Journal of Statistical Research *ISSN 0256 - 422 X*
2011, Vol. 45, No. 1, pp. 21-36
Bangladesh

SOME CLASSES OF ESTIMATORS FOR POPULATION MEAN AT CURRENT OCCASION IN TWO-OCCASION SUCCESSIVE SAMPLING

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SUMMARY

A problem related to the estimation of population mean on the current occasion based on the samples selected over two occasions is investigated. Some classes of estimators for estimating the population mean at the current occasion in two-occasion successive (rotation) sampling have been proposed. Properties of the proposed classes of estimators have been studied. Optimum replacement policies are discussed. Estimators in the proposed classes are compared with (i) the sample mean estimator when no information is used from the previous occasion (ii) the optimum estimator which is a linear combination of the means of the matched and unmatched portions of the sample at the current occasion and (iii) a chain type regression to ratio estimator when auxiliary information is used at both the occasions. Empirical comparisons are shown to justify the propositions of the estimators and suitable recommendations are made.

Keywords and phrases: Successive sampling, auxiliary information, chain-type ratio and regression, bias, mean square error, optimum replacement policy.

AMS Classification: 62D05

1 Introduction

In many social, demographic, industrial and agricultural surveys, the same population is sampled repeatedly and the same study variable is measured on each occasion so that development over time can be followed. For examples, labor force surveys are conducted monthly to estimate the number of people in employment, data on the prices of goods are collected monthly to determine a consumer price index, political opinion surveys are conducted at regular intervals to know the voter preferences, etc. In such studies, successive (rotation) sampling plays an important role to provide the reliable and the cost effective estimates of real life (practical) situations at different successive points of time (occasions). It also provides the effective (in terms of cost and precision) estimates of the patterns of change over the period of time.

The problem of successive (rotation) sampling with a partial replacement of sampling units was first considered by Jessen (1942) in the analysis of survey data related to agriculture farm. He pioneered using the entire information collected in the previous investigations (occasions). The theory of successive (rotation) sampling was further extended by Patterson (1950), Rao and Graham (1964), Gupta (1979), Das (1982) and Chaturvedi and Tripathi (1983), among others. Sen (1971) applied this theory with success in designing the estimator for the population mean on the current occasion using information on two auxiliary variables available on previous occasion. Sen (1972, 1973) extended his work for several auxiliary variables. Singh et al. (1991) and Singh and Singh (2001) used the auxiliary information available only at the current occasion and proposed estimators for the current population mean in two-occasion successive (rotation) sampling. Singh (2003) generalized their work for h-occasions successive sampling. Feng and Zou (1997) and Biradar and Singh (2001) used the auxiliary information on both the occasions for estimating the current population mean in successive sampling.

In many situations, information on an auxiliary variable may be readily available on the first as well as on the second occasions; for examples, tonnage (or seat capacity) of each vehicle or ship is known in survey sampling of transportation, number of beds in different hospitals may be known in hospital surveys, number of polluting industries and vehicles are known in environmental survey, nature of employment status, educational status, food availability and medical aids of a locality are well known in advance for estimating the various demographic parameters in demographic surveys. Many other situations in biological (life) sciences could be explored to show the benefits of the present study. Utilizing the auxiliary information on both the occasions, Singh (2005), Singh and Priyanka (2006, 2007, 2008), Singh and Karna (2009a,b) have proposed several estimators for estimating the population mean at the current (second) occasion in two-occasion successive (rotation) sampling. Following the above works and utilizing the information on a stable auxiliary variable readily available on both the occasions, the objective of the present work is to propose some more performing and relevant chain-type estimators for estimating the current population mean in two-occasion successive (rotation) sampling. Properties of the proposed estimators have been studied through empirical means of comparison and subsequently suitable recommendations are made.

2 Sample structures and notations

Let $U = (U_1, U_2, \ldots, U_N)$ be the finite population of N units, which has been sampled over two occasions. The character under study is denoted by $x(y)$ on the first (second) occasion, respectively. It is assumed that the information on an auxiliary variable z(stable over occasion), whose population mean is known, is available on both the occasions and is closely related(positively correlated)to x and y on the first and second occasions respectively. Consider a simple random sample (without replacement) of size n is drawn on the first occasion. A random sub-sample of size $m = n\lambda$ is retained (matched) from the sample selected on the first occasion for its use on the second occasion, while a fresh sample (un-matched sample) of size $u = (n - m) = n\mu$ is selected on the second occasion from the entire population by simple random sampling (without replacement) method so that the sample size on the second occasion is also n. λ and μ (λ+μ =1) are the fractions of the matched and fresh sample, respectively, at the current(second) occasion. The values of λ or μ should be chosen optimally. Hence, onwards, we use the following notations for their further use:
$\bar{X}$, $\bar{Y}$, $\bar{Z}$: The population means of x, y and z respectively.
$\bar{x}_n$, $\bar{x}_m$, $\bar{y}_u$, $\bar{y}_m$, $\bar{z}_u$, $\bar{z}_n$, $\bar{z}_m$:The sample means of the respective variables based on the sample sizes shown in suffices.

ρ_{yx}, ρ_{yz}, ρ_{xz} : The correlation coefficient between the variables shown in suffices.
$S_x^2 = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$: Population mean square of x.
S_y^2, S_z^2 : Population mean squares of y, z respectively.
$S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$: Covariance between y and x.
S_{yz} : Covariance between y and z.
β_{yx}, β_{yz}: Population regression coefficients between the variables shown in suffices.
C_x, C_y and C_z: Coefficients of variation for the variables shown in suffices.

3 Proposed classes of estimators

There are many ways for formulating the estimators. Different combinations of the available information in successive sampling over two occasions may be used to propose the estimators for the population mean but all such estimators may not be as precise as we desire, this require the need for new combinations which may produce desire results.

Motivated with above arguments, we intend to propose some classes of estimators for population mean of the study character y on the current (second) occasion. To estimate the population mean $\bar{Y}$ on the current(second)occasion, two different sets of estimators are considered. One set of estimators $S_u = \{T_{1u}, T_{2u}\}$ based on sample of size $u(= n\mu)$ drawn afresh on the second occasion and the second set of estimators $S_m = \{T_{1m}, T_{2m}\}$ based on the matched sample of size $m(= n\lambda)$, which is common to both the occasions. Estimators of the sets S_u and S_m are defined as:

$$T_{1u} = \bar{y}_u + b_{yz}(u)(\bar{Z} - \bar{z}_u), \qquad T_{2u} = \frac{\bar{y}_u}{\bar{z}_u}\bar{Z},$$
$$T_{1m} = \bar{y}_m^* + b_{yx}(m)(\bar{x}_n^* - \bar{x}_m^*), \qquad T_{2m} = \bar{y}_m^{**} + b_{yx}(m)(\bar{x}_n^* - \bar{x}_m^*),$$

where

$$\bar{y}_m^* = \bar{y}_m + b_{yz}(m)(\bar{Z} - \bar{z}_m), \bar{y}_m^{**} = \frac{\bar{y}_m}{\bar{z}_m}\bar{Z}, \bar{x}_m^* = \frac{\bar{x}_m}{\bar{z}_m}\bar{Z} \text{ and } \bar{x}_n^* = \frac{\bar{x}_n}{\bar{z}_n}\bar{Z},$$

and $b_{yz}(u), b_{yz}(m), b_{yx}(m)$ are the sample regression coefficients between the variables shown in suffices and based on the sample sizes shown in braces.

Considering the convex linear combinations of the estimators of sets S_u and S_m, we propose the following classes of estimators of population mean$\bar{Y}$ at the current (second) occasion:

$$T_{ij} = \varphi_{ij}T_{iu} + (1 - \varphi_{ij})T_{jm}(i, j = 1, 2), \tag{3.1}$$

where $\varphi_{ij}(i, j = 1, 2)$ are the unknown constants to be determined under certain criterion.

Remark 1. For estimating the population mean on each occasion the estimators $T_{iu}(i = 1, 2)$ are suitable, which implies that more belief on T_{iu} could be shown by choosing φ_{ij} as 1(or close to 1), while for estimating the change over the occasions, the estimators $T_{jm}(j = 1, 2)$ could be more useful, so φ_{ij} might be chosen as 0(or close to 0). For asserting both the problems simultaneously, the suitable(optimum)choice of φ_{ij} are required.

4 Properties of the estimators $T_{ij}(i, j = 1, 2)$

Since, T_{iu} and T_{jm} are simple linear regression, ratio or chain-type ratio and regression estimators, they are biased for population mean $\bar{Y}$. Therefore, the resulting classes of estimators T_{ij} defined in equation (3.1) are also biased estimators of $\bar{Y}$. The bias $B(.)$ and mean square errors $M(.)$ are derived up-to order $o(n^{-1})$ under the large sample approximations and using the following transformations: $\bar{y}_u = (1 + e_1)\bar{Y}$, $\bar{y}_m = (1 + e_2)\bar{Y}$, $\bar{x}_m = (1 + e_3)\bar{X}$, $\bar{x}_n = (1 + e_4)\bar{X}$, $\bar{z}_m = (1 + e_5)\bar{Z}$, $\bar{z}_n = (1 + e_6)\bar{Z}$, $\bar{z}_u = (1 + e_7)\bar{Z}$, $s_{yz}(u) = (1 + e_8)S_{yz}$, $s_{yz}(m) = (1 + e_9)S_{yz}$, $s_{yx}(m) = (1 + e_{10})S_{yx}$, $s_z^2(u) = (1 + e_{11})S_z^2$, $s_z^2(m) = (1 + e_{12})S_z^2$, $s_x^2(m) = (1 + e_{13})S_x^2$; Such that $E(e_k) = 0$ and $|e_k| \leq 1 \forall k$= 1,2,3,..., 13. Under the above transformation $T_{iu}(i = 1, 2)$ and $T_{jm}(j = 1, 2)$ take the following forms:

$$T_{1u} = \bar{Y}(1 + e_1) - \beta_{yz}\bar{Z}e_7(1 + e_8)(1 + e_{11})^{-1} \tag{4.1}$$

$$T_{2u} = \bar{Y}(1 + e_1)(1 + e_7)^{-1} \tag{4.2}$$

$$T_{1m} = \bar{Y}(1 + e_2) - \beta_{yz}\bar{Z}e_5(1 + e_9)(1 + e_{12})^{-1} + \beta_{yx}\bar{X}(1 + e_{10}) \times (1 + e_{13})^{-1}\left\{(1 + e_4)(1 + e_6)^{-1} - (1 + e_3)(1 + e_5)^{-1}\right\} \tag{4.3}$$

$$T_{2m} = \bar{Y}(1 + e_2)(1 + e_5)^{-1} + \beta_{yx}\bar{X}(1 + e_{10}) \times (1 + e_{13})^{-1}\left\{(1 + e_4)(1 + e_6)^{-1} - (1 + e_3)(1 + e_5)^{-1}\right\} \tag{4.4}$$

Thus, we have the following theorems:

Theorem 1. *Bias of the classes of estimators $T_{ij}(i, j = 1, 2)$ to the first order of approximations are obtained as*

$$B(T_{ij}) = \varphi_{ij}B(T_{iu}) + (1 - \varphi_{ij})B(T_{jm}); (i, j = 1, 2) \tag{4.5}$$

where

$$B(T_{1u}) = \beta_{yz}\left(\frac{1}{u}-\frac{1}{N}\right)\left(\frac{\alpha_{003}}{S_z^2}-\frac{\alpha_{012}}{S_{yz}}\right) \quad (4.6)$$

$$B(T_{2u}) = \bar{Y}\left(\frac{1}{u}-\frac{1}{N}\right)\left(C_z^2-\rho_{yz}C_yC_z\right) \quad (4.7)$$

$$B(T_{1m}) = \beta_{yz}\left(\tfrac{1}{m}-\tfrac{1}{N}\right)\left(\tfrac{\alpha_{003}}{S_z^2}-\tfrac{\alpha_{012}}{S_{yz}}\right)+\beta_{yx}\bar{X}\left(\tfrac{1}{m}-\tfrac{1}{n}\right)\{(\rho_{xz}C_xC_z-C_z^2)$$
$$+\frac{1}{\bar{X}}\left(\frac{\alpha_{300}}{S_x^2}-\frac{\alpha_{210}}{S_{yx}}\right)+\frac{1}{\bar{Z}}\left(\frac{\alpha_{111}}{S_{yx}}-\frac{\alpha_{201}}{S_x^2}\right)\} \quad (4.8)$$

and $B(T_{2m}) = \bar{Y}\left(\tfrac{1}{m}-\tfrac{1}{N}\right)\left(C_z^2-\rho_{yz}C_yC_z\right)+\beta_{yx}\bar{X}\left(\tfrac{1}{m}-\tfrac{1}{n}\right)\{(\rho_{xz}C_xC_z-C_z^2)$

$$+\frac{1}{\bar{X}}\left(\frac{\alpha_{300}}{S_x^2}-\frac{\alpha_{210}}{S_{yx}}\right)+\frac{1}{\bar{Z}}\left(\frac{\alpha_{111}}{S_{yx}}-\frac{\alpha_{201}}{S_x^2}\right)\} \quad (4.9)$$

where $\alpha_{rst} = E[(x-\bar{X})^r(y-\bar{Y})^s(z-\bar{Z})^t];(r,s,t\geq 0)$ *are integers.*

Proof. The bias of the classes of estimators $T_{ij}(i,j=1,2)$ are given by

$$B(T_{ij}) = E[T_{ij}-\bar{Y}] = \varphi_{ij}E(T_{iu}-\bar{Y})+(1-\varphi_{ij})E(T_{jm}-\bar{Y})$$
$$= \varphi_{ij}B(T_{iu})+(1-\varphi_{ij})B(T_{jm}) \quad (4.10)$$

where $B(T_{iu})$ = $E(T_{iu}-\bar{Y})$ and $B(T_{jm})$ = $E(T_{jm}-\bar{Y})$. Substituting the expressions of $T_{iu}(i=1,2)$ and $T_{jm}(j=1,2)$ from equations (4.1) - (4.4) in the equation (4.10), expanding the terms binomially and taking expectations up to o(n^{-1}), we have the expressions for the bias of the classes of estimators $T_{ij}(i,j=1,2)$ as described in equation (4.5). □

Theorem 2. *Mean square errors of the classes of estimators* $T_{ij}(i,j=1,2)$ *to the first order of approximations are obtained as*

$$M(T_{ij}) = \varphi_{ij}^2M(T_{iu})+(1-\varphi_{ij})^2M(T_{jm})+2\varphi_{ij}(1-\varphi_{ij})C_{ij};(i,j=1,2) \quad (4.11)$$

where

$$M(T_{1u}) = \left(\frac{1}{u}-\frac{1}{N}\right)(1-\rho_{yz}^2)S_y^2 \quad (4.12)$$

$$M(T_{2u}) = \left(\frac{1}{u}-\frac{1}{N}\right)(2(1-\rho_{yz}))S_y^2 \quad (4.13)$$

$$M(T_{1m}) = \left[\left(\frac{1}{m}-\frac{1}{N}\right)(1-\rho_{yz}^2)+\left(\frac{1}{m}-\frac{1}{n}\right)(2\rho_{xz}\rho_{yx}(\rho_{yz}-\rho_{yx}))\right]S_y^2 \quad (4.14)$$

$$M(T_{2m}) = \left[\left(\frac{1}{m}-\frac{1}{N}\right)2(1-\rho_{yz})+\left(\frac{1}{m}-\frac{1}{n}\right)(2\rho_{xz}\rho_{yx}+2\rho_{yz}\rho_{yx}-2\rho_{xz}\rho_{yx}^2-2\rho_{yx})\right]S_y^2 \quad (4.15)$$

$$C_{11} = -\frac{S_y^2}{N}(1-\rho_{yz}^2) \tag{4.16}$$

$$C_{12} = -\frac{S_y^2}{N}(1-\rho_{yz}^2) \tag{4.17}$$

$$C_{21} = -\frac{S_y^2}{N}(1-\rho_{yz}^2) \tag{4.18}$$

$$C_{22} = -\frac{S_y^2}{N}(2(1-\rho_{yz})) \tag{4.19}$$

Proof. It is obvious that mean square errors of the classes of estimators T_{ij}(i, j = 1, 2) are given by

$$\begin{aligned} M(T_{ij}) = E[T_{ij}-\bar{Y}]^2 = E[\varphi_{ij}(T_{iu}-\bar{Y}) + (1-\varphi_{ij})(T_{jm}-\bar{Y})]^2 \\ = \varphi_{ij}^2 M(T_{iu}) + (1-\varphi_{ij})^2 M(T_{jm}) + 2\varphi_{ij}(1-\varphi_{ij})C_{ij} \end{aligned} \tag{4.20}$$

where $M(T_{iu}) = E[T_{iu}-\bar{Y}]^2$, $M(T_{jm}) = E[T_{jm}-\bar{Y}]^2$ and $C_{ij} = E[(T_{iu}-\bar{Y})(T_{jm}-\bar{Y})]$ $(i,j=1,2)$. Substituting the expressions of T_{iu} $(i=1,2)$ and T_{jm} $(j=1,2)$ given in equations (4.1) - (4.4), in the equation(4.20), expanding the terms binomially and taking expectations up to o(n^{-1}), we have the expressions of mean square errors of the classes of estimators T_{ij} given in equation (4.11). □

Remark 2. The mean square errors of T_{ij} given in equation (4.11) are derived under the assumptions that the coefficients of variation of x, y and z are approximately equal.

5 Minimum MSEs of the classes of estimators $T_{ij}(i,j=1,2)$

Since, mean square errors of T_{ij} $(i,j=1,2)$ in equation (4.11) are functions of unknown constants $\varphi_{ij}(i,j=1,2)$, therefore, to get the optimum values of φ_{ij}, the mean square errors of T_{ij} given in equation (4.11)are differentiated with respect to φ_{ij} and equated to zero. The optimum values of φ_{ij} are obtained as

$$\varphi_{ij\,opt} = \frac{M(T_{jm}) - C_{ij}}{M(T_{iu}) + M(T_{jm}) - 2C_{ij}}; (i,j=1,2) \tag{5.1}$$

Now substituting the value of $\varphi_{ij\,opt}$ $(i,j=1,2)$ in equation (4.11), we get the optimum mean square errors of T_{ij} as

$$M(T_{ij})_{opt} = \frac{M(T_{iu}).M(T_{jm}) - C_{ij}^2}{M(T_{iu}) + M(T_{jm}) - 2C_{ij}}; (i,j=1,2) \tag{5.2}$$

Further, substituting the values from equations (4.12)-(4.19) in equations (5.1) and (5.2), the simplified values of $\varphi_{ij\,opt}$ and $M(T_{ij})_{opt}$ are shown below:

$$\varphi_{11\,opt} = \frac{\mu_{11}[A_1 + \mu_{11}A_2]}{A_1 + \mu_{11}^2 A_2} \tag{5.3}$$

$$M(T_{11})_{opt} = A_1 \left[\frac{A_1 + \mu_{11}A_2}{A_1 + \mu_{11}^2 A_2} - f \right] \frac{S_y^2}{n} \tag{5.4}$$

$$\varphi_{12\,opt} = \frac{\mu_{12}[A_6 + \mu_{12}A_8]}{[A_1 + \mu_{12}A_7 + \mu_{12}^2 A_8]} \tag{5.5}$$

$$M(T_{12})_{opt} = \left[\frac{A_{11} + \mu_{12}A_{10} + \mu_{12}^2 A_9}{A_1 + \mu_{12}A_7 + \mu_{12}^2 A_8} \right] \frac{S_y^2}{n} \tag{5.6}$$

$$\varphi_{21\,opt} = \frac{\mu_{21}[A_1 + \mu_{21}A_2]}{[A_3 + \mu_{21}A_{14} + \mu_{21}^2 A_{15}]} \tag{5.7}$$

$$M(T_{21})_{opt} = \left[\frac{A_{11} + \mu_{21}A_{12} + \mu_{21}^2 A_{13}}{A_3 + \mu_{21}A_{14} + \mu_{21}^2 A_{15}} \right] \frac{S_y^2}{n} \tag{5.8}$$

$$\varphi_{22\,opt} = \frac{\mu_{22}[A_3 + \mu_{22}A_4]}{A_3 + \mu_{22}^2 A_4} \tag{5.9}$$

$$M(T_{22})_{opt} = A_3 \left[\frac{A_3 + \mu_{22}A_4}{A_3 + \mu_{22}^2 A_4} - f \right] \frac{S_y^2}{n} \tag{5.10}$$

where $A_1 = (1 - \rho_{yz}^2), A_2 = 2\rho_{xz}\rho_{yx}(\rho_{yz} - \rho_{yx}), A_3 = 2(1 - \rho_{yz}), A_4 = 2(\rho_{xz}\rho_{yx} + \rho_{yz}\rho_{yx} - \rho_{xz}\rho_{yx}^2 - \rho_{yx}), A_5 = A_1 - A_3, A_6 = A_3 + fA_5, A_7 = -(1 - f)A_5, A_8 = A_4 - fA_5, A_9 = -fA_8A_1, A_{10} = (A_4 - f^2A_5)A_1, A_{11} = (1-f)A_1A_3, A_{12} = f^2A_1A_5 + A_3A_2, A_{13} = f(fA_1A_5 - A_2A_3), A_{14} = (1 + f)A_5, A_{15} = A_2 - fA_5$ and $f = \frac{n}{N}$.

Remark 3. The optimum values of $\varphi_{ij\,opt}(i, j = 1, 2)$ given in equations (5.3), (5.5), (5.7) and (5.9) are the functions of ρ_{xz}, ρ_{yz} and ρ_{yx} . The $\varphi_{ij\,opt}$ may be estimated with the help of corresponding sample correlation coefficients.

Remark 4. The optimum mean square errors of $T_{ij}(i, j = 1, 2)$ given in equations (5.4), (5.6), (5.8) and (5.10) are the functions of S_y^2, ρ_{xz}, ρ_{yz} and ρ_{yx} . Therefore, the mean square errors of T_{ij} can be estimated with the help of their corresponding sample estimates.

6 Optimum replacement policy

To determine the optimum values of $\mu_{ij}(i, j = 1, 2)$ (fraction of samples to be taken afresh at the current (second) occasion) so that population mean $\bar{Y}$ may be estimated with the maximum precision, we minimize mean square errors of T_{ij} $(i, j = 1, 2)$ given in equations (5.4), (5.6), (5.8) and (5.10) respectively with respect to μ_{ij}, which result in quadratic equations in μ_{ij}. The quadratic equations and respective solutions of μ_{ij} say $\hat{\mu}_{ij}$ $(i, j = 1, 2)$ are given below:

$$A_2\mu_{11}^2 + 2A_1\mu_{11} - A_1 = 0 \tag{6.1}$$

$$\hat{\mu}_{11} = \frac{-A_1 \pm \sqrt{A_1^2 + A_1 A_2}}{A_2} \tag{6.2}$$

$$Q_1 \mu_{12}^2 + 2Q_2 \mu_{12} + Q_3 = 0 \tag{6.3}$$

$$\hat{\mu}_{12} = \frac{-Q_2 \pm \sqrt{Q_2^2 - Q_1 Q_3}}{Q_1} \tag{6.4}$$

$$Q_4 \mu_{21}^2 + 2Q_5 \mu_{21} + Q_6 = 0 \tag{6.5}$$

$$\hat{\mu}_{21} = \frac{-Q_5 \pm \sqrt{Q_5^2 - Q_4 Q_6}}{Q_4} \tag{6.6}$$

$$A_4 \mu_{22}^2 + 2A_3 \mu_{22} - A_3 = 0 \tag{6.7}$$

$$\hat{\mu}_{22} = \frac{-A_3 \pm \sqrt{A_3^2 + A_3 A_4}}{A_4} \tag{6.8}$$

where $Q_1 = A_7 A_9 - A_8 A_{10}$, $Q_2 = A_1 A_9 - A_8 A_{11}$, $Q_3 = A_1 A_{10} - A_7 A_{11}$, $Q_4 = A_{14} A_{13} - A_{12} A_{15}$, $Q_5 = A_3 A_{13} - A_{11} A_{15}$ and $Q_6 = A_3 A_{12} - A_{11} A_{14}$.

From equations (6.2), (6.4), (6.6) and (6.8), it is obvious that real values of $\hat{\mu}_{ij}$ $(i, j = 1, 2)$ exist, iff, the quantities under square roots are greater than or equal to zero. For any combinations of correlations ρ_{yx}, ρ_{xz} and ρ_{yz}, which satisfy the conditions of real solutions; two real values of $\hat{\mu}_{ij}$ are possible. Hence, while choosing the values of $\hat{\mu}_{ij}$, it should be remembered that $0 \leq \hat{\mu}_{ij} \leq 1$. All the other values of $\hat{\mu}_{ij}$ are said to be inadmissible. Substituting the admissible values of $\hat{\mu}_{ij}$ say $\mu_{ij}^{(0)}$ $(i, j = 1, 2)$ from equations (6.2), (6.4), (6.6) and (6.8) into equations (5.4), (5.6), (5.8) and (5.10) respectively, we have the following optimum values of mean square errors of $T_{ij} (i, j = 1, 2)$.

$$\begin{aligned}
M(T_{11}^0)_{opt} &= A_1 \left[\frac{A_1 + \mu_{11}^{(0)} A_2}{A_1 + \mu_{11}^{(0)2} A_2} - f \right] \frac{S_y^2}{n} \\
M(T_{12}^0)_{opt} &= \frac{A_{11} + \mu_{12}^{(0)} A_{10} + \mu_{12}^{(0)2} A_9}{A_1 + \mu_{12}^{(0)} A_7 + \mu_{12}^{(0)2} A_8} \frac{S_y^2}{n} \\
M(T_{21}^0)_{opt} &= \frac{A_{11} + \mu_{21}^{(0)} A_{12} + \mu_{21}^{(0)2} A_{13}}{A_3 + \mu_{21}^{(0)} A_{14} + \mu_{21}^{(0)2} A_{15}} \frac{S_y^2}{n} \\
M(T_{22}^0)_{opt} &= A_3 \left[\frac{A_3 + \mu_{22}^{(0)} A_4}{A_3 + \mu_{22}^{(0)2} A_4} - f \right] \frac{S_y^2}{n}
\end{aligned}$$

7 Efficiency comparison

The percent relative efficiencies of the estimators $T_{ij} (i, j = 1, 2)$ with respect to (i) sample mean $\bar{y}_n$, when there is no matching (ii) $\hat{\bar{Y}} = \varphi^* \bar{y}_u + (1 - \varphi^*) \bar{y}_m^{'}$, when no auxiliary information is used at any occasion, where $\bar{y}_m^{'} = \bar{y}_m + \beta_{yx} (\bar{x}_n - \bar{x}_m)$, (Sukhatme et.al.(1984)) and (iii) $\Delta = \varphi^{**} \Delta_1 + (1 - \varphi^{**}) \Delta_2$, where $\Delta_1 = \frac{\bar{y}_u}{\bar{z}_u} \bar{Z}$ and $\Delta_2 = \bar{y}_m^{***} (\frac{\bar{Z}}{\bar{z}_m})$ where $\bar{y}_m^{***} = \bar{y}_m + b_{yx} (\bar{x}_n - \bar{x}_m)$, when auxiliary information is used at both the occasions (Singh and Karna(2009a)) have been obtained for different choices of ρ_{yx}, ρ_{xz} and ρ_{yz}. Since $\bar{y}_n$ and $\hat{\hat{Y}}$ are unbiased estimators of $\bar{Y}$. The variance of $\bar{y}_n$, the optimum variance of $\hat{\hat{Y}}$ and the optimum mean square error of Δ are given by

$$\begin{aligned} V(\bar{y}_n) &= \left(\frac{1}{n} - \frac{1}{N}\right) S_y^2 \\ V(\hat{\hat{Y}})_{opt^*} &= \left[1 + \sqrt{(1-\rho_{yx}^2)}\right] \frac{S_y^2}{2n} - \frac{S_y^2}{N} \\ M(\Delta)_{opt^*} &= B_1 \left[\frac{B_1 + \mu^{**} B_2}{B_1 + \mu^{**2} B_2} - f\right] \frac{S_y^2}{n} \end{aligned}$$

where $\mu^{**} = \{-B_1 \pm \sqrt{B_1^2 + B_1 B_2}\}B_2{}^{-1}$, $B_1 = 2(1-\rho_{yz})$, $B_2 = (2\rho_{xz}\rho_{yx} - \rho_{yx}^2)$ and μ^{**} is the optimum fraction of fresh sample at the current occasion for the estimator Δ. For different choices of ρ_{yx}, ρ_{yz} and ρ_{xz}, the optimum values of $\mu_{ij}^{(0)}$(i, j = 1, 2) and percent relative efficiencies $E_{ij}^{(1)}$, $E_{ij}^{(2)}$ and $E_{ij}^{(3)}$ of T_{ij}(i, j = 1, 2) with respect to $\bar{y}_n$, $\hat{\hat{Y}}$ and Δ respectively have been computed for f = 0.1 and shown in Tables 1-4, where

$$E_{ij}^{(1)} = \frac{V(\bar{y}_n)}{M(T_{ij}^0)_{opt}} \times 100, \quad E_{ij}^{(2)} = \frac{V(\hat{\hat{Y}})_{opt^*}}{M(T_{ij}^0)_{opt}} \times 100, \text{ and } E_{ij}^{(3)} = \frac{M(\Delta)_{opt^*}}{M(T_{ij}^0)_{opt}} \times 100; (i, j = 1, 2).$$

8 Conclusion

From Table 1 following interpretation may be read out: (a) For the fixed values of ρ_{xz} and ρ_{yz}, the values of $E_{11}^{(1)}$, $E_{11}^{(3)}$ and $\mu_{11}^{(0)}$ are increasing with the increasing values of ρ_{yx} while the values of $E_{11}^{(2)}$ are decreasing with the increasing values of ρ_{yx}. This behavior is in agreement with the Sukhatme et.al (1984) results, which explains that the more the value of ρ_{yx}, more the fraction of fresh sample is required at the current occasion. (b) For the fixed values of ρ_{xz} and ρ_{yx}, the values of $E_{11}^{(1)}$ and $E_{11}^{(2)}$ are increasing while the values of $\mu_{11}^{(0)}$ and $E_{11}^{(3)}$ are decreasing with the increasing values of ρ_{yz}. This behavior is highly desirable, since, it concludes that if the information on highly correlated auxiliary variable is available, it pays in terms of enhance precision of estimates as well as reduces the cost of the survey. (c) For the fixed values of ρ_{yz} and ρ_{yx}, the values of $E_{11}^{(1)}$ and $E_{11}^{(2)}$ are increasing for some choices of ρ_{xz} while decreasing pattern may also be seen for few choices ρ_{xz}. Similar behavior is visible for $\mu_{11}^{(0)}$ but the values of $E_{11}^{(3)}$ are increasing with the increasing values of ρ_{xz}. (d) Minimum value of $\mu_{11}^{(0)}$ is 0.3702, which indicates that only 37 percent of the total sample size is to be replaced at the current (second)occasion for the corresponding choices of correlations.

From Table 2 it may be observed that (a) For the fixed values of ρ_{xz} and ρ_{yz}, no definite patterns for $\mu_{12}^{(0)}$, $E_{12}^{(1)}$ and $E_{12}^{(2)}$ are observed, as the value of ρ_{yx} is increased while the values of $E_{12}^{(3)}$ are increasing with the increasing values of ρ_{yx} . (b) For the fixed values of ρ_{xz} and ρ_{yx}, the values of $E_{12}^{(1)}$ and $E_{12}^{(2)}$ are increasing while no definite trends are observed in $\mu_{12}^{(0)}$ with increasing choices of ρ_{yz} but the values of $E_{12}^{(3)}$ are decreasing with the increasing values of ρ_{yz} . (c) For the fixed values of ρ_{yz} and ρ_{yx}, the values of $E_{12}^{(1)}$ and $E_{12}^{(2)}$ are decreasing as the value of ρ_{xz} is increased while the values of $\mu_{12}^{(0)}$ are increasing for some choices of ρ_{xz} and decreasing for few choices of ρ_{xz} but the values of $E_{12}^{(3)}$ are increasing with the increasing values of ρ_{xz} . (d) $\mu_{12}^{(0)}$ attains minimum value 0.3607, which indicates that only 36 percent of the total sample size is to be replaced at the current (second) occasion.

From Table 3 we may conclude that (a) For the fixed values of ρ_{xz} and ρ_{yz}, $\mu_{21}^{(0)}$, $E_{21}^{(1)}$ and $E_{21}^{(2)}$ do not follow any pattern when the value of ρ_{yx} is increased while the values of $E_{21}^{(3)}$ are increasing with the increasing values of ρ_{yx} . (b) For the fixed values of ρ_{xz} and ρ_{yx}, the values of $E_{21}^{(1)}$ and $E_{21}^{(2)}$ are increasing while the values of $\mu_{21}^{(0)}$ and $E_{21}^{(3)}$ are decreasing with the increasing values of ρ_{yz}, which is highly desirable. (c) For the fixed values of ρ_{yz} and ρ_{yx}, the values of $\mu_{21}^{(0)}$, $E_{21}^{(1)}$ and $E_{21}^{(2)}$ do not follow any pattern when the value of ρ_{xz} is increased but the values of $E_{21}^{(3)}$ are increasing with the increasing values of ρ_{xz}. (d) Minimum value of $\mu_{21}^{(0)}$ is 0.0463, which indicates that the fraction to be replaced at the current occasion is low as about 5 percent of the total sample size, leading to appreciable reduction in cost.

From Table 4 it can be seen that: (a) For fixed values of ρ_{xz} and ρ_{yz}, the values of $\mu_{22}^{(0)}$, $E_{22}^{(1)}$ and $E_{22}^{(3)}$ are increasing while $E_{22}^{(2)}$ do not follow any pattern with the increasing value of ρ_{yx}. (b) For the fixed values of ρ_{xz} and ρ_{yx}, the values of $\mu_{22}^{(0)}$ and $E_{22}^{(3)}$ are decreasing while the values of $E_{22}^{(1)}$ and $E_{22}^{(2)}$ increase with the increasing trends of ρ_{yz}. This behavior is desirable. (c) For the fixed values of ρ_{yz} and ρ_{yx}, the values of $\mu_{22}^{(0)}$, $E_{22}^{(1)}$, $E_{22}^{(2)}$ and $E_{22}^{(3)}$ decrease with the increasing values of ρ_{xz}. (d) Minimum value of $\mu_{22}^{(0)}$ is 0.3762, which indicates that only 38 percent of the total sample size is to be replaced at the current (second)occasion.

Thus, it is clear that the use of an auxiliary variable is highly rewarding in terms of the proposed classes of estimators. It is also clear that if a highly correlated auxiliary variable is used, relatively, only a smaller fraction of the sample on the current (second) occasion is required to be replaced by a fresh sample, which is reducing the cost of the survey.

Acknowledgements

Authors are thankful to the referees for his valuable and inspiring suggestions. Authors are also thankful to the Indian School of Mines, Dhanbad for providing financial assistances to carry out the present research work.

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Table 1: Optimum values of μ_{11} and percent relative efficiencies of T_{11} with respect to $\bar{y}_n$, $\hat{\hat{Y}}$ and Δ.

$\rho_{xz}\downarrow$	$\rho_{yz}\downarrow$	$\rho_{yx}\rightarrow$	0.3	0.4	0.6	0.8	0.9
0.5	0.5	$\mu_{11}^{(0)}$	0.4904	0.4935	0.5104	0.5481	0.5810
		$E_{11}^{(1)}$	130.49	131.41	136.43	147.73	157.78
		$E_{11}^{(2)}$	127.15	125.32	121.27	114.89	108.33
		$E_{11}^{(3)}$	137.74	139.70	145.03	154.05	161.63
	0.7	$\mu_{11}^{(0)}$	0.4736	0.4736	0.4861	0.5213	0.5542
		$E_{11}^{(1)}$	184.65	184.65	190.04	205.41	219.98
		$E_{11}^{(2)}$	179.92	176.08	168.92	159.76	151.04
		$E_{11}^{(3)}$	120.75	122.06	125.63	131.84	137.29
	0.9	$\mu_{11}^{(0)}$	0.4175	0.4111	0.4175	0.4562	0.5556
		$E_{11}^{(1)}$	431.51	424.31	431.51	475.57	526.31
		$E_{11}^{(2)}$	420.47	404.63	383.56	369.89	361.37
		$E_{11}^{(3)}$	107.00	107.64	109.47	113.16	117.20
0.7	0.5	$\mu_{11}^{(0)}$	0.4867	0.4910	0.5148	0.5737	0.6358
		$E_{11}^{(1)}$	129.41	130.67	137.75	155.54	174.84
		$E_{11}^{(2)}$	126.10	124.61	122.44	120.98	120.04
		$E_{11}^{(3)}$	140.43	143.97	154.32	174.25	194.66
	0.7	$\mu_{11}^{(0)}$	0.4645	0.4645	0.4810	0.5310	0.5844
		$E_{11}^{(1)}$	180.72	180.72	187.81	209.66	233.54
		$E_{11}^{(2)}$	176.09	172.33	166.94	163.07	160.35
		$E_{11}^{(3)}$	123.18	125.95	134.07	149.67	165.26
	0.9	$\mu_{11}^{(0)}$	0.3960	0.3887	0.3960	0.4423	0.5952
		$E_{11}^{(1)}$	407.43	399.26	407.43	459.72	526.31
		$E_{11}^{(2)}$	397.00	380.74	362.16	357.56	361.37
		$E_{11}^{(3)}$	109.91	112.32	119.68	135.05	152.20
0.9	0.5	$\mu_{11}^{(0)}$	0.4832	0.4885	0.5194	0.6056	0.7306
		$E_{11}^{(1)}$	128.37	129.95	139.11	165.38	205.34
		$E_{11}^{(2)}$	125.09	123.92	123.66	128.63	140.99
		$E_{11}^{(3)}$	142.93	147.92	163.18	196.77	244.74
	0.7	$\mu_{11}^{(0)}$	0.4560	0.4560	0.4760	0.5414	0.6235
		$E_{11}^{(1)}$	177.08	177.08	185.68	214.27	251.39
		$E_{11}^{(2)}$	172.55	168.87	165.05	166.66	172.61
		$E_{11}^{(3)}$	125.30	129.29	141.31	166.24	195.50
	0.9	$\mu_{11}^{(0)}$	0.3781	0.3702	0.3781	0.4299	0.4630
		$E_{11}^{(1)}$	387.51	378.74	387.51	445.64	526.31
		$E_{11}^{(2)}$	377.59	361.17	344.45	346.61	361.37
		$E_{11}^{(3)}$	112.06	115.69	126.79	150.33	178.20

Table 2: Optimum values of μ_{12} and percent relative efficiencies of T_{12} with respect to $\bar{y}_n$, $\hat{\hat{Y}}$ and Δ.

$\rho_{xz} \downarrow$	$\rho_{yz} \downarrow$	$\rho_{yx} \rightarrow$	0.6	0.7	0.8	0.9
0.5	0.5	$\mu_{12}^{(0)}$	0.8423	0.7667	0.7405	0.7612
		$E_{12}^{(1)}$	135.26	140.72	150.60	169.19
		$E_{12}^{(2)}$	120.23	118.37	117.13	116.17
		$E_{12}^{(3)}$	143.79	148.54	157.04	173.32
	0.7	$\mu_{12}^{(0)}$	0.8669	0.7215	0.6852	0.7130
		$E_{12}^{(1)}$	196.97	204.83	220.35	205.31
		$E_{12}^{(2)}$	175.08	172.30	171.38	171.87
		$E_{12}^{(3)}$	130.21	133.95	141.42	156.22
	0.9	$\mu_{12}^{(0)}$	0.3932	0.3873	*	0.6116
		$E_{12}^{(1)}$	446.13	469.20	-	598.68
		$E_{12}^{(2)}$	396.56	394.69	-	411.05
		$E_{12}^{(3)}$	113.18	116.35	-	133.31
0.7	0.5	$\mu_{12}^{(0)}$	0.9708	0.8072	0.7461	0.7521
		$E_{12}^{(1)}$	133.37	136.87	145.86	164.21
		$E_{12}^{(2)}$	118.55	115.14	113.45	112.75
		$E_{12}^{(3)}$	149.42	153.65	163.41	182.84
	0.7	$\mu_{12}^{(0)}$	*	0.8490	0.6969	0.6988
		$E_{12}^{(1)}$	-	197.30	210.55	240.09
		$E_{12}^{(2)}$	-	165.97	163.76	164.84
		$E_{12}^{(3)}$	-	141.26	150.30	169.89
	0.9	$\mu_{12}^{(0)}$	0.3785	0.3893	0.3841	0.6122
		$E_{12}^{(1)}$	411.08	432.53	472.25	558.44
		$E_{12}^{(2)}$	365.40	363.84	367.31	383.42
		$E_{12}^{(3)}$	120.76	127.71	138.73	161.49
0.9	0.5	$\mu_{12}^{(0)}$	*	0.8818	0.7598	0.7457
		$E_{12}^{(1)}$	-	134.24	141.92	159.91
		$E_{12}^{(2)}$	-	112.92	110.38	109.81
		$E_{12}^{(3)}$	-	158.88	168.85	190.59
	0.9	$\mu_{12}^{(0)}$	0.3607	0.3744	0.3926	0.7672
		$E_{12}^{(1)}$	384.15	404.32	442.80	529.28
		$E_{12}^{(2)}$	341.46	340.11	344.40	363.40
		$E_{12}^{(3)}$	125.69	134.87	149.37	179.20

Note: "*" indicates that the admissible values of $\mu_{12}^{(0)}$ do not exist.

Table 3: Optimum values of μ_{21} and percent relative efficiencies of T_{21} with respect to $\bar{y}_n$, $\hat{\hat{Y}}$ and Δ.

$\rho_{xz}\downarrow$	$\rho_{yz}\downarrow$	$\rho_{yx}\rightarrow$	0.3	0.5	0.7	0.9
0.5	0.7	$\mu_{21}^{(0)}$	0.8061	0.8647	*	0.3814
		$E_{21}^{(1)}$	165.69	166.52	-	205.33
		$E_{21}^{(2)}$	161.45	154.13	-	140.98
		$E_{21}^{(3)}$	108.35	110.47	-	128.15
	0.9	$\mu_{21}^{(0)}$	0.4520	0.4431	0.4731	*
		$E_{21}^{(1)}$	419.93	412.96	435.23	-
		$E_{21}^{(2)}$	409.18	382.22	366.11	-
		$E_{21}^{(3)}$	104.13	105.53	107.92	-
0.7	0.5	$\mu_{21}^{(0)}$	*	*	0.0463	0.5256
		$E_{21}^{(1)}$	-	-	133.39	154.03
		$E_{21}^{(2)}$	-	-	112.21	105.76
		$E_{21}^{(3)}$	-	-	149.74	171.50
	0.7	$\mu_{21}^{(0)}$	0.7163	0.7623	*	0.4788
		$E_{21}^{(1)}$	163.19	164.70	-	216.93
		$E_{21}^{(2)}$	159.02	152.44	-	148.95
		$E_{21}^{(3)}$	111.24	116.54	-	153.51
	0.9	$\mu_{21}^{(0)}$	0.4232	0.4140	0.4448	*
		$E_{21}^{(1)}$	396.58	388.65	414.32	-
		$E_{21}^{(2)}$	386.43	359.72	348.52	-
		$E_{21}^{(3)}$	106.98	112.39	122.33	-
0.9	0.5	$\mu_{21}^{(0)}$	*	*	0.2006	0.6850
		$E_{21}^{(1)}$	-	-	134.74	180.20
		$E_{21}^{(2)}$	-	-	113.34	123.72
		$E_{21}^{(3)}$	-	-	159.47	214.77
	0.7	$\mu_{21}^{(0)}$	0.6614	0.7003	*	0.5566
		$E_{21}^{(1)}$	160.49	162.53	-	232.94
		$E_{21}^{(2)}$	156.38	150.43	-	159.94
		$E_{21}^{(3)}$	113.56	121.53	-	181.10
	0.9	$\mu_{21}^{(0)}$	0.4010	0.3916	0.4232	*
		$E_{21}^{(1)}$	377.23	368.71	396.58	-
		$E_{21}^{(2)}$	367.58	341.26	333.60	-
		$E_{21}^{(3)}$	109.09	117.23	132.29	-

Note: "*" indicates $\mu_{21}^{(0)}$ do not exist.

Table 4: Optimum values of μ_{22} and percent relative efficiencies of T_{22} with respect to $\bar{y}_n$, $\hat{\hat{\bar{Y}}}$ and Δ.

$\rho_{xz}\downarrow$	$\rho_{yz}\downarrow$	$\rho_{yx}\rightarrow$	0.3	0.5	0.7	0.9
0.5	0.5	$\mu_{22}^{(0)}$	0.5118	0.5359	0.5834	0.6964
		$E_{22}^{(1)}$	102.62	108.04	118.87	145.64
		$E_{22}^{(2)}$	100.00	100.00	100.00	100.00
		$E_{22}^{(3)}$	108.32	115.12	125.48	149.20
	0.7	$\mu_{22}^{(0)}$	0.4939	0.5109	0.5536	0.6667
		$E_{22}^{(1)}$	164.41	170.70	186.77	230.77
		$E_{22}^{(2)}$	160.20	157.99	157.11	158.45
		$E_{22}^{(3)}$	107.51	113.24	122.14	144.02
	0.9	$\mu_{22}^{(0)}$	0.4305	0.4305	0.4626	0.5742
		$E_{22}^{(1)}$	423.95	423.95	458.74	583.80
		$E_{22}^{(2)}$	413.10	392.39	385.89	400.84
		$E_{22}^{(3)}$	105.13	108.34	113.75	130.00
0.7	0.5	$\mu_{22}^{(0)}$	0.5008	0.5203	0.5647	0.6778
		$E_{22}^{(1)}$	100.16	104.53	114.60	141.13
		$E_{22}^{(2)}$	**	**	**	**
		$E_{22}^{(3)}$	108.69	116.38	128.65	157.14
	0.7	$\mu_{22}^{(0)}$	0.4783	0.4900	0.5294	0.6424
		$E_{22}^{(1)}$	158.65	162.97	177.64	221.11
		$E_{22}^{(2)}$	154.59	150.84	149.43	151.83
		$E_{22}^{(3)}$	108.15	115.32	127.17	156.46
	0.9	$\mu_{22}^{(0)}$	0.4044	0.4000	0.4291	0.5393
		$E_{22}^{(1)}$	395.93	391.30	422.45	544.00
		$E_{22}^{(2)}$	385.80	362.18	355.36	373.52
		$E_{22}^{(3)}$	106.80	113.16	124.73	157.32
0.9	0.5	$\mu_{22}^{(0)}$	0.4906	0.5064	0.5484	0.6614
		$E_{22}^{(1)}$	**	101.43	110.88	137.21
		$E_{22}^{(2)}$	**	**	**	**
		$E_{22}^{(3)}$	109.02	117.45	131.23	163.53
	0.7	$\mu_{22}^{(0)}$	0.4644	0.4721	0.5091	0.6218
		$E_{22}^{(1)}$	153.59	156.41	170.03	213.03
		$E_{22}^{(2)}$	149.65	144.77	143.03	146.26
		$E_{22}^{(3)}$	108.68	116.96	130.98	165.66
	0.9	$\mu_{22}^{(0)}$	0.3832	0.3762	0.4032	0.5118
		$E_{22}^{(1)}$	373.54	366.10	394.76	513.13
		$E_{22}^{(2)}$	363.98	338.85	332.07	352.32
		$E_{22}^{(3)}$	108.02	116.40	131.68	173.73

Note: "**" indicate no gain.