Journal of Statistical Research *ISSN 0256 - 422 X*
2010, Vol. 44, No. 2, pp. 267-288
Bangladesh

ON RECORD VALUES OF UNIVARIATE EXPONENTIAL DISTRIBUTION

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SUMMARY

Records arise naturally in many fields of studies such as climatology, sports, science, engineering, medicine, traffic, and industry, among others. The exponential distribution plays a pivotal role in the study of records because of its wide range of applicability in the modeling and analysis of life time data in these fields. In this paper, various properties of record values of univariate exponential distribution are reviewed. Most of the recent works are mentioned and some new results are presented. We hope that the findings of this paper will be a useful reference for the practitioners in various fields of studies and further enhancement of research in record value theory and its applications.

Keywords and phrases: Characterization, Exponential distribution, Mean residual life, Minimum variance linear unbiased estimator, Prediction, Reconstruction of past records, Record values.

1 Introduction

An observation is called a record if its value is greater than (or analogously, less than) all the preceding observations. Records arise naturally in many fields of studies such as climatology, sports, science, engineering, medicine, traffic, and industry, among others. For example, consider the weighing of objects on a scale missing its spring. An object is placed on this scale, and its weight is measured. The 'needle' indicates the correct value but does not return to zero when the object is removed. If various objects are placed on the scale, only the weights greater than the previous ones can be recorded. These recorded weights are the record value sequence. The development of the general theory of statistical analysis of record values began with the work of Chandler (1952). Further development on record value distributions such as estimation of parameters, prediction of record values, characterizations, reconstruction of past record values, etc., continued with the contributions of many authors and researchers, among them Foster and Stuart (1954), Renyi (1962), Shorrock (1972, 1973), Resnick (1973), Nagaraja (1977, 1978, 1988), Glick (1978), Dallas (1981), Nayak (1981), Dunsmore (1983), Gupta (1984), Houchens (1984), Lin (1987), Ahsanullah (1978, 1979, 1980, 1981, 1982, 1987, 1988, 1991, 1995, 2004, 2006), Samaniego and Whitaker (1986), Nevzorov (1988), Kamps (1995), Arnold et al. (1998), Rao and Shanbhag (1986, 1994, 1998), Awad and Raqab (2000), Al-Hussaini and Ahmad (2003), Klimczak and Rychlik (2005), Ahmadi et al. (2005), Ahsanullah and Aliev (2008) and Balakrishnan et al. (2009) are notable.

The exponential distribution plays a pivotal role in the study of records because of its wide range of applicability in modeling and analysis of life time data in these fields. In this paper, a review of results on records of univariate exponential distribution is presented. Some new results on mean residual life based on records and the minimum variance linear unbiased estimators of parameters in the reconstruction of past records from exponential distribution are also provided. The organization of this paper is as follows. Section 2 contains the distribution of record values. In Section 3, we provide distribution of record values and some distributional properties when the parent distribution is exponential. Here, we also obtain some new results on mean residual life based on records from univariate exponential distribution. Section 4 contains the use of records to estimate location and scale parameters of univariate exponential distribution. We have obtained some new results on the minimum variance linear unbiased estimators of parameters when the past records are reconstructed from exponential distribution. Section 5 contains prediction of future records. Characterizations of records from exponential distribution are provided in Section 6. The concluding remarks and direction for future research are provided in Section 7.

2 Distribution of Record Values

Suppose that $(X_n)_{n \geq 1}$ is a sequence of *i.i.d.* (independent and identically distributed) *rv*'s (random variables) with *cdf* (cumulative distribution function) F. Let $Y_n =$

max (min) $\{X_j \mid 1 \leq j \leq n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of $\{X_n | n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n)\ ,\ n > 1\}$, where $U(n) = \min\left\{j \mid j > U(n-1), X_j > X_{U(n-1)},\ n > 1\right\}$ and $U(1) = 1$. The record times of the sequence $(X_n)_{n\,\geq\,1}$ are the same as those for the sequence $(F(X_n))_{n\,\geq\,1}$. Since $F(X)$ has a uniform distribution for *rv* X, it follows that the distribution of $U(n)$, $n \geq 1$ does not depend on F. We will denote $L(n)$ as the indices where the lower record values occur. By our assumption $U(1) = L(1) = 1$. The distribution of $L(n)$ also does not depend on F. We say that F belongs to a class H^* if $r(x)$ (hazard rate, which will be formally defined in the next sub-section) is either monotone increasing or decreasing.

2.1 The Exact Distribution of Record Values

Many properties of the record value sequence can be expressed in terms of the function $R(x) = -\ln \overline{F}(x), 0 < \overline{F}(x) < 1$, called cumulative hazard rate. The function defined as $r(x) = (d/dx)R(x) = f(x)\left(\overline{F}(x)\right)^{-1}$, where f is *pdf* (probability density function) corresponding to F, is called the hazard rate. If we define $F_n(x)$ as the *cdf* of $X_{U(n)}$ for $n \geq 1$, then we have

$$F_n(x) = \int_{-\infty}^{x} \frac{(R(u))^{n-1}}{(n-1)!} dF(u), \quad -\infty < x < \infty\ . \tag{2.1}$$

Note that $\overline{F}_n(x) = \overline{F}(x) \sum_{j=0}^{n-1} \frac{(R(x))^{\,j}}{j!}$. The *pdf* of $X_{U(n)}$, denoted by f_n, is

$$f_n(x) = \frac{(R(x))^{n-1}}{(n-1)!}\ f(x), \quad -\infty < x < \infty\ . \tag{2.2}$$

Note that $\overline{F}_n(x) - \overline{F}_{n-1}(x) = \{\overline{F}(x)/f(x)\} f_n(x)$. Two *rv*'s X and Y with *cdf*'s F and G are said to be mutually symmetric if $F(x) = 1 - G(x)$ for all x, or equivalently if their corresponding *pdf*'s f and g exist, then $f(-x) = g(x)$ for all x. If a sequence of *i.i.d.* *rv*'s are symmetric about zero, then they are also mutually symmetric about zero, but not conversely. It is easy to show that for a symmetric or mutually symmetric (about zero) sequence $(X_n)_{n\,\geq\,1}$ of *i.i.d.* *rv*'s, $X_{U(n)}$ and $X_{L(n)}$ are identically distributed. The joint *pdf* $f(x_1, x_2, \ldots, x_n)$ of the n record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$f(x_1, x_2, \ldots, x_n) = \Pi_{j=1}^{n-1}\ r(x_j)\ f(x_n), \ -\infty < x_1 < \cdots < x_n < \infty. \tag{2.3}$$

The joint *pdf* of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i\,j}(x_i, x_j) = \frac{r(x_i)\,(R(x_i))^{i-1}}{(i-1)!} \quad \frac{f(x_j)\,[R(x_j) - R(x_i)]^{j-i-1}}{(j-i-1)!}\ , \quad -\infty < x_i < x_j < \infty. \tag{2.4}$$

The conditional *pdf* of $X_{U(j)}|X_{U(i)} = x_i$ is

$$f(x_j \mid x_i) = \frac{f_{i\ j}(x_i, x_j)}{f(x_i)}$$
$$= \frac{[R(x_j) - R(x_i)]^{j-i-1}}{(j-i-1)!} \cdot \frac{f(x_j)}{1 - F(x_i)}, \quad -\infty < x_i < x_j < \infty. \tag{2.5}$$

For $i > 0, 1 \le k < m$, the joint conditional *pdf* of $X_{U(i+k)}$ and $X_{U(i+m)}$ given $X_{U(i)}$ is

$$f_{i+k\ \ i+m\ |\ i}\left(x, y \mid X_{U(i)} = z\right) = \frac{1}{\Gamma(m-k)} \cdot \frac{1}{\Gamma(k)} [R(y) - R(x)]^{m-k-1} \times$$
$$[R(x) - R(z)]^{k-1} \frac{f(y)\ r(x)}{\overline{F}(z)}, \quad -\infty < z < x < y < \infty.$$

The marginal and conditional *pdf*'s of the n^{th} lower record value can be derived by using similar procedure as that of the n^{th} upper record value. For detailed treatment of the lower record see Ahsanullah et al. (2010).

3 Records of the Univariate Exponential Distribution

Here, we provide distribution of record values and some distributional properties when the parent distribution is exponential.

3.1 Pdf's of Single, Joint and Conditional Record Values

We will now consider the exponential distribution with *pdf* given by

$$f(x) = \begin{cases} \sigma^{-1} \exp\left(-\sigma^{-1}(x-\mu)\right), & x \ \ge \ \mu \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where μ and σ $(\sigma > 0)$ are parameters. The corresponding *cdf* F and the hazard rate r of the *rv* X with *pdf* (3.1) are respectively $F(x) = 1 - e^{-\sigma^{-1}(x-\mu)}$, $x \ge \mu$ *and* $r(x) = \{f(x)/\overline{F}(x)\} = \sigma^{-1}$.

We will denote the exponential distribution with *pdf* (3.1) by $E(\mu, \sigma)$, the exponential distribution $\left(\mu = 0\ ,\ \sigma = \frac{1}{\lambda}\right)$ by $E(\lambda)$, and the standard exponential distribution by $E(1)$. For $E(1)$, the *cdf* F_n $\left(\text{and } \overline{F}_n\right)$ of $X_{U(n)}$ can be expressed as

$$F_n(x) = \int_{-\infty}^{x} \frac{u^{n-1}}{(n-1)!} e^{-u} du, \quad 0 < x < \infty\ ,\ \overline{F}_n(x) = e^{-x} \sum_{j=0}^{n-1} \frac{x^j}{j!}.$$

The *pdf* f_n of $X_{U(n)}$ is

$$f_n(x) = \frac{x^{n-1}}{(n-1)!} e^{-x}, \quad 0 < x < \infty\ , \tag{3.2}$$

Note that $\overline{F}_n(x) - \overline{F}_{n-1}(x) = \{\overline{F}(x)/f(x)\} f_n(x)$, and for $E(\lambda)$,

$$\overline{F}_n(x) - \overline{F}_{n-1}(x) = \frac{\lambda^{n-1}\, x^{n-1}}{\Gamma(n)} \times \left(1 - e^{-\lambda x}\right).$$

For $E(\mu, \sigma)$, the joint *pdf* of $X_{U(m)}$ and $X_{U(n)}$, $m < n$ is

$$f_{m\ n}(x,y) = \begin{cases} \frac{\sigma^{-n}}{\Gamma(m)} (x-\mu)^{m-1} \frac{(y-x)^{n-m-1}}{\Gamma(n-m)} \exp\left(-\sigma^{-1}(y-\mu)\right), & \mu \le x < y < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

It is easy to see that, in this case, $\left(X_{U(n)} - X_{U(n-1)}\right)$ and $\left(X_{U(m)} - X_{U(m-1)}\right)$ are *i.d.* for $1 < m < n < \infty$. It can be shown that $X_{U(m)} \stackrel{d}{=} X_{U(m-1)} + U$, $(m > 1)$ where "$\stackrel{d}{=}$" means equality in distribution and U is independent of $X_{U(m)}$ and $X_{U(m-1)}$, and is identically distributed as X_1 if and only if $X_1 \sim E(\lambda)$. For $E(1)$ with $n \ge 1$,

$$P\left(\left(X_{U(n+1)} > w X_{U(n)}\right)\right) = \int_0^\infty \int_{wx}^\infty \frac{x^{n-1}}{\Gamma(n)}\, e^{-y} dy\, dx = \int_0^\infty \frac{x^{n-1}}{\Gamma(n)}\, e^{-wx} dx = w^{-n}.$$

The conditional *pdf* of $X_{U(n)}$ given $X_{U(m)} = x$ is

$$f_{n|m}(y|x) \quad = \quad \begin{cases} \sigma^{m-n} \frac{(y-x)^{n-m-1}}{\Gamma(n-m)} e^{-\sigma^{-1}(y-x)}, & \mu \le x < y < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Thus, $P\left(\left(X_{U(n)} - X_{U(m)}\right) = y \mid X_{U(m)} = x\right)$ does not depend on x. It can be shown that if $\mu = 0$, then $X_{U(n)} - X_{U(m)}$ is identically distributed as $X_{U(n-m)}$, $m < n$. We take $\mu = 0$ and $\sigma = 1$ and let $T_n = \sum_{j=1}^n X_{U(j)}$. Since

$$T_n = \left(X_{U(n)} - X_{U(n-1)}\right) + \cdots + (n-1)\left(X_{U(2)} - X_{U(1)}\right) + n X_{U(1)} = \sum_{j=1}^n j\ W_j,$$

where W_j's are *i.i.d.* $E(1)$, the characteristic function of T_n can be written as

$$\varphi_n\ (t) = \Pi_{j=1}^n\ \frac{1}{1-jt}. \tag{3.5}$$

Inverting (3.5), we obtain the *pdf* f_{T_n} of T_n as

$$f_{T_n}(u) = \sum_{j=1}^n \frac{1}{\Gamma(j)} \cdot \frac{(-1)^{n-j}}{\Gamma(n-j+1)}\ e^{-\frac{u}{j}}\ j^{n-2}. \tag{3.6}$$

Theorem 1. *Let* $(X_n)_{n \ge 1}$ *be a sequence of i.i.d. rv's with the standard exponential distribution. Suppose* $\xi_j = \frac{X_{U(j)}}{X_{U(j+1)}}$, $j = 1, 2, \ldots, m-1$, *then* ξ_j*'s are independent.*

Proof. We omit the proof here and refer the interested reader to Ahsanullah et al. (2010) for the proof. □

Corollary 3.1. Let $W_j = (\xi_j)^{\,j}$, $j = 1, 2, \ldots, m-1$, then W_j's are *i.i.d.* $U(0,1)$ (uniform over the unit interval) random variables.

Finally, the Fisher's Information for the n^{th} record of $E(\lambda)$ is $\frac{n}{\lambda^2}$.

3.2 Mean Residual Life of Records

Mean residual function, $M_X(t)$ of a random variable X is defined by $M_X(t) = E[X - t \mid X > t]$. For n^{th} upper record value $X_{U(n)}$, the mean residual function, $M_{X_{U(n)}}(t)$ is given by

$$M_{X_{U(n)}}(t) = E\left[X_{U(n)} - t \mid X_{U(n)} > t\right] = \frac{\sum_{j=0}^{n-1} \int_0^\infty F(x+t)\, \frac{(R(x+t))^j}{j!} dx}{\overline{F}(t) \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}}.$$

For the exponential distribution with $F(x) = 1 - e^{-\lambda\, x}$,

$$M_{X_{U(n)}}(t) = \frac{\sum_{j=0}^{n-1} \int_{\lambda t}^\infty e^{-u}\, \frac{u^j}{j!} du}{\lambda e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!}} = \frac{\sum_{j=0}^{n-1} \Gamma(\lambda t\,,\ j+1)}{\lambda e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!}},$$

where $\Gamma(a\,,\ k) = \int_a^\infty e^{-u}\, \frac{u^{k-1}}{(k-1)!} du$.

We omit the proof of the following theorem, which is a characterization of the exponential distribution, and refer the interested reader to Ahsanullah et al. (2010) for the proof.

Theorem 2. *Let* $(X_j)_{j\,\geq\,1}$ *be a sequence of absolutely continuous i.i.d. non-negative random variables with* $F(0) = 0$ *and* $F(x) > 0$ *for all* $x > 0$. *Then, the mean residual life,* $M_{X_{U(k+s\,\mid\,k)}}(t)$ *of* $X_{U(k+s)} - t \mid X_{U(k)} > t$ *is independent of* t *if and only if* X_1 *has the distribution function* $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$.

(This Theorem is a generalization of Theorem 1 of Raqab and Asadi (2008)).

We have

$$M_{X_{U(k+s\,\mid\,k)}}(t) = \int_t^\infty \sum_{j=0}^{s} \frac{(R_t(u))^{\,j}}{\Gamma(j+1)} \cdot \frac{\overline{F}(u)}{\overline{F}(t)}\; du\;, \tag{3.7}$$

If $F(x) = 1 - e^{-\lambda x}$, then $R_t(x) = R(x) - R(t) = \lambda(x - t)$ and

$$M_{X_{U(k+s\,\mid\,k)}}(t) = \int_t^\infty \sum_{j=0}^{s} \frac{(\lambda(u-t))^{\,j}}{\Gamma(j+1)}\; e^{-\lambda(u-t)}\; du = \frac{s+1}{\lambda}.$$

Differentiating both sides of (3.7) and simplifying, we obtain

$$r(t) = \frac{1 + M'_{X_{U(k+s\,\mid\,k)}}(t)}{M_{X_{U(k+s\,\mid\,k)}}(t) - M_{X_{U(k+s-1\,\mid\,k)}}(t)}.$$

Hence

$$F(x) = 1 - \exp\left\{ - \int_0^x \frac{1 + M'_{X_{U(k+s\,|\,k)}}(t)}{M_{X_{U(k+s\,|\,k)}}(t) - M_{X_{U(k+s-1\,|\,k)}}(t)}\, dt \right\}.$$

If $M_{X_{U(k+s\,|\,k)}}(t)$ is independent of t, then $M'_{X_{U(k+s\,|\,k)}}(t) = 0$, and $F(x) = 1 - e^{-\lambda x}$ for some $\lambda > 0$ and all $x \geq 0$.

3.3 Moments

Without any loss of generality we will consider in this sub-section the standard exponential distribution $E(1)$, with *pdf* $f(x) = \exp(-x)$, $x > 0$, for which we also have $f(x) = 1 - F(x)$. We already know that $X_{U(n)}$, in this setting, can be written as the sum of n *i.d. rv*'s $V_1, V_2, \ldots, V_n$ with common distribution $E(1)$. Further,

$$E\left[X_{U(n)}\right] = Var\left[X_{U(n)}\right] = n,\ Cov\left(X_{U(n)}, X_{U(m)}\right) = m\ ,\ m < n. \tag{3.8}$$

For $m < n$,

$$\begin{aligned} E\left[X_{U(n)}^{\,p}\ X_{U(m)}^{\,q}\right] &= \int_0^\infty \int_0^u \frac{1}{\Gamma(m)} \cdot \frac{1}{\Gamma(n-m)}\ u^q e^{-x}\ v^{m+p-1} (u-v)^{n-m-1}\, dvdu \\ &= \frac{\Gamma(m+p)\,\Gamma(n+p+q)}{\Gamma(m)\,\Gamma(n+p)}. \end{aligned}$$

Using (3.5), it can be shown that for $T_n = \sum_{j=1}^n X_{U(j)}$, we have

$$E[T_n] = n(n+1)\ /\ 2 \quad and \quad Var[T_n] = n(n+1)(2n+1)/6.$$

We omit the proofs of the theorems and corollaries of this sub-section and refer the interested reader to Ahsanullah et al. (2010) for the proofs. Some simple recurrence relations satisfied by single and product moments of record values are given by the following theorem.

Theorem 3. *For* $n \geq 1$ *and* $k = 0, 1, \ldots$

$$E\left[X_{U(n)}^{\,k+1}\right] = E\left[X_{U(n-1)}^{\,k+1}\right] + (k+1)\, E\left[X_{U(n)}^{\,k}\right]\ , \tag{3.9}$$

and consequently, for $0 \leq m \leq n-1$ *we can write*

$$E\left[X_{U(n)}^{\,k+1}\right] = E\left[X_{U(m)}^{\,k+1}\right] + (k+1) \sum_{j=m+1}^{n} E\left[X_{U(j)}^{\,k}\right]\ , \tag{3.10}$$

with $E\left[X_{U(0)}^{\,k+1}\right] = 0$ *and* $E\left[X_{U(n)}^{\,0}\right] = 1.$

Remark 1. The recurrence relation (3.9) can be used in a simple way to compute all the single moments of all the record values. Once again, using property that $f(x) = 1 - F(x)$, we can derive some simple recurrence relations for the product moments of record values.

Theorem 4. *For* $m \geq 1$ *and* $p,\ q = 0, 1, 2, \ldots$

$$E\left[X_{U(m)}^{\,p}\ X_{U(m+1)}^{\,q+1}\right] = E\left[X_{U(m)}^{\,p+q+1}\right] + (q+1)\,E\left[X_{U(m)}^{\,p}\ X_{U(m+1)}^{\,q}\right]\ , \tag{3.11}$$

and for $1 \leq m \leq n-2$ *and* $p,\ q = 0, 1, 2, \ldots$

$$E\left[X_{U(m)}^{\,p}\ X_{U(n)}^{\,q+1}\right] = E\left[X_{U(m)}^{\,p}\ X_{U(n-1)}^{\,q+1}\right] + (q+1)\,E\left[X_{U(m)}^{\,p}\ X_{U(n)}^{\,q}\right]\ . \tag{3.12}$$

Remark 2. By repeated application of the recurrence relation (3.12), with the help of the relation (3.11), we obtain, for $n \geq m+1$, that

$$E\left[X_{U(m)}^{\,p}\ X_{U(n)}^{\,q+1}\right] = E\left[X_{U(m)}^{\,p+q+1}\right] + (q+1)\sum_{j=m+1}^{n} E\left[X_{U(m)}^{\,p}\ X_{U(j)}^{\,q}\right]. \tag{3.13}$$

Corollary 3.2. For $n \geq m+1$, $Cov\left(X_{U(m)}\ ,\ X_{U(n)}\right) = Var\left[X_{U(m)}\right]$.

Corollary 3.3. By repeated application of the recurrence relations (3.11) and (3.12), we also obtain for $m \geq 1$

$$E\left[X_{U(m)}^{\,p}\ X_{U(m+1)}^{\,q+1}\right] = \sum_{j=0}^{q+1} (q+1)^{(j)}\,E\left[X_{U(m)}^{\,p+q+1-j}\right],$$

and for $1 \leq m \leq n-2$

$$E\left[X_{U(m)}^{\,p}\ X_{U(n)}^{\,q+1}\right] = \sum_{j=0}^{q+1} (q+1)^{(j)}\,E\left[X_{U(m)}^{\,p}\ X_{U(n-1)}^{\,q+1-j}\right],$$

and

$$(q+1)^{(0)} = 1 \ \ and \ \ (q+1)^{(j)} = (q+1)\,q\cdots(q+1-j+1)\,,\ for\ \ j \geq 1.$$

Remark 3. The recurrence relations (3.11) and (3.12) can be used in a simple way to compute all the product moments of all record values.

Theorem 5. *For* $m \geq 2$ *and* $p,\ q = 0, 1, 2, \ldots$

$$E[X_{U(m-1)}^{\,p+1}\ X_{U(m)}^{\,q}] = E[X_{U(m)}^{\,p+q+1}] - (p+1)\,E[X_{U(m)}^{\,p}\ X_{U(m+1)}^{\,q}], \tag{3.14}$$

and for $2 \leq m \leq n-2$ *and* $p,\ q = 0, 1, 2, \ldots$

$$E[X_{U(m-1)}^{\,p+1} X_{U(n-1)}^{\,q}] = E[X_{U(m)}^{\,p+1} X_{U(n-1)}^{\,q}] - (p+1)\,E[X_{U(m)}^{\,p} X_{U(m+1)}^{\,q}], \tag{3.15}$$

Corollary 3.4. By repeated application of the recurrence relation (3.15),with the help of (3.14), we obtain for $2 \leq m \leq n-1$ and $p,\ q = 0, 1, 2, \ldots$

$$E\left[X_{U(m-1)}^{\,p+1}\ X_{U(n-1)}^{\,q}\right] = E\left[X_{U(n-1)}^{\,p+q+1}\right] - (p+1)\sum_{j=m}^{n-1} E\left[X_{U(j)}^{\,p}\ X_{U(n)}^{\,q}\right].$$

Corollary 3.5. By repeated application of the recurrence relations (3.14) and (3.15), we obtain for $m \geq 2$ that

$$E\left[X_{U(m-1)}^{p+1}\ X_{U(m)}^{q}\right] = \sum_{j=0}^{p+1} (-1)^{j} (p+1)^{(j)} E\left[X_{U(m+j)}^{p+q+1-j}\right],$$

and for $2 \leq m \leq n-2$ that

$$E\left[X_{U(m-1)}^{p+1}\ X_{U(n-1)}^{q}\right] = \sum_{j=0}^{p+1} (-1)^{j} (p+1)^{(j)} E\left[X_{U(m-j)}^{p+1-j}\ X_{U(n+1-j)}^{q}\right],$$

where $(p+1)^{(j)}$ is as defined before.

It is also important to mention here that this approach can easily be adopted to derive recurrence relations for product moments involving more than two record values.

3.4 Limiting Distribution of Record Values

We have seen that for $\mu = 0$ and $\sigma = 1$, $E\left[X_{U(n)}\right] = Var\left[X_{U(n)}\right] = n$. Hence

$$P\left(\frac{X_{U(n)} - n}{\sqrt{n}} \leq x\right) = \int_0^{n+x\sqrt{n}} \frac{x^{n-1}e^{-x}}{\Gamma(n)} dx = p_n(x) \ , \ say \ . \tag{3.16}$$

Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$. Then following table gives values of $p_n(x)$ for various values of n and x and values of $\Phi(x)$ for comparison.

Table 1: Values of $p_n(x)$

$n \setminus x$	-2	-1	0	1	2
5	0.0002	0.1468	0.5575	0.8475	0.9590
10	0.0046	0.1534	0.5421	0.8446	0.9601
15	0.0098	0.1554	0.5343	0.8436	0.9653
25	0.0122	0.1568	0.5243	0.8427	0.9684
45	0.0142	0.1575	0.5198	0.8423	0.9698
$\Phi(x)$	0.0228	0.1587	0.5000	0.8413	0.9772

Thus for large values of n, $\Phi(x)$ is a good approximation of $p_n(x)$.

Finally, the entropy of n^{th} upper record value $X_{U(n)}$, for $E(\lambda)$, is $n + \ln(\Gamma(n)) - \ln\lambda - (n-1)\psi(n)$, where $\psi(n)$ is the digamma function, $\psi(n) = \Gamma'(n)/\Gamma(n)$. See Ahsanullah et al. (2010) for derivation of this result.

4 Estimation of Parameters

We shall consider here the linear estimations of μ and σ.

4.1 Minimum Variance Linear Unbiased Estimator (MVLUE)

Here we derive the MVLUEs $\hat{\mu}$, $\hat{\sigma}$ of μ and σ, when the underlying distribution is exponential. For details on the methods of deriving MVLUE, one can visit Lloyd (1952), and David (1981), among others. Suppose $X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}$ are the m record values from an *i.i.d.* sequence of *rv*'s with common *cdf* $E(\mu, \sigma)$. Let $Y_i = \sigma^{-1}\left(X_{U(i)} - \mu\right)$, $i = 1, 2, \ldots, m$. Then

$$E[Y_i] = Var[Y_i] = i\ ,\ i = 1, 2, \ldots, m, \quad \text{and} \quad Cov(Y_i\ ,\ Y_j) = \min\{i\ ,\ j\}$$

Let $\mathbf{X} = \left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}\right)$, then $E[\mathbf{X}] = \mu\mathbf{L} + \sigma\boldsymbol{\delta}$, $Var[\mathbf{X}] = \sigma^2\mathbf{V}$, where

$$\begin{aligned} \mathbf{L} &= (1, 1, \ldots, 1)'\ ,\ \boldsymbol{\delta} = (1, 2, \ldots, m)'\ , \\ \mathbf{V} &= (V_{ij})\ , \quad V_{ij} = \min\ \{i\ ,\ j\}\ ,\ \ i\ ,\ j = 1, 2, \ldots, m. \end{aligned}$$

The inverse $\mathbf{V}^{-1} = \left(V^{ij}\right)$ can be expressed as

$$V^{ij} = \begin{cases} 2 & \text{if } i = j = 1, 2, \ldots, m-1 \\ 1 & \text{if } i = j = m \\ -1 & \text{if } |i - j| = 1, \quad i\ ,\ j = 1, 2, \ldots, m \\ 0, & \text{otherwise.} \end{cases}$$

The MVLUEs $\hat{\mu}$, $\hat{\sigma}$ of μ and σ respectively are

$$\hat{\mu} = -\boldsymbol{\delta}'\mathbf{V}^{-1}\left(\mathbf{L}\boldsymbol{\delta}' - \boldsymbol{\delta}L'\right)\ \mathbf{V}^{-1}\mathbf{X}\ /\ \boldsymbol{\Delta}\ ,\ \hat{\sigma} = \mathbf{L}'\mathbf{V}^{-1}(\mathbf{L}\boldsymbol{\delta}' - \boldsymbol{\delta}\mathbf{L})V^{-1}\mathbf{X}\ /\ \boldsymbol{\Delta},$$

where $\boldsymbol{\Delta} = (\mathbf{L}'\mathbf{V}^{-1}\mathbf{L})(\boldsymbol{\delta}'\mathbf{V}^{-1}\boldsymbol{\delta}) - (\mathbf{L}'\mathbf{V}^{-1}\boldsymbol{\delta})^2$, $Var[\hat{\mu}] = \sigma^2\mathbf{L}'\mathbf{V}^{-1}\boldsymbol{\delta}/\boldsymbol{\Delta}$, $Var[\ \hat{\sigma}] = \sigma^2\mathbf{L}'\mathbf{V}^{-1}\mathbf{L}/\boldsymbol{\Delta}$, and $Cov(\hat{\mu}\ ,\ \hat{\sigma}) = -\sigma^2\mathbf{L}'\mathbf{V}^{-1}\boldsymbol{\delta}/\boldsymbol{\Delta}$.

It can be shown that

$$\mathbf{L}'\mathbf{V}^{-1} = (1, 0, 0, \ldots, 0)\ ,\ \ \boldsymbol{\delta}'\mathbf{V}^{-1} = (0, 0, \ldots, 0, 1)\ ,\boldsymbol{\delta}'\mathbf{V}^{-1}\boldsymbol{\delta} = m \quad \textit{and} \quad \boldsymbol{\Delta} = m - 1.$$

Upon simplification we get

$$\hat{\mu} = \frac{\left(mX_{U(1)} - X_{U(m)}\right)}{(m-1)}\ ,\ \hat{\sigma} = \frac{\left(X_{U(m)} - X_{U(1)}\right)}{(m-1)}\ , \tag{4.1}$$

with

$$Var[\hat{\mu}] = \frac{m\sigma^2}{(m-1)}\ ,\ Var[\hat{\sigma}] = \frac{\sigma^2}{(m-1)} \quad \textit{and} \quad Cov(\hat{\mu},\ \hat{\sigma}) = -\frac{\sigma^2}{(m-1)}. \tag{4.2}$$

4.2 Best Linear Invariant Estimator (BLIE)

The best linear invariant (in the sense of minimum mean squared error and invariant with respect to the location parameter μ) estimators, BLIEs , $\widetilde{\mu}$, $\widetilde{\sigma}$ of μ and σ are

$$\widetilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{1\,2}}{1+E_{2\,2}}\right) \ , \ \ and \ \ \widetilde{\sigma} = \frac{\hat{\sigma}}{(1+E_{2\,2})} \ ,$$

where $\hat{\mu}$ and $\hat{\sigma}$ are MVLUEs of μ and σ and

$$\begin{pmatrix} Var[\hat{\mu}] & Cov(\hat{\mu}\ ,\ \hat{\sigma}) \\ Cov(\hat{\mu}\ ,\ \hat{\sigma}) & Var[\hat{\sigma}] \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{1\,1} & E_{1\,2} \\ E_{2\,1} & E_{2\,2} \end{pmatrix} \ .$$

The mean squared errors of these estimators are

$$MSE[\widetilde{\mu}] = \sigma^2(E_{1\,1} - E_{1\,2}^2(1+E_{2\,2})^{-1}), \ and \ \ MSE[\widetilde{\sigma}] = \sigma^2 E_{2\,2}(1+E_{2\,2})^{-1}.$$

We have

$$E\left[(\widetilde{\mu}-\mu)\,(\widetilde{\sigma}-\sigma)\right] = \sigma^2 E_{1\,2}(1+E_{2\,2})^{-1}.$$

Using the values of $E_{1\,1}$, $E_{1\,2}$ and $E_{2\,2}$ from (4.2), we obtain

$$\widetilde{\mu} = ((m+1)\ X_{U(1)} - X_{U(m)})\ /\ m\ ,\ \widetilde{\sigma} = (X_{U(m)} - X_{U(1)})\ /\ m\ ,$$
$$Var[\widetilde{\mu}] = \sigma^2(m^2+m-1)\ /\ m^2\ ,\ and\ \ Var[\widetilde{\sigma}] = \sigma^2(m-1)\ /\ m^2.$$

4.3 Reconstruction of Past Records

Here we derive the minimum variance linear unbiased estimators of past records, when the underlying distribution is exponential.

Suppose that we have observed the $n-m$ $(n>m)$ upper record values $\underline{r} = (r_{m+1}, \ldots, r_n)$ from an exponential population $E(\mu,\sigma)$. Balakrishnan et al. (2009) investigated the problem of reconstructing past records $Y = R_l$, $l \le m$, having observed records $\underline{r}$. They obtained the maximum likelihood estimate $\hat{R}_{l\ ,\ M}$ of R_l and $MSE[\hat{R}_{l\ ,\ M}] = E[(\hat{R}_{l\ ,\ M} - R_l)^2]$ as

$$\hat{R}_{l\ ,\ M} = \left(\frac{n-l+1}{n-m+1}\right)\ R_{m+1} + \left(\frac{l-m}{n-m+1}\right)\ R_m\ ,$$

and

$$MSE[\ \hat{R}_{l\ ,\ M}] = \frac{\sigma^2}{(n-m+1)^2}\Big\{(n-m+1)^2\ (m+1-l)(m+2-l) + (l-m)(n-m-1)$$
$$\times\ \ [(m+1-l)(n-m+2) + (n-m)]\Big\}.$$

It is easy to see that the estimate $\hat{R}_{l\ ,\ M}$ of R_l is biased. The bias of this estimate is

$$Bias = \sigma\left[\frac{n-l+1}{n-m+1}\,(\mu+(m+1)\,\sigma) + \frac{l-m}{n-m+1}\,(\mu+n\sigma) - (\mu+l\sigma)\right]$$
$$= \frac{1}{n-m+1}\,(n+m+1-2l)\,\sigma.$$

Thus, we have $\frac{Bias}{\sigma} = \frac{n+m+1-2l}{n-m+1}$. If $n = 8$, $m = 5$ and $l = 1$, then $\frac{Bias}{\sigma} = 3$ and $\frac{MSE[\hat{R}_{l\ ,\ M}]}{\sigma^2} = 16$.

Since R_{m+1} and R_n are sufficient statistics, the minimum variance linear unbiased estimate $\overset{*}{R}_l$ of R_l is given by

$$\overset{*}{R}_l = \left(\frac{n-l}{n-m-1}\right) R_{m+1} + \left(\frac{l-m-1}{n-m-1}\right) R_n \ .$$

The variance of $\overset{*}{R}_l$ is given by

$$\begin{aligned} Var[\overset{*}{R}_l] &= \left(\frac{\sigma}{n-m-1}\right)^2 [(n-l)^2 (m+1) + (l-m-1)^2 n + \\ & \quad 2(n-l)(l-m-1)(m+1)] \ . \end{aligned}$$

For $n = 8$, $m = 5$, $l = 1$, we have $\frac{Var[\overset{*}{R}_l]}{\sigma^2} = 18.5$. Thus, MVLUE is worth considering.

5 Prediction of Record Values

Here we consider the prediction of future records based on past records. For details on these, see, for example, Ahsanullah (1980) , Dunsmore (1983) , Awad and Raqab (2000) , Al-Hussaini and Ahmad (2003) , Ahmadi et al. (2005) and Klimczak and Rychlik (2005), among others.

5.1 Prediction of the sth Upper Record Value

We will predict the s^{th} upper record value based on the first m record values for $s > m$. Let $\mathbf{W}' = (W_1, W_2, \ldots, W_m)$, where

$$\sigma^2 W_i = Cov(X_{U(i)}\ ,\ X_{U(s)}),\ i = 1, 2, \ldots, m \ \ and \ \ \alpha^* = \frac{E[X_{U(s)} - \mu]}{\sigma}.$$

The best linear unbiased predictor of $X_{U(s)}$ is $\hat{X}_{U(s)}$, where $\hat{X}_{U(s)} = \hat{\mu} + \hat{\sigma}\alpha^* + \mathbf{W}'\mathbf{V}^{-1}(\mathbf{X} - \hat{\mu}\mathbf{L} - \hat{\sigma}\boldsymbol{\delta})$, $\hat{\mu}$ and $\hat{\sigma}$ are MVLUEs of μ and σ respectively. It can be shown that $\mathbf{W}'\mathbf{V}^{-1}(\mathbf{X} - \hat{\mu}\mathbf{L} - \hat{\sigma}\boldsymbol{\delta}) = 0$, and hence

$$\begin{aligned} \hat{X}_{U(s)} &= ((s-1)\ X_{U(m)} + (m-s)\ X_{U(1)})/(m-1) \qquad (5.1) \\ E[\hat{X}_{U(s)}] &= \mu + s\sigma\ ,\ Var[\hat{X}_{U(s)}] = \sigma^2(m + s^2 - 2s)\ /\ (m-1) \\ MSE[\hat{X}_{U(s)}] &= E[(\hat{X}_{U(s)} - X_{U(s)})^2] = \sigma^2(s-m)(s-1)\ /\ (m-1). \end{aligned}$$

Let $\widetilde{X}_{U(s)}$ be the best linear invariant predictor of $X_{U(s)}$. Then it can be shown that

$$\widetilde{X}_{U(s)} = \hat{X}_{U(s)} - C_{1\ 2}\ (1 + E_{2\ 2})^{-1}\ \hat{\sigma}\ , \tag{5.2}$$

where $C_{1\ 2}\ \sigma^2 = Cov(\hat{\sigma}, (\mathbf{L} - \mathbf{W}'\mathbf{V}^{-1}\mathbf{L})\ \hat{\mu} + (\alpha^* - \mathbf{W}'\mathbf{V}^{-1}\boldsymbol{\delta})\ \hat{\sigma})$ and $\sigma^2 E_{2\ 2} = Var[\hat{\sigma}]$.
Upon simplification, we get

$$\begin{aligned}
\widetilde{X}_{U(s)} &= \frac{m-s}{m}\ X_{U(1)} + \frac{s}{m}\ X_{U(m)}\ , \\
E[\widetilde{X}_{U(s)}] &= \mu + \left(\frac{ms+m-s}{m}\right)\ \sigma\ , \\
Var[\widetilde{X}_{U(s)}] &= \frac{\sigma^2\left(m^2+ms^2-s^2\right)}{m^2}\ , \\
MSE[\widetilde{X}_{U(s)}] &= MSE[\hat{X}_{U(s)}] - \frac{(s-m)^2}{m\,(m-1)}\ \sigma^2 = \frac{s\,(s-m)}{m}\ \sigma^2\ .
\end{aligned}$$

It is well-known that the best (unrestricted) least square predictor of $X_{U(s)}$ is

$$\hat{\overline{X}}_{U(s)} = E[X_{U(s)} \mid X_{U(1)}, \ldots, X_{U(m)}] = X_{U(m)} + (s-m)\ \sigma. \tag{5.3}$$

But $\hat{\overline{X}}_{U(s)}$depends on the unknown parameter σ. If we substitute the minimum variance unbiased estimate $\hat{\sigma}$ for σ, then $\hat{\overline{X}}_{U(s)}$ becomes equal to $\hat{X}_{U(s)}$. Now

$$\begin{aligned}
E[\hat{\overline{X}}_{U(s)}] &= \mu + s\sigma = E[X_{U(s)}]\ ,\ Var[\hat{\overline{X}}_{U(s)}] = m\sigma^2, \quad \textit{and} \\
MSE[\hat{\overline{X}}_{U(s)}] &= E[(\hat{\overline{X}}_{U(s)} - X_{U(s)})^2] = (s-m)\sigma^2.
\end{aligned}$$

6 Characterization of Exponential Distribution Based on Record Values

In this section, we review the characterization results related to the exponential distribution based on record values. The problem of characterizing exponential distribution based on record values started in late sixties by Tata (1969) and followed in seventies by Nagaraja (1977), Srivastava (1978) and Ahsanullah (1978, 1979). Further development continued with the contributions of many authors and researchers, among them Dallas (1981), Nayak (1981), Pfeifer (1982), Gupta (1984), Deheuvel (1984), Iwińska (1985), Rao and Shanbhag (1986, 1994, 1998), Nagaraja (1988), Ahsanullah and Kirmani (1991), Huang and Li (1993), Grudzień and Szynal (1996, 1997), Basak (1996), López-Bláquez and Moreno-Rebollo (1997), Bairamov and Ahsanullah (2000), Lee et al. (2002), Bairamov et al. (2005), Chang and Lee (2006) and Yanev et al. (2008) are notable. Various characterizations of the exponential distribution based on record values can be found in Ahsanullah (2004). In what follows, we discuss the characterization results (without proofs) in the chronological order rather than their importance.

Theorem 6. *(Tata, 1969) If F is absolutely continuous, then F is exponential if and only if $X_{U(0)}$ and $X_{U(1)} - X_{U(0)}$ are independent.*

As pointed out by Gupta (1984) , the following characterizations appeared in the literature for the i.i.d. case.

1. *The independence of $X_{U(j+1)} - X_{U(j)}$ and $X_{U(j)}$ characterizes the exponential distribution, Srivastava (1978) , Ahsanullah (1979) and Pfeifer (1982).*

2. *$E\left[\left(X_{U(j+1)} - X_{U(j)}\right) \mid X_{U(j)}\right]$ is independent of $X_{U(j)}$ characterizes the exponential distribution, Srivastava (1978) , Ahsanullah (1978) and Nagaraja (1977).*

3. *$Var\left[\left(X_{U(j+1)} - X_{U(j)}\right) \mid X_{U(j)}\right]$ is independent of $X_{U(j)}$ characterizes the exponential distribution, Ahsanullah (1981).*

Subsequently, two characterization results were presented by Ahsanullah (1979) which generalize Tata's (1969) result as well as other characterizations reported in this direction.

Nayak (1981) also presented a generalization of Tata's result by showing that for an absolutely continuous distribution the independence of $X_{U(r)}$ and $X_{U(n)} - X_{U(s)}$ for some $0 \leq r < s < n$ is already sufficient to characterize the exponential distribution. This result was independently proved by Dallas (1981) whose method of proof can be used to characterize other distributions. Ahsanullah (1981) , which was referred to in (3) above, as well as Pfeifer (1982) employed the independence of certain functions of record values (or record increments) to establish further characterizations of the exponential distribution. The similarity between characterizations of exponential distribution based on order statistics and based on record values motivated Gupta (1984) to investigate the relationship between record values and order statistics. The same problem was taken up by Deheuvels (1984) independently. Gupta showed that the conditional distributions of $X_{i+1\ ,\ i+1} - X_{i\ ,\ i+1}$ given $X_{i\ ,\ i+1}$ and $X_{U(i+1)} - X_{U(i)}$ given $X_{U(i)}$ are the same. Gupta also investigated a relation between conditional distributions of $X_{j+1\ ,\ n}$ given $X_{j\ ,\ n}$ and $X_{U(j)}$ given $X_{U(j-1)}$. For different set of conditions, see Deheuvels (1984). Gupta (1984) established two characterization results from which all the results mentioned in $(1) - (3)$ above follow as special cases.

Note that, as pointed out by Deheuvels (1984) and Gupta (1984), much of the characterizations of exponential distribution based on order statistics can be expressed equivalently based on record values and vice versa.

Iwińska (1985) presented a characterization of the exponential distribution by a distributional property of the difference of two arbitrary record values, and also based on the conditional distributions of the difference of two, not necessarily consecutive, record values. A characterization of the exponential distribution based on the expectation of spacings between two, not necessarily consecutive, record values is established by Iwińska (1986). Further, if $(Y_j)_{j \geq 1}$ is the sequence of (upper) record values from a sequence of *i.i.d.* random variables $(X_j)_{j \geq 1}$ having a continuous *cdf* F, for which $E[Y_{m+1}]$ is finite, then it has been shown by Nagaraja (1978) that this (expectation) condition holds if $E[X_1]$ and $E\left[X_1^{+} \left(\ln \left(X_1^{+}\right)\right)^m\right]$ are finite. Using this fact, Nagaraja (1988) established the following characterization result in Theorem 7 below.

Theorem 7. *$E[Y_{m+1} \mid Y_m]$ and $E[Y_m \mid Y_{m+1}]$ are both linear in the conditioning random variable for some m if and only if F is an exponential (type) cdf . (A dual result holds for lower record values).*

Using concepts of NBU (NWU) and IHR (DHR) , Ahsanullah presented certain characterizations of the exponential distribution based on the record values.

Huang and Li (1993) investigated some extensions of various results given in Ahsanullah (1978 , 1979) , Dallas (1981) and Gupta (1984) characterizing the exponential distribution based on record values.

The following theorem is a generalization of Gupta's (1984) and Rao and Shanbhag's (1986) results where Huang and Li (1993) will consider the difference of any two adjacent record values after $X_{U(j)}$ instead of the difference of $X_{U(j+1)}$ and $X_{U(j)}$.

Theorem 8. *Assume that F has pdf f and $F(x) > 0$ for $x > 0$. Let G be a non-decreasing function having non-lattice support on $x > 0$ with $G(0) = 0$ and $E[G(X_1)] < \infty$. If, for some fixed non-negative integers j and k,*

$$E\left[G\left(X_{U(j+k+1)} - X_{U(j+k)}\right) \mid X_{U(j)} = x\right] = c, \textit{for every } x > 0,$$

where $c > 0$ is a constant, and if for some $\xi > 0$, $c < \int_0^{\infty} e^{-\xi x}\, dG(x) < \infty$, then $c = E[G(X_1)]$ and X_1 is exponentially distributed.

Remark 4. Considering a sequence of populations and sequences of random variables $(X_i^{(n)})_{i \geq 1}$ (stemming from the n^{th} population) , White (1990 , 1993) characterized the exponential distribution based on the equidistribution of $X_{U(n)} - X_{U(n-1)}$ and $X_1^{(n)}$. His results are very interesting, but they are not in the same directions as the ones explained here so far.

Grudzień and Szynal (1996) characterized the exponential distribution in terms of record statistics with random index. In their (1997) paper, they mentioned that Lin and Too (1989) characterized the exponential distribution via moments of record values. Grudzień and Szynal (1997) extended the results of Lin and Too's (1989) in Theorems 9 and 10 below. For a fixed integer $k \geq 1$, define the sequence $(Y_n^{(k)})_{n \geq 1}$ of the k^{th} record values as follows:

$$Y_n^{(k)} = X_{U_k(n)\ ,\ L_k(n) + k - 1}\ , \qquad n = 1, 2, \ldots,$$

where the sequence $(U_k(n))_{n \geq 1}$ of the k^{th} record times is given by

$$\begin{aligned} U_k(1) &= 1\ , \\ U_k(n+1) &= \min\{j \mid j > U_k(n)\ ,\ X_{j,\ j+k-1} > X_{U_k(n),\ L_k(n)+k-1}\}, n = 1, 2, \ldots \end{aligned}$$

Note that for $k = 1$ the sequence $({Y_n}^{(1)})_{n \geq 1}$ is the sequence $\left(X_{U(n)}\right)_{n \geq 1}$ for record values define earlier.

Theorem 9. *Assume that* $E\left[|\min \ \{X_1, X_2, \ldots, X_k\}|^{2p}\right] < \infty$ *for a fixed integer* $k \geq 1$ *and some* $p > 1$*. Suppose that* N *is a positive integer valued random variable independent of* $(X_n)_{n \geq 1}$*. Then* $F \sim E(1)$ *if and only if*

$$E\left[\left(Y_N^{(k)}\right)^2\right] - 2k^{-1}E\left[N \ Y_{N+1}^{(k)}\right] + k^{-2}E\left[N \ (N+1)\right] = 0 \ ,$$

provided $E\left[N^2\right] < \infty$.

Theorem 10. *Assume that* $E\left[|\min \ \{X_1, X_2, \ldots, X_k\}|^{2p}\right] < \infty$ *for a fixed integer* $k \geq 1$ *and some* $p > 1$*. Then* $F \sim E(1)$ *if and only if*

$$E\left[\left({Y_1}^{(k)}\right)^2\right] - 2k^{-1}E\left[\ {Y_2}^{(k)}\right] + 2k^{-2} = 0 \ ,$$

proving that each set $\left\{E\left[\left({Y_1}^{(k)}\right)^2\right] \ , \ E\left[\ {Y_2}^{(k)}\right]\right\}$, $k \geq 1$, *characterizes the exponential distribution.*

López-Bláquez and Moreno-Rebollo (1997) presented certain characterizations of distributions based on linear regression of record values. One of these distributions is exponential distribution which extends Nagaraja's (1988) result.

Rao and Shanbhag (1998) extend the results of Dallas (1981) , Gupta (1984) , Rao and Shanbhag (1986 , 1994) , White (1998) and Huang and Li (1993) all of which, except Rao and Shanbhag (1994) , were mentioned earlier.

The following two results on the characterization of exponential distribution in Theorem 11 and 12 below are due to Bairamov and Ahsanullah (2000). Define two sequences of random variables $(\xi_n)_{n \geq 1}$ and $(\eta_n)_{n \geq 1}$ by

$$\xi_n = \begin{cases} 1, & \text{if } X_{n+1} \ < \ \sum_{i=1}^n X_i \ , \\ 0, & \text{otherwise} \end{cases} \qquad \eta_n = \begin{cases} 1, & \text{if } X_{L(n)+1} \ < \ X_{L(n)} \ , \\ 0, & \text{otherwise,} \end{cases}$$

where $n = 1, 2, \ldots$.

Theorem 11. *Let* X_1 *be a non-negative random variable having a continuous cdf* F *satisfying* $\inf\{x \mid F(x) > 0\} = 0$*. Then, the following statements are equivalent:*

(a) X_1 *has an exponential distribution.*

(b) For some $n > 1$, $E[\xi_n] = E[\eta_n]$, *and* F *is either NBU or NWU.*

Theorem 12. *Let* F *be absolutely continuous satisfying* $\inf\{x \mid F(x) > 0\} = 0$*. Then the following properties are equivalent:*

(a) X_1 *has an exponential distribution.*

(b) For some $n > 1$, $\sum_{i=1}^{n} X_i \sim X_{L(n)}$, *and* F *is either NBU or NWU.*

Two characterizations of the exponential distribution due to Lee (2001) , are given in terms of conditional expectations of record values improving similar characterizations mentioned before. Lee et al. (2002) reported two more characterizations of the exponential distribution similar to those in Lee (2001). The newer results and the ones in Lee (2001) can be combined in the following differently worded statements.

Theorem 13. *Let* F *be an absolutely continuous cdf with* $F(x) < 1$ *for all* x. *Then* F *is exponential if and only if for some* n *and* m, $m \leq n-1$ *and some integer* i, $1 \leq i \leq 4$, $E\left[X_{U(n+i)} - X_{U(n)} \mid X_{U(m)} = y\right] = i\ c$, $c > 0$.

For more recent characterizations of the exponential distribution we refer the reader to Ahsanullah (2004). He presented various characterizations of the exponential distribution in different directions.

The following is due to Basak (1996) , which is based on lower k-records. This result generalizes the work of Ahsanullah and Kirmani (1991).

Theorem 14. *Let* $(X_n)_{n \geq 1}$ *be a sequence of i.i.d. rv's with cdf* F *such that* $F(0) = 0$ *and* $\lim_{x \to 0^+} \frac{F(x)}{x} = \lambda$, $\lambda > 0$. *If* $(L(n,k) - k + 1)\, X_{L(n,k)}$ *and* $X_{1,n}$, $k \geq 1$, *are identically distributed, then* $X_1 \sim E(\lambda)$.

Iwińska (2005) presented characterizations of the exponential distribution based on the distributional properties and expected values of the record values. It is assumed throughout the paper that the random variables are continuous and non-negative and their *cdf* F has the property that $\lim_{x \to 0^+} \frac{F(x)}{x}$ exists and is finite. Bairamov et al. (2005) characterized the exponential distribution in terms of the regression of a function of a record value with its adjacent record values as covariate. Some of the characterizations mentioned before are similar in nature. Yanev et al. (2008) extended Bairamov et al.'s (2005) results to truncated exponential distributions with support $(l_F\ ,\ \infty)$ where $l_F = \inf\{x \mid F(x) > 0\}$. Clearly, in case $l_F = 0$, their results reduce to Bairamov et al.'s (2005) for the exponential distribution. For more details of the truncated exponential distributions, we refer the reader to Yanev et al. (2008).

7 Concluding Remarks

In this paper, various properties of record values of univariate exponential distribution are reviewed. Most of the recent works and some new results are presented. Some new results on mean residual life based on records from univariate exponential distribution are obtained. We have also derived some new results on the minimum variance linear unbiased estimators of parameters when the past records are reconstructed from exponential distribution. We hope that the findings of this paper will be a useful reference for practitioners in various fields of studies and further enhancement of research in record value theory and its applications.

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