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REDUCTION OF BIAS AND MEAN SQUARE ERROR IN ESTIMATING AR(1) MODEL PARAMETER BASED ON QUENOUILLE–TYPE AND OPTIMUM OVERLAPPING SERIES SPLITTING ESTIMATORS

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SUMMARY

The autocorrelation parameter of AR(1) model is estimated very often by the ordinary least squares estimator (OLSE) due to its simplicity. The present investigation aims at deriving the algebraic expression of the covariance between two OLSE's obtainable from two overlapping (OS) or non-overlapping (NOS) or gapping (GS) series whatsoever choosing from the given whole series. Such expression is used to obtain the expressions of bias, mean square error and variance of Quenouille's estimator (1956). Based on OS splitting, a Quenouille-type family and another competent family of estimators are suggested. Their comparative performances are discussed in respect of bias and mean square error.

Keywords and phrases: AR(1), Bias, Mean Square Error, OLSE, Overlapping series(OS), Quenouille's estimator.

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1 Introduction

Let us consider an AR(1) model arising from a Markovian stochastic process

$$u_t = \rho u_{t-1} + e_t, \tag{1.1}$$

where $t = 1, 2, \ldots$, $e_t \sim N_1(0, \sigma_e^2)$ and $u_0 \sim N_1(0, \sigma_e^2/\theta)$. e_t's are independently distributed and also independent of u_0. $\theta = 1 - \xi, \xi = \rho^2, |\rho| < 1$. Clearly, $E(u_t) = 0, Cov(u_{t-k}, u_t) = \rho^k(\sigma_e^2/\theta)$, but $Cov(u_{t-k}, e_t) = 0$ for $k > 0$. The parameter ρ is customarily called autocorrelation parameter. Given a series of observations, ρ is usually estimated by the ordinary least squares estimator (OLSE) having the simplest expression (viz. Koutsoyiannis, 1996, §10.1). Expression of OLSE is also a good approximation to maximum likelihood estimate based on approximated likelihood function under normal set-up as assumed in (1.1) (Kendall and Stuart, 1968, §50.4). Quite competently first order serial correlation coefficient is also an well known estimator where 1^{st} order bias may be eliminated by Quenouille's method (Kendall and Stuart, 1968, §48.4).

The present investigation aims at obtaining the covariance between two OLSE's of ρ (say, R_1 and R_2) based on two series originated from (1.1), *e.g.*

$$\begin{aligned} 1^{st}\text{series} &: \{u_1, u_2, \ldots, u_m; u_{m+1}, u_{m+2}, \ldots, u_{m+n}\} \equiv \{u_t\}_1^{m+n} \\ 2^{nd}\text{series} &: \{u_{m+1}, u_{m+2}, \ldots, u_{m+n}; u_{m+n+1}, u_{m+n+2}, \ldots, u_{m+n+p}\} \equiv \{u_t\}_{m+1}^{m+n+p} \end{aligned}$$

which comprise $(m+n)$ and $(n+p)$ observations respectively. We define

$$R_1 = \frac{\sum_1^{m+n-1} u_t u_{t+1}}{\sum_1^{m+n-1} u_t^2}, \qquad R_2 = \frac{\sum_{m+1}^{m+n+p-1} u_t u_{t+1}}{\sum_{m+1}^{m+n+p-1} u_t^2}. \tag{1.2}$$

Clearly, the two series as stated above would be overlapping (OS) or non-overlapping (NOS) according as $n > 0$ or $n = 0$. Additionally, there may arise another situation when the two series are quite separated *e.g.* $\{u_1, u_2, \ldots, u_m\}$ and $\{u_{m+n+1}, u_{m+n+2}, \ldots, u_{m+n+p}\}$. In the case of such gapping series (GS), R_1 and R_2 in (1.2) would have $(m+n-1)$ and $(m+1)$ to be replaced by $(m-1)$ and $(m+n+1)$ respectively.

Covering the algebraic derivation of $Cov(R_1, R_2)$ for OS, NOS and GS series in §2, we show its application in §3 to obtain the bias, mean square error and variance of Quenouille's estimator (R') (1956) for ρ which is actually aimed at reducing the bias in estimating ρ.

Since two subseries on which R_1 and R_2 are based are clearly correlated due to their generation from AR(1) model, R_1 and R_2 must have some statistical association which could be measured by usual correlation coefficient between R_1 and R_2. One alternative measure of association based on their MSE's and Mean Product Error (MPE) has been suggested in §4.

Recalling Quenouille's estimator (Quenouille, 1956), we, next, demarcate a family of estimators for ρ, named Quenouille-type family (or simply Q-family) based on OS splitting series. Trivially, Quenouille's estimator is the leading member of these family having minimum second order bias. Members of this family are adjudged in view of their biasedness and mean square error (MSE) in §5. In §6 one new family of OS splitting estimators (called M-family) is suggested, with an intuition to reduce the influence of superimposing part common to two sub-series. Relevant comparison between Q-family and M-family is also shown in respect of bias and MSE criteria. In §7, the findings of a Monte Carlo experiment are recorded in a brief form. Finally, some concluding remarks over the present investigation are placed in §8.

2 Derivation of Cov(R_1, R_2)

Let us denote

$$A_1 = \sum_1^{m+n-1} u_t u_{t+1}, \qquad A_2 = \sum_{m+1}^{m+n+p-1} u_t u_{t+1}. \tag{2.1}$$

A_1 and A_2 may be conveniently split into six components each under model (1.1).

$$\begin{aligned} A_1 &= \rho \sum_{1}^{m-1} u_t^2 + \sum_{1}^{m-1} u_t e_{t+1} + \rho u_m^2 + u_m e_{m+1} + \rho \sum_{m+1}^{m+n-1} u_t^2 + \sum_{m+1}^{m+n-1} u_t e_{t+1}. \\ A_2 &= \rho \sum_{m+1}^{m+n-1} u_t^2 + \sum_{m+1}^{m+n-1} u_t e_{t+1} + \rho u_{m+n}^2 + u_{m+n} e_{m+n+1} \\ &\quad + \rho \sum_{m+n+1}^{m+n+p-1} u_t^2 + \sum_{m+n+1}^{m+n+p-1} u_t e_{t+1}. \end{aligned}$$

Clearly,

$$E(A_1) = \frac{(m+n-1)\rho}{\theta}\sigma_e^2, \qquad E(A_2) = \frac{(n+p-1)\rho}{\theta}\sigma_e^2. \tag{2.2}$$

In order to derive the expression of $E(A_1A_2)$, we are to make an algebraic sum of the expectations of as many as 36 terms where typically k^{th} term (T_k) is the product of r^{th} component of A_1 and s^{th} component A_2, $k = 6(r-1)+s$. We have derived $E(T_k)$, $k = 1(1)36$; r, $s = 1(1)6$. Two accessories often used for such purpose are presented here in the form of a lemma.

Lemma 2.1. *Under (1.1)*

(*i*) $E(u_j^2 u_i^2) = (\sigma_e^4/\theta^2)(1 + 2\xi^{j-i}), j \geq i.$

(*ii*) $E(u_j^2 u_i e_{i+1}) = 2(\sigma_e^4/\theta)(\xi^{j-i}/\rho), j > i.$ *Q.E.D.*

We furnish here the expressions of $E(T_k)$ excluding their common multiplier $(\xi/\theta^2)\sigma_e^4$.

$E(T_1) = (m-1)(n-1) + \dfrac{2\xi^2(1-\xi^{m-1})(1-\xi^{n-1})}{\theta^2}$ $\qquad E(T_3) = (m-1) + \dfrac{2\xi^{n+1}(1-\xi^{m-1})}{\theta}.$

$E(T_5) = (m-1)(p-1) + \dfrac{2\xi^{n+2}(1-\xi^{m-1})(1-\xi^{p-1})}{\theta^2}.$ $\qquad E(T_7) = \dfrac{2\xi}{\theta}(1-\xi^{m-1})(1-\xi^{n-1}).$

$E(T_9) = 2\xi^n(1-\xi^{m-1}).$ $\qquad E(T_{11}) = \dfrac{2\xi^{n+1}(1-\xi^{m-1})(1-\xi^{p-1})}{\theta}.$

$E(T_{13}) = E(T_{27}) = (n-1) + \dfrac{2\xi}{\theta}(1-\xi^{n-1}).$ $\qquad E(T_{15}) = 1 + 2\xi^n.$

$E(T_{17}) = (p-1) + \dfrac{2\xi^{n+1}(1-\xi^{p-1})}{\theta}.$ $\qquad E(T_{19}) = E(T_{33}) = 2(1-\xi^{n-1}).$

$E(T_{21}) = 2(1-\xi^{n-1})(1-\xi).$ $\qquad E(T_{23}) = 2(1-\xi^n)(1-\xi^{p-1}).$

$E(T_{25}) = (n^2-1) + \dfrac{4(n-2)\xi}{\theta} - \dfrac{4\xi^2(1-\xi^{n-2})}{\theta^2}.$ $\qquad E(T_{26}) = E(T_{31}) = 2(n-2) - \dfrac{2\xi}{\theta}(1-\xi^{n-2}).$

$E(T_{29}) = (n-1)(p-1) + \dfrac{2\xi^2}{\theta^2}(1-\xi^{n-1})(1-\xi^{p-1}).$ $\qquad E(T_{32}) = (n-1)\dfrac{\theta}{\xi}.$

$E(T_{35}) = \dfrac{2\xi}{\theta}(1-\xi^{n-1})(1-\xi^{p-1}).$ $\qquad E(T_i) = 0, \quad i = 2, 4, \ldots, 24, 28, 30, 34, 36.$

for $m \geq 1$, $n \geq 1$, $p \geq 1$.

Proposition 2.1.

(i) For OS,

$$Cov\,(A_1, A_2) = \frac{\sigma_e^4}{\theta^2}\left[(n-1)(1+\xi+\frac{4\xi}{\theta}) + \frac{2}{\theta^2}(\xi^n - \xi^{m+1} - \xi^{p+1} + \xi^{m+n+p})\right], m, n, p \geq 1.$$

For GS,

$$Cov\,(A_1, A_2) = \frac{2\sigma_e^4}{\theta^2}\left[\frac{\xi^{n+2}(1-\xi^{m-1})(1-\xi^{p-1})}{\theta^2}\right], m, n, p \geq 1.$$

For NOS,

$$Cov\,(A_1, A_2) = \frac{2\sigma_e^4}{\theta^2}\left[\frac{\xi^{2}(1-\xi^{m-1})(1-\xi^{p-1})}{\theta^2}\right],\ m, p \geq 1.$$

(ii) For $A = \sum_1^{n-1} u_t u_{t+1}$,

$$V(A) = \frac{\sigma_e^4}{\theta^2}\left[(n-1)(1+\xi+\frac{4\xi}{\theta}) - \frac{4\xi}{\theta^2} + \frac{4\xi^n}{\theta^2}\right], for\ n \geq 2.$$

(iii) For $B = \sum_1^{n-1} u_t^2$,

$$V(B) = \frac{2\sigma_e^4}{\theta^2}\left[(n-1)\left(1+\frac{2\xi}{\theta}\right) - \frac{2\xi}{\theta^2} + \frac{2\xi^n}{\theta^2}\right],\ for\ \ n \geq 2.$$

(iv) For A and B as defined above,

$$Cov(A, B) = \frac{2\rho\sigma_e^4}{\theta^2}\left[(2n-3)\left(1+\frac{\xi}{\theta}\right) - \frac{2\xi}{\theta^2} + \frac{\xi^{n-1}}{\theta} + \frac{2\xi^n}{\theta^2}\right],\ \ for\ \ n \geq 2.$$

Proof.

(i) Assembling all the non-vanishing terms of $E(T_k)$, $k = 1(1)36$, we obtain $E(A_1A_2)$ from which subtracting $E(A_1)E(A_2)$ (vide (2.2)), $Cov(A_1,\ A_2)$ may be expressed with some algebraic effort. For GS($m \geq 1, n \geq 1, p \geq 1$) and for NOS ($m \geq 1,\ n = 0,\ p \geq 1$), only $E(T_5)$, $E(T_6)$, $E(T_{11})$ and $E(T_{12})$ would be necessary to obtain $Cov(A_1,\ A_2)$.

(ii) Once we choose $m = 0$ and $p = 0$ in A_1 and A_2 , we get $A = \sum_1^n u_t u_{t+1}$, $n \geq 1$ so that $E(A^2)$ requires the sum of only four terms, *e.g.* $E(T_{25})$, $E(T_{26})$, $E(T_{31})$ and $E(T_{32})$ to get at the expression of $V(A)$. It may be remarked that mere substituting $m = 0$ and $p = 0$ in the expression of $Cov(A_1,\ A_2)$ will not really do.

(iii)

$$E(B^2) = E\left(\sum_1^{n-1} u_t^2\right)^2 = \frac{\sigma_e^4}{\theta^2}\left[(n^2-1) + (n-2)\frac{4\xi}{\theta} - \frac{4\xi^2}{\theta^2} + \frac{4\xi^n}{\theta^2}\right].$$

[vide $E(T_{25})$] *Q.E.D.*

(iv)

$$Cov\left(\sum_1^{n-1} u_t e_{t+1}, B\right) = E\left(\sum_{j=1}^{n-1} u_j^2 \sum_{i=1}^{n-1} u_i e_{i+1}\right) = \frac{2\sigma_e^4}{\theta^2}\rho\left[(n-2) - \frac{\xi}{\theta} + \frac{\xi^{n-1}}{\theta}\right].$$

[vide $E(T_{31})$]

Noting that $Cov(A,B) = \rho V(B) + Cov(\sum_1^{n-1} u_t e_{t+1}, B)$, the proof is now simple. *Q.E.D.*

Proposition 2.2. *Given* $A = \sum_1^{n-1} u_t u_{t+1}$ *and* $B = \sum_1^{n-1} u_t^2$, *define* $R = \frac{A}{B}$, $B(R) =$ *bias in estimating* ρ *by* $R = E(R) - \rho$ *and* $MSE(R) =$ *Mean square error in* $R = E(R-\rho)^2$. *Then*

(i) $B(R) = -\frac{2\rho}{n-1}\left[1 - \frac{1-\xi^{n-1}}{(n-1)\theta}\right] \approx -\frac{2\rho}{n-1}\left[1 - \frac{1}{(n-1)\theta}\right]$

(ii) $MSE(R) = \frac{\theta}{n-1} - \frac{8\xi^{n+2}}{\theta^2(n-1)^2} \approx \frac{\theta}{n-1}$

(iii) $V(R) = \frac{\theta}{n-1} - \frac{8\xi^{n+2}}{\theta^2(n-1)^2} - \frac{4\xi}{(n-1)^2}\left[1 - \frac{1-\xi^{n-1}}{(n-1)\theta}\right]^2 \approx \frac{\theta}{n-1} - \frac{4\xi}{(n-1)^2}\left[1 - \frac{1}{(n-1)\theta}\right]^2$

Proof. Treating R as a function $f(A,B)$ of A and B, let us expand $f(A,B)$ at the expectations EA and EB of A and B respectively by Taylor series:

$$f(A,B) = f(EA,EB) + (A-EA)\frac{\partial f}{\partial A}|_E + (B-EB)\frac{\partial f}{\partial B}|_E + \frac{1}{2}[(A-EA)^2\frac{\partial^2 f}{\partial A^2}|_E + (B-EB)^2\frac{\partial^2 f}{\partial B^2}|_E + 2(A-EA)(B-EB)\frac{\partial^2 f}{\partial A \partial B}|_E] + \cdots$$

The notation E is meant for replacing A and B by EA and EB in the expressions of the partial derivatives. $EA = \rho EB$ and $EB = (n-1)\frac{\sigma_e^2}{\theta}$.

Now,

$$\frac{\partial f}{\partial A} = \frac{1}{B}, \frac{\partial f}{\partial B} = -\frac{A}{B^2}, \frac{\partial^2 f}{\partial A^2} = 0, \frac{\partial^2 f}{\partial B^2} = \frac{2A}{B^3}, \frac{\partial^2 f}{\partial A \partial B} = -\frac{1}{B^2}.$$

Let us assume that the terms involving higher than 2^{nd} order partial derivatives have negligible contributions to the Taylor series. Then $B(R)$ and $MSE(R)$ would have the following approximated formulae.

$$B(R) = Ef(A,B) - \rho = Ef(A,B) - f(EA,EB) \approx V(B)\frac{EA}{(EB)^3} - \frac{Cov(A,B)}{(EB)^2}$$

$$MSE(R) = E(R-\rho)^2 \approx \frac{V(A)}{(EB)^2} + \frac{V(B)(EA)^2}{(EB)^4} - 2\frac{Cov(A,B)(EA)}{(EB)^3}$$

Remembering further that

$$V(R) = MSE(R) - [B(R)]^2,$$

we may apply the Prop.2.1 (ii), (iii) and (iv) to complete the proof. *Q.E.D.*

Note 2.1 Defining first order sample serial correlation by

$$r_1 = \frac{\sum_1^{n-1} u_t u_{t+1} - \frac{1}{n-1}\left(\sum_1^{n-1} u_t\right)\left(\sum_1^{n-1} u_{t+1}\right)}{\sqrt{\sum_1^{n-1} u_t^2 - \frac{1}{n-1}\left(\sum_1^{n-1} u_t\right)^2}\sqrt{\sum_1^{n-1} u_{t+1}^2 - \frac{1}{n-1}\left(\sum_1^{n-1} u_{t+1}\right)^2}}.$$

Marriott and Pope (1954) derived (up to 1^{st} order bias)

$$B(r_1) = -\frac{1+3\rho}{n-1}$$

whereas due to Prop.2.2(up to 2^{nd} order bias)

$$B(R) = -\frac{2\rho}{n-1}\left[1 - \frac{1}{(n-1)\theta}\right]$$

showing that 1^{st} order bias in r_1 is more than that in R. □

We now set up the following results based on two OLSE's, *viz.* R_1 and R_2 as defined in (1.2).

Proposition 2.3.

(i) For OS, the mean product error(MPE) and covariance between R_1 and R_2 are given respectively as,

$$\begin{aligned} MPE(R_1,R_2) &= E(R_1-\rho)(R_2-\rho) = \frac{n-1}{(n+m-1)(n+p-1)}\theta, \\ Cov(R_1,R_2) &= \frac{n-1}{(n+m-1)(n+p-1)}\theta \\ &\quad -\frac{4\xi}{(n+m-1)(n+p-1)}\left[1 - \frac{1-\xi^{m+n-1}}{(m+n-1)\theta}\right]\left[1 - \frac{1-\xi^{n+p-1}}{(n+p-1)\theta}\right]. \end{aligned}$$

(ii) For GS and NOS , $MPE(R_1, R_2) = 0$,

$$Cov(R_1, R_2) = -\frac{4\xi}{(m-1)(p-1)}\left[1 - \frac{1-\xi^{m-1}}{(m-1)\theta}\right]\left[1 - \frac{1-\xi^{p-1}}{(p-1)\theta}\right].$$

Proof.

(i) Re-expressing (2.1) as $A_1 = \rho B_1 + A_1'$ and $A_2 = \rho B_2 + A_2'$, where

$$A_1' = \sum_{1}^{m+n-1} u_t e_{t+1} \quad and \quad A_2' = \sum_{m+1}^{m+n+p-1} u_t e_{t+1};$$

$$R_i = \rho + \frac{A_i'}{B_i} \quad and \quad B(R_i) = E(R_i - \rho) = E\left(\frac{A_i'}{B_i}\right); i = 1, 2.$$

So that

$$Cov(R_1, R_2) = E\left(\frac{A_1'}{B_1}\frac{A_2'}{B_2}\right) - B(R_1)B(R_2). \tag{2.3}$$

Now, $E(A_1'A_2')$ is obtained by summing the expectations of only 9 terms, namely T_8, T_{10}, T_{12}, T_{20}, T_{22}, T_{24}, T_{32}, T_{34} and T_{36} among which $E(T_{32})$ is the only non-vanishing one.

Thus $E(A_1'A_2') = E(T_{32}) = (n-1)(\sigma_e^4/\theta)$. Applying Taylor series expansion (as used in proving Prop.2.2)

$$\begin{aligned} MPE(R_1, R_2) &= E(R_1 - \rho)(R_2 - \rho) = E\left(\frac{A_1'}{B_1}\frac{A_2'}{B_2}\right) \\ &\approx \frac{E(A_1'A_2')}{(EB_1)(EB_2)} = \frac{(n-1)\theta}{(m+n-1)(n+p-1)}. \end{aligned} \tag{2.4}$$

Using (2) in (2.4) and applying Prop.2.2(i) the result is obvious.

(ii) For $n = 0$, $E(A_1'A_2')$ reduces to only $E(T_{12})$ which is equal to 0. *Q.E.D*

3 Application to Quenouille's Estimator

In order to eliminate 1st order bias, Quenouille (1956) split a single series $\{u_1, u_2, \ldots, u_{2n}\}$ into two halves, e.g.$\{u_1, u_2, \ldots, u_n\}$ and $\{u_{n+1}, u_{n+2}, \ldots, u_{2n}\}$. Let R, R_1 and R_2 be the OLSE's of ρ based on the series as a whole and on 1^{st} and 2^{nd} halves respectively. Quenouille's estimator is given by

$$R' = 2R - \frac{(R_1 + R_2)}{2}. \tag{3.1}$$

Following proposition concerns with the approximate expressions of the bias, MSE and variance of R'.

Proposition 3.1.

$$(i)\quad B(R') \approx \frac{4\rho}{(2n-1)(2n-2)}\left[1 - \frac{(2n)^2-2}{(2n-1)(2n-2)\theta}\right].$$

$$(ii)\quad MSE(R') \approx \frac{\theta}{2n-2}.$$

$$(iii)\quad V(R') \approx \frac{\theta}{2n-2} - \frac{16\xi}{(2n-1)^2(2n-2)^2}\left[1 - \frac{(2n)^2-2}{(2n-1)(2n-2)\theta}\right]^2.$$

Proof. (i) The proof is based exclusively on Prop.2.2.

(ii) To derive MSE(R') , we shall make use of the following results based on Prop.2.2 and Prop.2.3.

Due to Prop.2.2(ii),

$$MSE(R) = \frac{\theta}{2n-1}, MSE(R_1) = \frac{\theta}{n-1}, MSE(R_2) = \frac{\theta}{n-1}.$$

Due to Prop.2.3(ii),

$$MPE(R_1, R_2) = 0,$$

Choosing $m = n$ and $p = 0$ in Prop.2.3(i),

$$MPE(R_2, R) = \frac{\theta}{2n-1}.$$

However, we shall pay a special attention to the case of deriving $MPE(R_1,\, R)$.

$$\begin{aligned} A'_1 &= \sum_1^{n-1} u_t e_{t+1}, B_1 = \sum_1^{n-1} u_t^2, \\ A'_2 &= \sum_1^{2n-1} u_t e_{t+1} = A'_1 + \sum_n^{2n-1} u_t e_{t+1}, \\ B_2 &= \sum_1^{2n-1} u_t^2 = B_1 + \sum_n^{2n-1} u_t^2 \end{aligned}$$

so that using E(T_{32}) with $m = 0$, $E(A'_1 A'_2) = E(A'_1)^2 = \frac{\sigma_e^4}{\theta}(n-1)$.

Subsequently,

$$MPE(R_1, R) \approx \frac{E(A'_1 A'_2)}{(EB_1)(EB_2)} = \frac{\theta}{2n-1}.$$

Combining the above MSE's and MPE's, we obtain

$$MSE(R') = 4MSE(R) + \frac{1}{4}\left[MSE(R_1) + MSE(R_2) + 2MPE(R_1, R_2)\right] - 2\left[MPE(R_1, R) + MPE(R_2, R)\right] \approx \frac{\theta}{2n-2}.$$

(iii) The proof is simple. *Q.E.D.*

Note 3.1 From Prop.2.2 and Prop.3.1 it is asserted again that the first order bias is clearly removed in R'. But no appreciable improvement is noticed in $MSE(R')$ compared to $MSE(R)$. □

4 Association between R_1 and R_2

The statistical relationship between R_1 and R_2 may be studied by installing two kinds of measure of association, e.g.

$$\varphi_{R_1,R_2} = \frac{Cov(R_1, R_2)}{\sqrt{V(R_1)V(R_2)}}$$

and

$$\varphi^*_{R_1,R_2} = \frac{MPE(R_1, R_2)}{\sqrt{MSE(R_1)MSE(R_2)}}$$

Clearly, $|\varphi_{R_1,R_2}| < 1$ and $\left|\varphi^*_{R_1,R_2}\right| < 1$. φ_{R_1,R_2} is indeed the "Classical" (correlation coefficient) measure between R_1 and R_2. Alternatively, in comparison, $\varphi^*_{R_1,R_2}$ may be treated as a "Conventional" measure. Applying Prop.2.3 the following proposition on φ and φ^* is obvious.

Proposition 4.1. *(i) For overlapping series (OS)*

$$\varphi_{R_1,R_2} = \frac{(n-1)\theta - 4\xi\left[1 - \frac{1}{(m+n-1)\theta}\right]\left[1 - \frac{1}{(n+p-1)\theta}\right]}{\sqrt{(m+n-1)\theta - 4\xi\left[1 - \frac{1}{(m+n-1)\theta}\right]^2}\sqrt{(n+p-1)\theta - 4\xi\left[1 - \frac{1}{(n+p-1)\theta}\right]^2}}$$

$$\approx \frac{(n-1) - 4\left(\frac{\xi}{\theta}\right)}{\sqrt{(m+n-1) - 4\left(\frac{\xi}{\theta}\right)}\sqrt{(n+p-1) - 4\left(\frac{\xi}{\theta}\right)}}, n \geq 1,$$

$$\varphi^*_{R_1,R_2} = \frac{(n-1)}{\sqrt{(m+n-1)(n+p-1)}}.$$

(ii) For non-overlapping as well as for gapping series (NOS & GS)

$$\varphi_{R_1,R_2} = \frac{-4\left(\frac{\xi}{\theta}\right)}{\sqrt{(m-1) - 4\left(\frac{\xi}{\theta}\right)}\sqrt{(p-1) - 4\left(\frac{\xi}{\theta}\right)}}, \qquad \varphi^*_{R_1,R_2} = 0.$$

□

5 A family of Quenouille-type Estimator

Let us consider an OS splitting of a time series $\{u_t\}_1^{2n}$ into two parts as $\{u_t\}_1^{n+q}$ and $\{u_t\}_{n-q+1}^{2n}$ consisting of $(n+q)$ observations each but with 2q observations, namely $\{u_t\}_{n-q+1}^{n+q}$ in common. We define a family of Quenouille's estimators $\{R'_q\}_0^{n-1}$, called Q-family as follows:

$$R'_q = 2R - (R_{1q} + R_{2q})/2 \tag{5.1}$$

where R, R_{1q}, R_{2q} are the OLSE's of ρ based on the three series $\{u_t\}_1^{2n}$, $\{u_t\}_1^{n+q}$ and $\{u_t\}_{n-q+1}^{2n}$ respectively. Clearly, the leading member of (5.1), R'_0 (when $q = 0$) corresponds to Quenouille's estimator R' (vide(3.1)). Of course, R'_0 is trivially based on $\{u_t\}_1^n$, and $\{u_t\}_{n+1}^{2n}$ comprising an NOS series.

Since every member of (5.1) is aiming at estimating the same parameter ρ, their performances may be adjudged under the purview of two well-known criteria, *e.g.* squared bias B_q^2 and mean square error (M_q) where

$$B_q^2 = \left[E(R'_q - \rho)\right]^2 \qquad and \qquad M_q = E(R'_q - \rho)^2.$$

Applying Prop.2.2 (i) and 2.3 (i), we derive

$$B_q^2 = \frac{4\xi}{(2n-1)^2}\left[\frac{2q-1}{n+q-1} + \frac{2n^2+2q^2-4nq+4q-1}{\theta(2n-1)(n+q-1)^2}\right]^2, \tag{5.2}$$

for $q = 0, 1, 2, \cdots, n-1$.

Recalling Prop.3.1 (ii), for $q = 0$,

$$M_0 = MSE(R'_0) = MSE(R') = \frac{\theta}{2n-2}. \tag{5.3}$$

For $q > 0$,

$$\begin{aligned} M_q &= MSE(R'_q) \\ &= 4MSE(R) + \frac{1}{4}\left[MSE(R_{1q}) + MSE(R_{2q}) + 2MPE(R_{1q}, R_{2q})\right] \\ &- 2\left[MPE(R_{1q}, R) + MPE(R_{2q}, R)\right]. \end{aligned}$$

Along the proof of Prop. 3.1(ii), we note that

$$MSE(R) = \frac{\theta}{2n-1}, \qquad MSE(R_{1q}) = \frac{\theta}{n+q-1} = MSE(R_{2q}),$$

$$MPE(R_{1q}, R_{2q}) = \frac{2q-1}{(n+q-1)^2}\theta, \qquad MPE(R, R_{1q}) = \frac{\theta}{2n-1} = MPE(R, R_{2q}).$$

Finally,

$$M_q = \frac{n+3q-2}{2(n+q-1)^2}\theta, \quad for \quad q = 1, 2, \cdots, n-1. \tag{5.4}$$

So far as MSE criterion is concerned, the following proposition is quite easy to establish algebraically.

Proposition 5.1. *(i)* $M_q \downarrow_n$ *uniformly over* ξ *for* $\xi \in (0,1)$.

(ii) $M_q \uparrow_\xi$ *uniformly over* n *for* $n = 2, 3, 4, \cdots$.

(iii) $M_q \uparrow_q$ *for* $q = 1, 2, \cdots, \left[\frac{n+1}{3}\right]$ *and*
$M_q \downarrow_q$ *for* $q = \left[\frac{n+1}{3}\right] + 1, \cdots, (n-1)$ *uniformly over any pair* (ξ, n).

(iv) $Max_q\ M_q = \frac{9}{8(2n-1)}$ *at* $q = \left[\frac{n+1}{3}\right]$
$Min_q\ M_q = \frac{4n-5}{8(n-1)^2}$ *at* $q = n-1$

(v) Interlace inequality:

$$\begin{aligned} M_{n-1} &< M_{n-2} < \mathbf{M_1} < \mathbf{M_0} < M_{n-3} < M_{n-4} < M_{n-5} < \mathbf{M_2} \\ &< M_{n-6} < \cdots < M_{n-9} < \mathbf{M_3} < M_{n-10} < \cdots < M_{\left[\frac{n+1}{3}\right]}. \end{aligned}$$ □

However, while studying squared bias criterion, the expression (5.2) of B_q^2 is hard to tackle algebraically. But its monotonicity properties have been revealed through numerical computations based on some selected triplets (n, ξ, q) and stated below in the form of a conjecture.

Conjecture 5.1

(i) For any specific value of q , $B_q^2 \downarrow_n$ uniformly over ξ and $B_q^2 \downarrow_\xi$ uniformly over n,

(ii) $B_q^2 \downarrow_q$ uniformly over both ξ and n. □

Note 5.1 Quenouille's estimator $R'(= R'_0)$ has <u>minimum</u> squared bias but its $MSE(= M_0)$ is not the least. Least MSE is possessed by the last member of the Q-family, namely by R'_{n-1}. □

6 Optimum OS Splitting Estimator

6.1 Formulation of a family of OS splitting estimators

Recalling OS series on which R_1 and R_2 (vide (1.2)) are based, the common part $\{u_t\}_{m+1}^{m+n}$ may also provide an OLSE for ρ as

$$R_3 = \left\{ \sum_{m+1}^{m+n-1} u_t u_{t+1} \right\} \Big/ \left\{ \sum_{m+1}^{m+n-1} u_t^2 \right\}$$

On the basis of R_1, R_2 and R_3, a family of intuitive estimators for ρ, called M-family may be suggested as

$$R^* = R_1 + R_2 - R_3$$

aiming at reducing bias through eliminating the "superimposing" influence of the common series $\{u_t\}_{m+1}^{m+n}$. Applying Props. 2.2 and 2.3, associated bias and MSE of R^* would have the following expressions.

$$\begin{aligned} B(R^*) &= B(R_1) + B(R_2) - B(R_3) \\ &\approx -2\rho\left[\left(\frac{1}{m+b} + \frac{1}{p+b} - \frac{1}{b}\right) - \frac{1}{\theta}\left(\frac{1}{(m+b)^2} + \frac{1}{(p+b)^2} - \frac{1}{b^2}\right)\right] \\ MSE(R^*) &= MSE(R_1) + MSE(R_2) + MSE(R_3) + 2MPE(R_1, R_2) \\ &\quad - 2MPE(R_1, R_3) - 2MPE(R_2, R_3) \approx \frac{mp + b^2}{b(m+b)(p+b)}\theta \end{aligned} \tag{6.1}$$

where $b = n - 1$ for brevity.

Clearly, $B(R^*)$ and $MSE(R^*)$ are functions of the triplet (m, n, p) with total number of observations $N = m + n + p$. Proposition 6.1 concerns with their minimizations in respect of m, n and p.

Proposition 6.1. *(i) For* $(m,\, n,\, p) = (m,\, [\sqrt{mp}]+1,\, p)$, *corresponding* R^* *possesses no first order bias.* $B(R^*) \approx -\frac{4\rho}{\theta}.\frac{1}{\sqrt{mp}(\sqrt{m}+\sqrt{p})^2}$

(ii) For $(m,\, n,\, p) = ([\frac{N-1}{3}],\, ([\frac{N-1}{3}]+1,\, ([\frac{N-1}{3}])$, *corresponding* R^* *(say,* R_0^**) satisfies (i) and possesses minimum second order bias.* $B(R_0^*) \approx -\frac{9\rho}{\theta}.\frac{1}{(N-1)^2}$.

(iii) $MSE(R^*) \downarrow_n$. *For* $(m,\, n,\, p) = (1, N - 2, 1)$, *corresponding* R^* *(say,* R_a^**) possesses minimum MSE.* $MSE(R_a^*) = \frac{1+(N-3)^2}{(N-3)(N-2)^2}\theta$.

(The notation $[v]$ *stands for the greatest integer contained in* v*.)*

Proof. (i) Setting $\frac{1}{m+b} + \frac{1}{p+b} - \frac{1}{b} = 0$, solution of b would be $\sqrt{mp}$.

(ii) Minimization of $\frac{1}{(m+\sqrt{mp})^2} + \frac{1}{(p+\sqrt{mp})^2} - \frac{1}{(\sqrt{mp})^2}$ with respect to m and p under the restriction $m + (\sqrt{mp} + 1) + p = N$ will show the result.

(iii) Simple to prove. *Q.E.D.*

Note 6.1

(a) In view of Prop. 6.1(i), only integer value of $\sqrt{mp}$ will show the 1^{st} order bias completely eliminated. In case $\sqrt{mp} \in (k, k+1)$, k being an integer, 1^{st} order bias will not be eliminated but reducible to a minimum in which case $(m,\, k,\, p)$ and $(m,\, k+1,\, p)$ are both used for numerical comparison.

(b) As regards Prop. 6.1(ii), when $N - 1$ is a multiple of 3(say, $3k$), $(k,\, k+1,\, k)$ will provide unique R_0^*. Contrarily, for $N - 1 = 3k + 1$, $(k+1,\, k,\, k+1)$ and $(k,\, k+2,\, k)$ or for $N - 1 = 3k + 2$, $(k+1,\, k+1,\, k+1)$, $(k,\, k+2,\, k+1)$ and $(k+1,\, k+2,\, k)$ may be tried with.

6.2 Performance Comparison

(a) **Biasedness Criterion**

For comparison purpose we consider "minimum biased" estimators $R'_0(=R')$ and R^*_0 belonging to Q- and M-families respectively. Ratio of their bias is given by

$$\left|\frac{B(R'_0)}{B(R^*_0)}\right| = \frac{4}{9}\left[\frac{2N-3}{(N-1)(N-2)} + \left(\frac{N-1}{N-2}\right)\xi\right].$$

whose maximum value would never exceed unity. Indeed bias in Quenouille's estimator (R'_0) is quite 50% lower than that in R^*_0. Clearly ,Quenouille's estimator is superior to any member of M-family so far as biasedness criterion is concerned.

(b) **MSE criterion**

Recalling (5.3), (5.4), (6.1) and Prop. 6.1(iii),

$$MSE(R') = MSE(R'_0) = M_0 \approx \frac{\theta}{(N-2)}, \qquad MSE(R'_{n-1}) = M_{n-1} \approx \frac{2N-5}{2(N-2)^2}\theta$$

$$MSE(R^*_0) \approx \frac{3}{2(N-1)}\theta, \qquad MSE(R^*_a) \approx \frac{1+(N-3)^2}{(N-3)(N-2)^2}\theta.$$

from which

$$MSE(R^*_a) < MSE(R'_{n-1}) < MSE(R') < MSE(R^*_0).$$

Recalling interlace inequality,

$$MSE(R^*_a) < MSE(R'_{n-1}) < MSE(R'_1) < MSE(R'_{n-2}) < MSE(R'_0) < \cdots$$
$$\cdots < MSE(R'_{[\frac{n+1}{3}]}) < MSE(R^*_0).$$

Thus R^*_a belonging to M-family has better performance than every member of Q-family so far as MSE criterion is concerned.

7 Monte Carlo Results

A model sampling experiment is conducted for $\rho = \pm 0.05,\ \pm 0.3,\ \pm 0.5,\ \pm 0.8$ over each of artificially constructed AR(1) series comprising $10,000$ randomly generated observations being started up with a random u_0 from $N_1(0,\ \sqrt{1-\rho^2})$ succeeded by $10,000$ random e_t's from $N_1(0,\ 1)$ (choosing $\sigma_e^2 = 1$).

We deliberately take the sample size N ($= 22,\ 52,\ 100$) so as to get $\frac{N}{2}$ and $\frac{N-1}{3}$ as pure numbers. The samples of size N each are now collected from the generated series, dropping 5 observations between each sample so as to ensure the samples to be free from seasonal effects etc. The number (k) of such samples is 370, 175, 95 respectively for $N = 22, 52, 100$.

For comparison purpose we confine ourselves to computing simulated bias and MSE of only five competing estimates, *e.g.* R, R', R'_{n-1}, R^*_0 and R^*_a for each sample. As for example, simulated bias and MSE of R are computed as the averages of $(R-\rho)$'s and $(R-\rho)^2$'s over k samples of size N each. The comparison with corresponding approximated theoretical values of bias and MSE are shown in the tables A.1,A.2, A.3 and A.4 in the appendix.

Tables A.1 and A.2 relate to computation of bias respectively for +ve and -ve ρ-values. In respect of every pair (N, ρ) , 1^{st} row show the theoretical biases (as were approximated up to 2^{nd} order in the derivation)while the 2^{nd} corresponds to simulated estimate of bias related to each of five estimators. Standard error(s.e.) of every simulated estimate of bias is next shown in the 3^{rd} row. To study the discrepancy, the entries of 4^{th} row show the ratios being defined by difference between the simulated and observed bias divided by twice the standard error(V in Appendix A).As the ratios (V)are less than unity almost everywhere, simulated and observed biases may be stated to be in a good agreement. But for $(N, \rho) = (22, \pm 0.05)$, such ratios are quite larger than unity. It may be remarked here that the approximated theoretical bias up to 2^{nd} order is not enough for the agreement when N is too small and/or $|\rho|$ is nearing unity. However, these five estimators may be ordered in respect of amount of bias from the least to the largest as (R, R', R'_{n-1}, R^*_0,R^*_a).

As regards the contents of tables A.3 and A.4,the first row shows the approximated theoretical values of the MSE's (vide subsection 6.2(b))while the 2^{nd} row indicates the simulated MSE's in respect of every pair (N, ρ). As expected, simulated MSE's are decreasing with increasing N and $|\rho|$ both, maintaining a good resemblance with theoretical MSE's. The differences between the simulated and theoretical MSE's are shown in the 3^{rd}row. In most of the cases these differences are positive indicating that approximated formula for MSE is lowering the actual values. Furthermore, the largest differences (as shown in the last row of each table)for all the estimators occur at $N = 22$ when $\rho = \pm 0.8$ or $+0.5$. Such evidence indicates that the term(s) in the approximated formula for MSE(which was indeed up to first order only)is not sufficient enough for smaller N and larger threshold values of $|\rho|$. However, the ordering of the estimators in respect of their MSE's from the least to the largest is satisfactorily taken as (R,R^*_a,R'_{n-1},R',R^*_0).

8 Concluding Remarks

While estimating ρ of Gaussian AR(1) model, first order serial correlation (r_1) or least squares estimator (R) has dominating first order bias (vide Note 2.1). Quenouille (1956) suggested an estimator (R') just by splitting the whole series (having even number of observations) into two equal halves, in which case 1^{st} order bias was completely eliminated.

Present investigation establishes two families,*e.g.* Q-family and M-family (vide §5 and §6) whose members are all estimating the same parameter ρ on the basis of two overlapping subseries chosen from the whole series. Their performances are compared with reference to both Biasedness and Mean Square Error. Q-family comprises members having no 1^{st} order bias (unlike M-family). Eventually, Quenouille's estimator (R') coincides with the leading member (R'_0) of Q-family, having least amount of 2^{nd} order bias with maximum MSE. Contrarily, in view of M-family there exists a member (R^*_0) having least bias (not of course less than that of R'). However, its another member (R^*_a) has not only no first order bias but also the least MSE (up to 1^{st} order only) in the two families. Thus both the families are worthy of respective superiority due to R'_0 and R^*_a referring to biasedness and MSE for the same parameter (ρ).

It may be mentioned in this connection that there is a "rough and ready" process of removing 1^{st} order bias by what is named as jack-knifing (Quenouille, 1949; Tukey, 1958). Removing some observation(s) each time from the sample, pseudo-estimates are created and averaged to get a jack-knife estimate. In contrast, members of Q- or M-family are the creation from the pairs of overlapping subseries with varying number of "superimposed" observations.

So far as the derivation of covariance between R_1 and R_2 is concerned, we have been able to produce the leading terms only. Obviously, the expression are never exact, but correct up to second order for bias and up to 1^{st} order only for MSE. So further refinement is essential to distinguish between MSE and variance. However, such refinement requires cumbersome expression of higher-order moments as well as of partial derivatives as involved in Taylor's expansion. Vinod et. al(1996) may be referred to for some allied results on autoregressive model.

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Appendix A:

Comparison of Theoretical bias and MSE's with bias and MSE's from Monte Carlo simulation. The simulation exercise is performed using S-PLUS software.

Table 1: Bias (for +ve ρ)

N	ρ	R'	R_0^*	R'_{n-1}	R_a^*	R	Remarks
22	0.05	0.0001	-0.001	-0.0023	-0.005	-0.0045	Theoretical
		-0.0079	-0.0093	-0.0065	-0.0079	-0.0077	Simulated
		0.0122	0.0142	0.0111	0.0109	0.0108	Standard error
		0.3279	0.2923	0.1892	0.133	0.1481	V
	0.3	0.0007	-0.0067	-0.0136	-0.0301	-0.0271	Theoretical
		-0.0098	-0.0217	-0.0248	-0.0276	-0.0275	Simulated
		0.012	0.0137	0.0109	0.0106	0.0106	Standard error
		0.4375	0.5474	0.5138	0.1179	0.0189	V
	0.5	0.0025	-0.0136	-0.0229	-0.0497	-0.0446	Theoretical
		-0.0129	-0.0331	-0.0411	-0.0444	-0.0446	Simulated
		0.0117	0.0129	0.0104	0.0101	0.0101	Standard error
		0.6581	0.7558	0.875	0.2624	0	V
	0.8	0.0167	-0.0454	-0.039	-0.0743	-0.0661	Theoretical
		-0.0207	-0.0477	-0.0639	-0.0667	-0.0674	Simulated
		0.0101	0.0108	0.0085	0.0084	0.0084	Standard error
		1.8515	0.1065	1.4647	0.4524	0.0774	V
52	0.05	0	-0.0002	-0.001	-0.002	-0.0019	Theoretical
		-0.0003	-0.0094	-0.0023	-0.0014	-0.0014	Simulated
		0.0116	0.0128	0.0107	0.0106	0.0106	Standard error
		0.0129	0.3594	0.0607	0.0283	0.0236	V
	0.3	0.0001	-0.0011	-0.0058	-0.012	-0.0115	Theoretical
		-0.0011	-0.0111	-0.0115	-0.0111	-0.0111	Simulated
		0.0112	0.0126	0.0104	0.0103	0.0103	Standard error
		0.0536	0.3968	0.274	0.0437	0.0194	V
52	0.5	0.0003	-0.0023	-0.0097	-0.0199	-0.0191	Theoretical
		-0.0023	-0.0115	-0.0194	-0.0194	-0.0195	Simulated
		0.0105	0.0122	0.0098	0.0097	0.0097	Standard error
		0.1238	0.377	0.4949	0.0258	0.0206	V
	0.8	0.0024	-0.0077	-0.0159	-0.0309	-0.0297	Theoretical
		-0.004	-0.0125	-0.0309	-0.0311	-0.0312	Simulated
		0.0086	0.0109	0.0076	0.0075	0.0075	Standard error
		0.3721	0.2202	0.9868	0.0133	0.1	V
100	0.05	0	0	-0.0005	-0.001	-0.001	Theoretical
		-0.0042	-0.0086	-0.0031	-0.0034	-0.0034	Simulated
		0.0118	0.0156	0.0114	0.0114	0.0114	Standard error
		0.178	0.2756	0.114	0.1053	0.1053	V
	0.3	0	-0.0003	-0.003	-0.0061	-0.006	Theoretical
		-0.0053	-0.0076	-0.0083	-0.0083	-0.0083	Simulated
		0.0111	0.0147	0.0108	0.0108	0.0108	Standard error
		0.2387	0.2483	0.2454	0.1019	0.1065	V
	0.5	0.0001	-0.0006	-0.005	-0.0102	-0.01	Theoretical
		-0.0065	-0.007	-0.0128	-0.0126	-0.0126	Simulated
		0.01	0.0131	0.0098	0.0097	0.0097	Standard error
		0.33	0.2443	0.398	0.1237	0.134	V
	0.8	0.0006	-0.002	-0.0081	-0.016	-0.0157	Theoretical
		-0.0065	-0.0083	-0.0182	-0.0182	-0.0182	Simulated
		0.0075	0.01	0.007	0.007	0.007	Standard error
		0.4733	0.315	0.7214	0.1571	0.1786	V

Table 2: Bias (for $-$ve ρ)

N	ρ	R'	R_0^*	R'_{n-1}	R_a^*	R	Remarks
22	-0.05	-0.0001	0.001	0.0023	0.005	0.0045	Theoretical
		-0.0072	-0.0043	0.0004	-0.0003	-0.0001	Simulated
		0.0122	0.0143	0.0111	0.0108	0.0108	Standard error
		0.291	0.1853	0.0856	0.2454	0.213	V
	-0.3	-0.0007	0.0067	0.0136	0.0301	0.0271	Theoretical
		-0.0046	0.0089	0.0181	0.0187	0.019	Simulated
		0.0117	0.0141	0.0106	0.0104	0.0104	Standard error
		0.1667	0.078	0.2123	0.5481	0.3894	V
	-0.5	-0.0025	0.0136	0.0229	0.0497	0.0446	Theoretical
		-0.001	0.0193	0.0335	0.0346	0.0349	Simulated
		0.0109	0.0136	0.0098	0.0096	0.0096	Standard error
		0.0688	0.2096	0.5408	0.7865	0.5052	V
	-0.8	-0.0167	0.0454	0.039	0.0743	0.0661	Theoretical
		0.0156	0.0506	0.0604	0.0609	0.0607	Simulated
		0.009	0.0114	0.0081	0.0079	0.0079	Standard error
		1.7944	0.2281	1.321	0.8481	0.3418	V
52	-0.05	0	0.0002	0.001	0.002	0.0019	Theoretical
		-0.0005	-0.0084	0.0011	0.0021	0.0021	Simulated
		0.0116	0.0128	0.0106	0.0106	0.0106	Standard error
		0.0216	0.3359	0.0047	0.0047	0.0094	V
	-0.3	-0.0001	0.0011	0.0058	0.012	0.0115	Theoretical
		-0.0019	-0.0059	0.0086	0.0098	0.0098	Simulated
		0.0112	0.0123	0.0102	0.0102	0.0102	Standard error
		0.0804	0.2846	0.1373	0.1078	0.0833	V
	-0.5	-0.0003	0.0023	0.0097	0.0199	0.0191	Theoretical
		-0.0022	-0.0041	0.0145	0.0157	0.0158	Simulated
		0.0102	0.0115	0.0093	0.0093	0.0093	Standard error
		0.0931	0.2783	0.2581	0.2258	0.1774	V
52	-0.8	-0.0024	0.0077	0.0159	0.0309	0.0297	Theoretical
		0.0049	-0.0011	0.0265	0.0276	0.0277	Simulated
		0.0078	0.0103	0.0069	0.0069	0.0069	Standard error
		0.4679	0.4272	0.7681	0.2391	0.1449	V
100	-0.05	0	0	0.0005	0.001	0.001	Theoretical
		-0.0042	-0.0089	-0.0014	-0.0018	-0.0019	Simulated
		0.0118	0.0156	0.0114	0.0114	0.0114	Standard error
		0.178	0.2853	0.0833	0.1228	0.1272	V
	-0.3	0	0.0003	0.003	0.0061	0.006	Theoretical
		-0.0054	-0.0091	0.0016	0.0011	0.0011	Simulated
		0.0111	0.0145	0.0107	0.0107	0.0107	Standard error
		0.2432	0.3241	0.0654	0.2336	0.229	V
	-0.5	-0.0001	0.0006	0.005	0.0102	0.01	Theoretical
		-0.0064	-0.0086	0.0037	0.0032	0.0033	Simulated
		0.0099	0.0128	0.0094	0.0095	0.0095	Standard error
		0.3182	0.3594	0.0691	0.3684	0.3526	V
	-0.8	-0.0006	0.002	0.0081	0.016	0.0157	Theoretical
		-0.0039	-0.0024	0.0102	0.0104	0.0104	Simulated
		0.0072	0.0087	0.0064	0.0065	0.0065	Standard error
		0.2292	0.2529	0.1641	0.4308	0.4077	V

Table 3: MSE (for +ve ρ)

N	ρ	R'	R_0^*	R'_{n-1}	R_a^*	R	Remarks
22	0.05	0.0475	0.0475	0.0486	0.0499	0.0713	Theoretical
		0.0436	0.0436	0.0456	0.0549	0.0752	Simulated
		0.0039	0.0039	0.003	0.005	0.0039	Diff
	0.3	0.0433	0.0434	0.0444	0.0455	0.065	Theoretical
		0.0425	0.0424	0.0442	0.0533	0.0703	Simulated
		0.0008	0.001	0.0002	0.0078	0.0053	Diff
22	0.5	0.0357	0.0358	0.0366	0.0375	0.0536	Theoretical
		0.04	0.0398	0.0414	0.0503	0.0627	Simulated
		0.0043	0.004	0.0048	0.0128	0.0091	Diff
	0.8	0.0171	0.0172	0.0176	0.018	0.0257	Theoretical
		0.0307	0.0304	0.0307	0.0378	0.0454	Simulated
		0.0136	0.0132	0.0131	0.0198	0.0197	Diff
52	0.05	0.0196	0.0196	0.0198	0.02	0.0293	Theoretical
		0.0198	0.0198	0.0199	0.0234	0.0289	Simulated
		0.0002	0.0002	0.0001	0.0034	0.0004	Diff
	0.3	0.0178	0.0178	0.018	0.0182	0.0268	Theoretical
		0.0187	0.0187	0.019	0.0218	0.0278	Simulated
		0.0009	0.0009	0.001	0.0036	0.001	Diff
	0.5	0.0147	0.0147	0.0149	0.015	0.0221	Theoretical
		0.0167	0.0167	0.017	0.0191	0.0261	Simulated
		0.002	0.002	0.0021	0.0041	0.004	Diff
	0.8	0.0071	0.0071	0.0071	0.0072	0.0106	Theoretical
		0.0109	0.0109	0.0111	0.0129	0.0211	Simulated
		0.0038	0.0038	0.004	0.0057	0.0105	Diff
100	0.05	0.0101	0.0101	0.0101	0.0102	0.0151	Theoretical
		0.0124	0.0124	0.0124	0.0132	0.0233	Simulated
		0.0023	0.0023	0.0023	0.003	0.0082	Diff
	0.3	0.0092	0.0092	0.0092	0.0093	0.0138	Theoretical
		0.0112	0.0112	0.0112	0.0118	0.0207	Simulated
		0.002	0.002	0.002	0.0025	0.0069	Diff
	0.5	0.0076	0.0076	0.0076	0.0077	0.0114	Theoretical
		0.0091	0.0091	0.0092	0.0095	0.0162	Simulated
		0.0015	0.0015	0.0016	0.0018	0.0048	Diff
	0.8	0.0036	0.0036	0.0037	0.0037	0.0055	Theoretical
		0.005	0.005	0.005	0.0053	0.0096	Simulated
		0.0014	0.0014	0.0013	0.0016	0.0041	Diff

Table 4: MSE (for $-$ve ρ)

N	ρ	R'	R_0^*	R'_{n-1}	R_a^*	R	Remarks
22	-0.05	0.0475	0.0475	0.0486	0.0499	0.065	Theoretical
		0.0432	0.0433	0.0453	0.0547	0.0738	Simulated
		0.0043	0.0042	0.0033	0.0048	0.0088	Diff
	-0.3	0.0433	0.0434	0.0444	0.0455	0.0536	Theoretical
		0.04	0.0401	0.0419	0.0509	0.0688	Simulated
		0.0033	0.0033	0.0025	0.0054	0.0152	Diff
	-0.5	0.0357	0.0358	0.0366	0.0375	0.0257	Theoretical
		0.0352	0.0353	0.0367	0.0439	0.0509	Simulated
		0.0005	0.0005	0.0001	0.0064	0.0252	Diff
	-0.8	0.0171	0.0172	0.0176	0.018	0.0014	Theoretical
		0.0265	0.027	0.028	0.0301	0.0156	Simulated
		0.0094	0.0098	0.0104	0.0121	0.0142	Diff
52	-0.05	0.0196	0.0196	0.0198	0.02	0.0268	Theoretical
		0.0198	0.0197	0.0198	0.0235	0.0266	Simulated
		0.0002	0.0001	0	0.0035	0.0002	Diff
	-0.3	0.0178	0.0178	0.018	0.0182	0.0221	Theoretical
		0.0183	0.0182	0.0182	0.0218	0.023	Simulated
		0.0005	0.0004	0.0002	0.0036	0.0009	Diff
	-0.5	0.0147	0.0147	0.0149	0.015	0.0106	Theoretical
		0.0154	0.0153	0.0153	0.0182	0.0185	Simulated
		0.0007	0.0006	0.0004	0.0032	0.0079	Diff
	-0.8	0.0071	0.0071	0.0071	0.0072	0.0006	Theoretical
		0.009	0.009	0.009	0.0106	0.0043	Simulated
		0.0019	0.0019	0.0019	0.0034	0.0037	Diff
100	-0.05	0.0101	0.0101	0.0101	0.0102	0.0138	Theoretical
		0.0124	0.0124	0.0123	0.0132	0.0201	Simulated
		0.0023	0.0023	0.0022	0.003	0.0063	Diff
	-0.3	0.0092	0.0092	0.0092	0.0093	0.0114	Theoretical
		0.0109	0.0109	0.0108	0.0116	0.0157	Simulated
		0.0017	0.0017	0.0016	0.0023	0.0043	Diff
	-0.5	0.0076	0.0076	0.0076	0.0077	0.0055	Theoretical
		0.0086	0.0086	0.0084	0.0094	0.0073	Simulated
		0.001	0.001	0.0008	0.0017	0.0018	Diff
	-0.8	0.0036	0.0036	0.0037	0.0037	0.0003	Theoretical
		0.0041	0.0041	0.004	0.0049	0.0019	Simulated
		0.0005	0.0005	0.0003	0.0012	0.0016	Diff