.026
		-0.221	-1.263	0.306	-0.059	1.342	-0.099	-0.562	0.136	-0.026	0.597
		0.037	0.212	-0.051	0.010	-0.226	0.016	0.093	-0.023	0.004	-0.099
		0.212	1.211	-0.293	0.056	-1.287	0.093	0.532	-0.129	0.025	-0.566
MSEM($\hat{\beta}_{SRK}$)	(1.25, 1.25, 1.25)	-0.051	-0.293	0.071	-0.014	0.312	-0.023	-0.129	0.031	-0.006	0.137
		0.010	0.056	-0.014	0.003	-0.060	0.004	0.025	-0.006	0.001	-0.026
		-0.226	-1.287	0.312	-0.060	1.368	-0.099	-0.566	0.137	-0.026	0.601

Appendix A.

Define

$$\left.\begin{aligned} \Sigma_x &:= n^{-1}M'M + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p \\ H &:= \sqrt{n}(n^{-1}S - \Sigma_x) \\ \text{and} \qquad h &:= \sqrt{n}\{n^{-1}X'(\epsilon - \Delta\beta) + \sigma_\delta^2\beta\} \end{aligned}\right\}. \tag{A.1}$$

We now present some lemmas which are useful in proving main results of this section.

Lemma A.1. *As $n \to \infty$,*

(i) $\lim_{n\to\infty} \Sigma_x = \Sigma$, $\lim_{n\to\infty} f_R(\Sigma_x) = f_R(\Sigma)$,

(ii) $\Sigma_x = O(1)$, $f_R(\Sigma_x) = O(1)$,

(iii) $H = O_P(1)$, $h = O_P(1)$.

Proof. This lemma can be proved using the assumptions described in Section 2. □

Define function $f : \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times p} \to \mathbb{R}^{p\times p}$ as

$$f(Z_1, Z_2) = Z_1(Z_2 * I_p), \qquad Z_1, Z_2 \in \mathbb{R}^{p\times p}. \tag{A.2}$$

Lemma A.2. *Let d be a $p \times 1$ non-stochastic vector. Then*

$$\begin{aligned}
&(i)\ \ E(h) = 0\ , \\
&(ii)\ \ E(H) = 0\ , \\
&(iii)\ E(Hdd'H) &=&\ (\sigma_\phi^2 + \sigma_\delta^2)[\Sigma_x\{dd' + (d'd)I_p\} \\
& &&+dd'(n^{-1}M'M) + (n^{-1}d'M'Md)I_p] \\
& &&+(\gamma_{1\phi}\sigma_\phi^3 + \gamma_{1\delta}\sigma_\delta^3)\{f(n^{-1}M'e_ne_p', dd') + (f(n^{-1}M'e_ne_p', dd'))' \\
& &&+2f(I_p, n^{-1}dd'M'e_ne_p')\} + (\gamma_{2\phi}\sigma_\phi^4 + \gamma_{2\delta}\sigma_\delta^4)f(I_p, dd'), \\
&(iv)\ E(hd'H) &=&\ -\sigma_\delta^2[\Sigma_x(d\beta' + (d'\beta)I_p) + \gamma_{1\delta}\sigma_\delta\{f(n^{-1}Me_ne_p', \beta d') \\
& &&+f(I_p, n^{-1}\beta d'M'e_ne_p) + (f(n^{-1}M'e_ne_p', d\beta'))'\} \\
& &&+\gamma_{2\delta}\sigma_\delta^2 f(I_p, \beta d')], \\
&(v)\ E(hh') &=&\ (\sigma_\epsilon^2 + \sigma_\delta^2(\beta'\beta))\Sigma_x + \sigma_\delta^4\beta\beta' + \gamma_{1\delta}\sigma_\delta^3\{f(M'e_ne_p', \beta\beta') \\
& &&+(f(M'e_ne_p', \beta\beta'))'\} + \gamma_{2\delta}\sigma_\delta^4 f(I_p, \beta\beta').
\end{aligned}$$

Proof. This lemma can be proved using the assumptions described in Section 2. □

Lemma A.3. *Let $\{d_n\}$ be a sequence of $p\times 1$ non-stochastic vectors such that* $\lim_{n\to\infty} d_n = d$. *Then, as $n \to \infty$,*

$$\begin{pmatrix} h \\ Hd_n \end{pmatrix} \xrightarrow{d} N_{2p}\left(0, \begin{pmatrix} \Omega_h & \Omega_{hH}(d) \\ \Omega_{hH}'(d) & \Omega_H(d) \end{pmatrix}\right),$$

where

$$\begin{aligned}\Omega_h &= (\sigma_\epsilon^2+\sigma_\delta^2(\beta'\beta))\Sigma+\sigma_\delta^4\beta\beta'+\gamma_{1\delta}\sigma_\delta^3\{f(\boldsymbol{\sigma}_\mu e_p',\beta\beta') \\ &\quad +\big(f(\boldsymbol{\sigma}_\mu e_p',\beta\beta')\big)'\}+\gamma_{2\delta}\sigma_\delta^4 f(I_p,\beta\beta'), \qquad (A.3)\\ \Omega_{hH}(d) &= -\sigma_\delta^2[\Sigma(d\beta'+(d'\beta)I_p)+\gamma_{1\delta}\sigma_\delta\{f(\boldsymbol{\sigma}_\mu e_p',\beta d') \\ &\quad +f(I_p,\beta d'\boldsymbol{\sigma}_\mu e_p)+(f(\boldsymbol{\sigma}_\mu e_p',d\beta'))'\}+\gamma_{2\delta}\sigma_\delta^2 f(I_p,\beta d')], \qquad (A.4)\end{aligned}$$

and

$$\begin{aligned}\Omega_H(d) &= (\sigma_\delta^2+\sigma_\phi^2)[\Sigma\{dd'+(d'd)I_p\}+dd'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu'+(d'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu'd)I_p] \\ &\quad +(\gamma_{1\phi}\sigma_\phi^3+\gamma_{1\delta}\sigma_\delta^3)[f(\boldsymbol{\sigma}_\mu e_p',dd')+\{f(\boldsymbol{\sigma}_\mu e_p',dd')\}' \\ &\quad +2f(I_p,e_p\boldsymbol{\sigma}_\mu dd')]+(\gamma_{2\phi}\sigma_\phi^4+\gamma_{2\delta}\sigma_\delta^4)f(I_p,dd'). \qquad (A.5)\end{aligned}$$

Proof. Let x_i',δ_i',ϕ_i' and μ_i' be the ith rows of the matrices X,Δ,Φ and M respectively. From (A.1), (2.1), and (2.2), we have

$$\begin{aligned}h &= \frac{1}{\sqrt{n}}X'\epsilon-\frac{1}{\sqrt{n}}(X'\Delta-n\sigma_\delta^2 I_p)\beta \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\{x_i\epsilon_i-(x_i\delta_i'-\sigma_\delta^2 I_p)\beta\} \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\{\mu_i\epsilon_i+\delta_i\epsilon_i+\phi_i\epsilon_i-\mu_i\delta_i'\beta-\phi_i\delta_i'\beta-(\delta_i\delta_i'-\sigma_\delta^2 I_p)\beta\}.\end{aligned}$$

Using the vec operator and Hadamard product of matrices, we can write

$$\begin{aligned}\mu_i\delta_i'\beta &= (\beta'\otimes I_p)\mathrm{vec}(\mu_i\delta_i')=(\beta'\otimes I_p)(I_p\otimes\mu_i)\delta_i, \\ \phi_i\delta_i'\beta &= (\beta'\otimes I_p)\mathrm{vec}(\phi_i\delta_i'), \\ (\delta_i\delta_i'-\sigma_\delta^2 I_p)\beta &= (\beta'\otimes I_p)\mathrm{vec}(\delta_i\delta_i'-\sigma_\delta^2 I_p).\end{aligned}$$

Therefore

$$\begin{aligned}h &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\{\mu_i\epsilon_i+\delta_i\epsilon_i+\phi_i\epsilon_i-(\beta'\otimes I_p)(I_p\otimes\mu_i)\delta_i-(\beta'\otimes I_p)\mathrm{vec}(\phi_i\delta_i') \\ &\qquad -(\beta'\otimes I_p)\mathrm{vec}(\delta_i\delta_i'-\sigma_\delta^2 I_p)\} \\ &= \sum_{i=1}^n F_{in}w_i, \qquad (A.6)\end{aligned}$$

where, for $i=1,2,\ldots,n$,
$F_{in} = \frac{1}{\sqrt{n}}\begin{pmatrix}\mu_i, & I_p, & I_p, & -(\beta'\otimes I_p)(I_p\otimes\mu_i), & -(\beta'\otimes I_p), & -(\beta'\otimes I_p)\end{pmatrix}$ are $p\times(1+$

$3p + 2p^2)$ non-stochastic matrices and $w_i = \begin{pmatrix} \epsilon_i \\ \delta_i \epsilon_i \\ \phi_i \epsilon_i \\ \delta_i \\ \mathrm{vec}(\phi_i \delta_i') \\ \mathrm{vec}(\delta_i \delta_i' - \sigma_\delta^2 I_p) \end{pmatrix}$ are $(1 + 3p + 2p^2) \times 1$ independent and identically distributed vectors.

Now, using vec operator and Hadamard product, we can write

$$\begin{aligned} H d_n &= (d_n' \otimes I_p)\mathrm{vec}(H) \\ &= \sum_{i=1}^{n} G_{in} u_i, \end{aligned} \tag{A.7}$$

where, for $i = 1, 2, \ldots, n$, $G_{in} = n^{-\frac{1}{2}}(d_n' \otimes I_p)\begin{pmatrix} (I_p \otimes \mu_i), & (\mu_i' \otimes I_{p^2}), & I_{p^2}, & I_{p^2}, & I_{p^2} \end{pmatrix}$ are $p \times (p + 3p^2 + p^3)$ non-stochastic matrices and $u_i = \begin{pmatrix} (\phi_i + \delta_i) \\ \mathrm{vec}\big(I_p \otimes (\phi_i + \delta_i)\big) \\ \mathrm{vec}(\phi_i \delta_i' + \delta_i \phi_i') \\ \mathrm{vec}(\phi_i \phi_i' - \sigma_\phi^2 I_p) \\ \mathrm{vec}(\delta_i \delta_i' - \sigma_\delta^2 I_p) \end{pmatrix}$ are $(p + 3p^2 + p^3) \times 1$ independent and identically distributed vectors.

From (A.6) and (A.7), we get

$$\begin{pmatrix} h \\ H d_n \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} F_{in} & 0 \\ 0 & G_{in} \end{pmatrix} \begin{pmatrix} w_i \\ u_i \end{pmatrix}, \tag{A.8}$$

where, for $i = 1, 2, \ldots, n$, $\begin{pmatrix} F_{in} & 0 \\ 0 & G_{in} \end{pmatrix}$ are $2p \times (1 + 4p + 5p^2 + p^3)$ non-stochastic matrices and $\begin{pmatrix} w_i \\ u_i \end{pmatrix}$ are $(1 + 4p + 5p^2 + p^3) \times 1$ random vectors. From assumption (v), described in Section 2, we note that the elements of $\sqrt{n}\begin{pmatrix} F_{in} & 0 \\ 0 & G_{in} \end{pmatrix}$ are bounded. Also note that, $\begin{pmatrix} w_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ u_2 \end{pmatrix}, \ldots, \begin{pmatrix} w_n \\ u_n \end{pmatrix}$ independent and identically distributed and from

Lemma A.2

$$E\begin{pmatrix} h \\ Hd_n \end{pmatrix} = 0,$$

$$\begin{aligned}
\lim_{n\to\infty} E(hh') &= (\sigma_\epsilon^2 + \sigma_\delta^2(\beta'\beta))\Sigma + \sigma_\delta^4\beta\beta' + \gamma_{1\delta}\sigma_\delta^3\{f(\boldsymbol{\sigma}_\mu e_p', \beta\beta') \\
&\quad + \big(f(\boldsymbol{\sigma}_\mu e_p', \beta\beta')\big)'\} + \gamma_{2\delta}\sigma_\delta^4 f(I_p, \beta\beta') = \Omega_h, \\
\lim_{n\to\infty} E(Hd_nd_n'H) &= (\sigma_\delta^2 + \sigma_\phi^2)[\Sigma\{dd' + (d'd)I_p\} + dd'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu' + (d'\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu' d)I_p] \\
&\quad + (\gamma_{1\phi}\sigma_\phi^3 + \gamma_{1\delta}\sigma_\delta^3)[f(\boldsymbol{\sigma}_\mu e_p', dd') + \{f(\boldsymbol{\sigma}_\mu e_p', dd')\}' \\
&\quad + 2f(I_p, e_p\boldsymbol{\sigma}_\mu dd')] + (\gamma_{2\phi}\sigma_\phi^4 + \gamma_{2\delta}\sigma_\delta^4)f(I_p, dd') = \Omega_H(d),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} E(hd_n'H) &= -\sigma_\delta^2[\Sigma(d\beta' + (d'\beta)I_p) + \gamma_{1\delta}\sigma_\delta\{f(\boldsymbol{\sigma}_\mu e_p', \beta d') \\
&\qquad + f(I_p, \beta d'\boldsymbol{\sigma}_\mu e_p) + (f(\boldsymbol{\sigma}_\mu e_p', d\beta'))'\} + \gamma_{2\delta}\sigma_\delta^2 f(I_p, \beta d')] \\
&= \Omega_{hH}(d).
\end{aligned}$$

Therefore, on applying central limit theorem (see Malinvaud, 1966, p. 212), for the choice of function $\varphi(n) = \sqrt{n}$, we conclude that $\begin{pmatrix} h \\ Hd_n \end{pmatrix}$ has a $2p-$variate limiting normal distribution with mean vector 0 and covariance matrix $\begin{pmatrix} \Omega_h & \Omega_{hH}(d) \\ \Omega_{hH}'(d) & \Omega_H(d) \end{pmatrix}$. □

The following corollary is an immediate consequence of Lemma A.3.

Corollary A.1. Under the assumptions of Lemma A.3, as $n \to \infty$,

$$h + Hd_n \xrightarrow{d} N_p\big(0, \Omega_h + \Omega_{hH}(d) + \Omega_{hH}'(d) + \Omega_H(d)\big),$$

where Ω_h, $\Omega_{hH}(d)$, and $\Omega_H(d)$ are defined in Lemma A.3.

Lemma A.4. *As $n \to \infty$,*

(i) $\sqrt{n}(b_\delta^{(1)} - \beta) \xrightarrow{d} N_p\big(0, (\Sigma - \sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma - \sigma_\delta^2 I_p)^{-1}\big)$,

(ii) $\sqrt{n}(b_\delta^{(2)} - \beta) \xrightarrow{d} N_p\big(0, f_R(\Sigma)(\Sigma - \sigma_\delta^2 I_p)^{-1}\Omega_h(\Sigma - \sigma_\delta^2 I_p)^{-1}f_R'(\Sigma)\big)$.

Here where $f_R'(\Sigma)$ denotes the transpose of the matrix $f_R(\Sigma)$.

Proof. We have, from (3.8),

$$\sqrt{n}(b_\delta^{(1)} - \beta) = (n^{-1}S - \sigma_\delta^2 I_p)^{-1}h. \tag{A.9}$$

Since $\operatorname*{plim}_{n\to\infty}(n^{-1}S-\sigma_\delta^2 I_p)=(\Sigma-\sigma_\delta^2 I_p)$, therefore from (A.9), Lemma A.3 and Slutzky's Lemma (see Arnold, 1990, p. 451) we obtain the first assertion.

Now from (3.10), we have

$$\sqrt{n}(b_\delta^{(2)}-\beta) \quad = \quad f_R(S)\{\sqrt{n}(b_\delta^{(1)}-\beta)\}, \tag{A.10}$$

since $r=R\beta$. Using the fact that $\operatorname*{plim}_{n\to\infty} f_R(S)=f_R(\Sigma)$, (A.10) and assertion (i), we prove the second assertion. □

Lemma A.5. *As $n\to\infty$,*

$$\sqrt{n}\big(b-(I_p-\sigma_\delta^2\Sigma_x^{-1})\beta\big)\xrightarrow{d}N_p(0,\Omega_b),$$

where

$$\Omega_b=\Sigma^{-1}\{\Omega_h+\Omega_{hH}(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_{hH}'(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_H(\sigma_\delta^2\Sigma^{-1}\beta)\}\Sigma^{-1}.$$

Proof. From (2.1) and (A.1), we have

$$\begin{aligned} b &= S^{-1}X'y \\ &= (I_p-n\sigma_\delta^2 S^{-1})\beta+(nS^{-1})(n^{-\frac{1}{2}}h). \end{aligned} \tag{A.11}$$

Therefore

$$\begin{aligned} \sqrt{n}\big(b-(I_p-\sigma_\delta^2\Sigma_x^{-1})\beta\big) &= (nS^{-1})h+\sqrt{n}\{(I_p-n\sigma_\delta^2S^{-1})-(I_p-\sigma_\delta^2\Sigma_x^{-1})\}\beta \\ &= (nS^{-1})(h+\sigma_\delta^2 H\Sigma_x^{-1}\beta). \end{aligned} \tag{A.12}$$

Since $\lim_{n\to\infty}\Sigma_x=\Sigma$, we have, from Corollary A.1,

$$h+\sigma_\delta^2H\Sigma_x^{-1}\beta\xrightarrow{d}N_p\big(0,\Omega_h+\Omega_{hH}(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_{hH}'(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_H(\sigma_\delta^2\Sigma^{-1}\beta)\big), \tag{A.13}$$

as $n\to\infty$. Now using (A.12), $\operatorname*{plim}_{n\to\infty} n^{-1}S=\Sigma$, $\operatorname*{plim}_{n\to\infty} f_R(S)=f_R(\Sigma)$ and $\operatorname*{plim}_{n\to\infty} f_R(S-n\sigma_\delta^2 I_p)=f_R(\Sigma-\sigma_\delta^2I_p)$, and (A.13), we obtain the required result. □

Lemma A.6. *As $n\to\infty$,*

(i) $\sqrt{n}(b_K^{(1)}-\beta)\xrightarrow{d}N_p\big(0,(\Sigma-\sigma_\delta^2I_p)^{-1}\Omega_K(\Sigma-\sigma_\delta^2I_p)^{-1}\big)$,

(ii) $\sqrt{n}(b_K^{(2)}-\beta)\xrightarrow{d}N_p\big(0,f_R(\Sigma)(\Sigma-\sigma_\delta^2I_p)^{-1}\Omega_K(\Sigma-\sigma_\delta^2I_p)^{-1}f_R'(\Sigma)\big)$,

where $\Omega_K=\Omega_h+\Omega_{hH}(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_{hH}'(\sigma_\delta^2\Sigma^{-1}\beta)+\Omega_H(\sigma_\delta^2\Sigma^{-1}\beta)$; Ω_h, $\Omega_{hH}(\cdot)$, and $\Omega_H(\cdot)$ are given in Lemma A.3.

Proof. We have, from (3.13),

$$\begin{aligned} \sqrt{n}(b_K^{(1)}-\beta) &= \sqrt{n}(K_x^{-1}b-\beta) \\ &= K_x^{-1}[\sqrt{n}\{b-(I_p-\sigma_\delta^2\Sigma_x^{-1})\beta\}]. \end{aligned} \tag{A.14}$$

Since $\lim_{n\to\infty} K_x = I_p - \sigma_\delta^2\Sigma^{-1}$, therefore from (A.14) and Lemma A.5, we obtain the first assertion.

Now we have, from (3.15),

$$\sqrt{n}(b_K^{(2)} - \beta) \quad = \quad f_R(S)\{\sqrt{n}(b_K^{(1)} - \beta)\}. \tag{A.15}$$

Using the fact that $\underset{n\to\infty}{\text{plim}}\, f_R(S) = f_R(\Sigma)$, (A.15), assertion (i) and Slutzky's Lemma (see Arnold, 1990, p. 451), we prove the second assertion. □

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