Journal of Statistical Research *ISSN 0256 - 422 X*
2011, Vol. 45, No. 1, pp. 71-76
Bangladesh

A LOWER CONFIDENCE LIMIT FOR RELIABILITY OF A COHERENT SYSTEM WITH INDEPENDENT COMPONENTS VIA THE CHA ALGORITHM

G. CHAUDHURI
Indian Institute of Planning & Management Kolkata, India
Email: firstauthor@domain.net

SUMMARY

In this paper, we solve the problem of finding interval estimate for system reliability via the CHA algorithm (Chaudhuri et al., 2001) following the Easterling (1972) approach. We consider a coherent system composed of independent components. No distributional assumption is made for the component life times. A closed form expression for the standard error of the system reliability, for a given mission of duration, is obtained. The method of calculating the $100(1-\alpha)$ % lower confidence limit for the system reliability is illustrated for a simple low-pressure coolant injection system (LPCI) with two pumps (Blischke and Murthy, 2000). Both the CHA algorithm and the usual variance method are used for calculations. Some simulation results are also reported. This paper basically extends the results of Easterling to any coherent system.

Keywords and phrases: Structure function, Coherent Structure, the CHA algorithm, Confidence Interval, System Reliability, LPCI system

1 Introduction

In this paper, we consider a coherent structure composed of independent components. The problem is to find a lower confidence limit for the system reliability. We confine our attention to the lower confidence limit of system reliability, since it is of most interest to the reliability practitioners in the context of interval estimation of system reliability.

We consider the problem of predicting system reliability from the knowledge of component reliabilities. This situation arises when one has to estimate the reliability of a large complex system. Component reliabilities are not known in practice, particularly, in early stage of system design. Therefore, assumption of a particular parametric distribution for component life does not make sense. Thus, no functional form of distribution is assumed for the component life times. The component reliabilities can be estimated from (accelerated) life tests data.

An excellent account on the topic of interval estimation of system reliability is available in Crowder et al. (1991). The first step in obtaining the lower confidence limit of a coherent system is to get a point estimate of the system reliability R(t). The basis of estimation of system reliability is the following model (under the assumption of independence of components, see Chaudhuri et al. 2001):

$$R(t) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n}, r_i(t)^{D(i,j)} \tag{1.1}$$

which connects the system reliability R(t) with the component reliabilities $r_i(t)$, $i = 1, \ldots, n$, for a mission of duration t. With n being the number of components of the system under consideration, let m denote number of minimal path sets for the system which is assumed to be coherent. If $\hat{r}_i(t)$ is an estimate of ith component reliability, then an obvious estimate of R(t) is given by:

$$\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n} .\hat{r}_i(t)^{D(i,j)} \tag{1.2}$$

In principle, one can obtain Var($\hat{R}$) from (1.2), and thus, the lower $100(1-\alpha)\%$ confidence limit is calculated as

$$\hat{R}(t) - z_\alpha\sqrt{\text{Var}(\hat{R})}, \tag{1.3}$$

where z_α is the α-fractile of the standard normal distribution. Unfortunately, this does not work well, because this is basically an asymptotic normal approximation used to construct the lower confidence limit. Dissatisfaction with this approximation led Easterling (1972) to consider the use of a binomial distribution with parameters $\hat{n}$ and $\hat{R}$ where $\hat{n} = \hat{R}(1-\hat{R})/\text{Var}(\hat{R})$. From this consideration one can now easily obtain a lower confidence limit for R(t). See Section 3 for details. In this paper we address the problem of finding the lower confidence interval for the reliability of a complex system in the light of the CHA algorithm presented in Chaudhuri et al. (2001). We consider the coherent system composed of independent components. No distributional assumption is made for the component life times. A closed form expression for the standard error of the system reliability, for a given mission of duration, is obtained in Section 2. The method of calculating the $100(1-\alpha)\%$ lower confidence limit for the system reliability is given in Section 3 and is illustrated for a simple low-pressure coolant injection system (LPCI) with two pumps (Blischke and Murthy, 2000) in Section 4. Both the CHA algorithm and the usual variance method are used for calculations. Some simulation results are also reported in Section 4. This paper basically extends the results of Easterling to any coherent system without distributional assumption for component life-lengths.Section 5 contains some discussions.The problem of finding a lower confidence limit of a coherent system with independent exponential components is treated in Chaudhuri (2004).

2 Variance of $\hat{R}$

We have from equation (1.2),

$$E(\hat{R}(t)^2) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n} E(\hat{r}_i(t)^{2D(i,j)}) + \sum_{j\neq k}^{2^m-1} 1(j).1(k)\prod_{i=1}^{n} E(\hat{r}_i(t)^{D(i,j)+D(i,k)}) \tag{2.1}$$

and

$$E(\hat{R}(t)) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n} E(\hat{r}_i(t)^{D(i,j)}). \tag{2.2}$$

Since

$$\text{Var}(\hat{R}(t) = E(\hat{R}(t)^2) - (E\hat{R}(t))^2, \tag{2.3}$$

Var($\hat{R}$) can be computed from (2.1) and (2.2). The computations involved in (2.3) are straightforward in the light of the CHA algorithm described in Chaudhuri et al. (2001).

Suppose we have a binomial test data where n_i number of items of component i tested for t hours. Let f_i denote the number of failures of component i during the test. If we record the status (failed/ not failed) of an item during test, then we have a sequence of n_i Bernoulli trials with success probability $p_i = r_i(t)$,$i = 1, \ldots, n$. Thus,

$$\hat{r}_i(t) = 1 - \hat{q}_i(t), \quad \hat{q}_i(t) = \frac{f_i}{n_i}, \quad \text{Var}(\hat{r_i}(t) = \frac{\hat{r}_i(t)(1-\hat{r}_i(t))}{n_i} \tag{2.4}$$

3 A lower confidence limit

Consider a hypothetical single binomial experiment (see Crowder et al. 1991) where Y denotes the number of system survivals for mission time t such that Y follows a binomial distribution with parameters $\hat{n}$ and $\hat{R}$ where

$$\hat{n} = \frac{\hat{R}(1-\hat{R})}{\text{Var}(\hat{R})}. \tag{3.1}$$

Thus, we can determine a $100(1-\alpha)\%$ lower confidence interval limit for R that satisfies

$$\sum_{i=0}^{x} \binom{\hat{n}}{i} R^i (1-R)^{\hat{n}-i} = 1-\alpha, \tag{3.2}$$

where $x = \hat{n}\hat{R}$. Using a relationship between the binomial CDF and the incomplete beta function one can solve the equation (3.2). Hence, the required lower confidence limit is obtained from:

$$R_L = B\left(1-\alpha; x, \hat{n} - x + 1\right), \tag{3.3}$$

where $B\left(\gamma; a, b\right)$ is the γ -fractile of the beta distribution with parameters a and b not being necessarily integers.

4 Illustrative examples

Example 1: Consider a low-pressure coolant injection system (LPCI) described by Martz and Waller (1990). This type of system acts as a coolant to the nuclear reactor when an accident breaks out resulting in low pressure in the reactor vessel. In the study by Martz et al. (1990), it was used in 1150 -megawatt U.S. nuclear-powered boiling-water electric power plants.

The system is composed of pumps, check valves and motor-operated valves (see Blischke and Murthy, 2000). For the functioning of the LPCI a minimal number of components must work. For the sake of illustration, let the system consist of two pumps A and C such that the PUMP C is a backup to PUMP A. With this configuration, the LPCI becomes a parallel system so that the system fails if and only if both pumps A and C fail.

A 240 success-fail type tests for each component provides the following data (see Blischke and Murthy, 2000):

Table 1: LPCL system component data

Component	Number of tests	Number of non-failures
1 (PUMP A)	240	236
2 (PUMP C)	240	238

Thus, we have

$$\hat{r}_1 = 0.983333 \quad \text{Var}(\hat{r}_1) = 0.00006842 \tag{4.1}$$

$$\hat{r}_2 = 0.9917 \quad \text{Var}(\hat{r}_2) = 0.000034433 \tag{4.2}$$

Since

$$\hat{R} = 1 - (1 - \hat{r}_1)(1 - \hat{r}_2). \tag{4.3}$$

Blischke and Murthy (2000) compute $\text{Var}(\hat{R}) = 1.6634 \times 10^{-08}$ using the relation:

$$\text{Var}(\hat{R}) = (E\hat{r}_1)^2 \text{Var}(\hat{r}_2) + (E\hat{r}_2)^2 \text{Var}(\hat{r}_1) + \text{Var}(\hat{r}_1)\text{Var}(\hat{r}_2). \tag{4.4}$$

Now we shall employ the CHA algorithm to find $\text{Var}(\hat{R})$:

For a parallel system:

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad , 1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad , m = n = 2 \tag{4.5}$$

Thus, the equation (2.1) yields:

$$E(\hat{R}^2) = E(\hat{r}_1^2) + E(\hat{r}_2^2) + E(\hat{r}_1^2)E(\hat{r}_2^2) + 2\left[E(\hat{r}_1)E(\hat{r}_2) - E(\hat{r}_1^2)E(\hat{r}_2) - E(\hat{r}_1)E(\hat{r}_2^2)\right] \tag{4.6}$$

and

$$E(\hat{R}) = E(\hat{r}_1) + E(\hat{r}_2) - E(\hat{r}_1)E(\hat{r}_2). \tag{4.7}$$

Hence, using the CHA algorithm, we obtain, Var$(\hat{R}) = 1.6634 \times 10^{-08}$, which, as expected, is in agreement with Blischke and Murthy (2000)'s method. Thus, the CHA algorithm gives the same result.

Note that the 95% lower confidence limit R_L is obtained by using the Excel function

$$BETAINV(0.05; x, \hat{n} - x + 1),$$

where x and $\hat{n} - x + 1$ are as in Section 3. The R_L can be shown to be 0.9994 which is (as well as one obtained by Easterling approach)less than the calculated lower bound (=0.9997) using the equation (1.3).

Example 2 (Simulation): Consider a parallel system composed of two independent exponential components with reliability function $r_i(t) = e^{-}\lambda_i t, i = 1, 2$ with $\lambda_1 = 1$ and $\lambda_2 = 2$. We generate respectively 3000 and 5000 Bernoulli sequences for component 1 and 2 for various values of time parameter shown in the Table 2.

Table 2: Simulation results

t	Proposed R_L	R_L(equation(1.3))
0.3	0.8762	0.8765
0.4	0.8096	0.8098
0.5	0.7407	0.7410
0.6	0.6744	0.6745
0.7	0.6081	0.6083
0.8	0.5474	0.5475
0.9	0.4913	0.4914

We can draw the same conclusion (as in Example 1) in this simulation study also.

5 Discussion

The CHA algorithm is simple and easy-to-use. As expected, our approach here does fairly well in this simple study of parallel systems with two components. These examples are considered for illustrative purpose only. The CHA algorithm can handle any coherent structure of any complexity. The advantages of the Easterling approach (1972) over the Likelihood Ratio approach (Madansky, 1965) was established by Easterling in 1972 paper where he demonstrated that his approach appears to be as accurate as the LR method and is considerably easier to implement to obtain the confidence limits.

Acknowledgement

I wish to thank the referee for some valuable comments and suggestions that substantially improved the presentation of the paper.

References

[1] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Winston, Rinehart, USA.

[2] Blischke, W.R. and Murthy, D.N.P. (2000). *Reliability: Modeling, Prediction and Optimization*, John Wiley and Sons, Inc., New York

[3] Chaudhuri, G. (2004). A Lower Confidence Limit for Reliability of a Coherent System with Independent Exponential Components, *Calcutta Statistical Association Bulletin*, **55**, 119–127.

[4] Chaudhuri, G. , Hu, K., and Afshar, N. (2001). A new approach to system reliability, *IEEE Transaction on Reliability,* **50**(1), 75–84.

[5] Crowder, M.J., Kimber, A.C., Smith, R.L. and Sweeting, T.J. (1991). *Statistical analysis of reliability data*, Chapman and Hall, New York.

[6] Easterling, R.G. (1972). Approximate confidence intervals for system reliability, *J. American Statistical Association*, **67**, 220–222.

[7] Martz, H.F. and Waller, R.A. (1990). Bayesian reliability analysis of complex series/parallel systems of binomial subsystems and components, *Technometrics*, **32**, 407–416.