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OPTIMAL STEIN-RULE ESTIMATOR UNDER QUARTIC LOSS FUNCTION

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SUMMARY

In this approach, we construct the necessary and sufficient conditions in order a James-Stein type estimator outperforms the minimax estimator of the mean. Particularly we consider a class of scale mixture of multivariate normal distributions and derive the dominating conditions under the quartic loss function. Some leading examples are also exhibited for checking the efficiency of the proposed model and specifying theoretical computations.

Keywords and phrases: James-Stein estimator, Minimax estimator, Quartic loss function, Scale mixture of multivariate normal distributions

1 Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_p)$ be a random vector in $\mathbb{R}^p$ distributed as a variance mixture of multivariate normal distributions with un-known mean vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$. Thus we assume that the density function of $\boldsymbol{X}$ is of the form

$$f(\boldsymbol{X}) = \int_0^\infty \frac{1}{(2\pi t^{-q})^{\frac{p}{2}}} \ \exp\Big(-\frac{t^q}{2}\ \|\boldsymbol{x}-\boldsymbol{\theta}\|^2\Big)\, dG(t), \tag{1.1}$$

where G is the distribution of a known non-negative random variable V. In other words, we consider the conditional structure in which $\boldsymbol{X}|V = t \sim N_p(\boldsymbol{\theta}, t^{-q}\boldsymbol{I}_p)$, $q > 0$ and $V \sim G(.)$. See Gupta and Varga (1993, Theorem 2.7 3, pp. 78-79) for more details.

It is well-known, that $\boldsymbol{X}$ is an inadmissible estimator of $\boldsymbol{\theta}$ under quadratic loss function for $p \geq 3$; we refer to the couple of works due to Strawderman (1974), Berger (1975), Brandwein (1978) and Bock (1985). They considerably proposed improved estimators $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_p)$ under quadratic loss. However, Berger (1978) considered loss functions L which are polynomials in the coordinates of $\boldsymbol{\delta}-\boldsymbol{\theta}$. He gave a developed example when $L(\boldsymbol{\theta},\boldsymbol{\delta}) = \|\boldsymbol{\delta}-\boldsymbol{\theta}\|^4 = (\sum_{i=1}^{p}(\delta_i-\theta_i)^2)^2$ (that is the square of the usual quadratic loss).

Let $\boldsymbol{X} = (X_1,\ldots,X_p)$ distributed according to the model (1.1). In this paper following Fourdrinier et al. (2008) we basically engage with the problem of estimating $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_p)$ under the quartic loss function

$$L\big(\boldsymbol{\theta},\boldsymbol{\delta}(\boldsymbol{X})\big) = \sum_{i=1}^{p}\big(\delta_i(\boldsymbol{X})-\theta_i\big)^4, \tag{1.2}$$

where $\boldsymbol{\delta}(\boldsymbol{X}) = (\delta_1(\boldsymbol{X}),\ldots,\delta_p(\boldsymbol{X}))$ estimates $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_p)$, and investigate the conditions for which an estimator $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X}+\boldsymbol{g}(\boldsymbol{X})$ dominates $\boldsymbol{X}$ for $p \geq 3$. We give an extension to the earlier work for the class of scale mixture of multivariate normal distributions. More important we show the robustness of the superiority conditions for the classes under study.

As it is noted in the earlier work, the quartic loss is neither quadratic nor spherically symmetric. Hence our results represent an interesting example of the stein effect whereby reasonably explicit dominating estimators can be obtained in a setting that is somewhat unusual. To be honest we must say that all fundamental computations are followed by the utilities given in Fourdrinier et al. (2008).

We organize our paper as follows: In section 2, we survey on minimaxity conditions under quartic loss function. Section 3, is devoted to specifying the Stein-type class of shrinkage estimators as well as main results, while some practical models are included in section 4 for discussing on the superiority conditions.

2 Minimax estimators under quartic loss

The measurement associated with the quartic loss given by (1.2) is $R(\boldsymbol{\theta},\boldsymbol{\delta}) = E_\theta[L(\boldsymbol{\theta},\boldsymbol{\delta}(\boldsymbol{X}))]$, Where E_θ denotes the expectation with respect to the sampling distribution (1.1).

It is easy to show that for the minimax estimator $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$, $R(\boldsymbol{\theta},\boldsymbol{\delta}^0(\boldsymbol{X})) = 3p$, since the integral of G is equal to 1. Following Stein (1981) we take the class of shrinkage estimators $\boldsymbol{\delta}(\boldsymbol{X})$ of $\boldsymbol{\theta}$ of the form $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X}+\boldsymbol{g}(\boldsymbol{X})$ into consideration where $\boldsymbol{g}$ is a function from $\mathbb{R}^p$ into $\mathbb{R}^p$.

Note that from Stein lemma for the weakly differentiable function $\boldsymbol{g}$ we can immediately conclude that

$$\begin{aligned} E_{\boldsymbol{\theta}}\Big[(X_i-\theta_i)g_i(\boldsymbol{X})\Big] &= \int_0^\infty E_{\boldsymbol{\theta}}\Big[(X_i-\theta_i)g_i(\boldsymbol{X})\Big|V=t\Big]dG(t) \\ &= \int_0^\infty t^{-q}\, E_{\boldsymbol{\theta}}\Big[\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})\Big|V=t\Big]dG(t) \\ &= E_{\boldsymbol{\theta}}\kappa_q^{(1,1,1)}, \end{aligned} \tag{2.1}$$

where

$$\kappa_q^{(l,j,k)} = \int_0^\infty \Big(\frac{1}{t^q}\Big)^l \Big[\frac{\partial^j}{\partial X_i^j}g_i^k(\boldsymbol{X})\Big|V=t\Big]dG(t). \tag{2.2}$$

More precisely

$$\kappa_q^{(i)} = \int_0^\infty \left(\frac{1}{t^q}\right)^i dG(t). \tag{2.3}$$

Lemma 2.1. Assume that $\boldsymbol{g}$ is a three times weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$ satisfying $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right] < \infty$, for every $1 \leq i \leq p$. Then, under quartic loss (1.2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is

$$\eth g(\boldsymbol{X}) = \sum_{i=1}^p \left[g_i^4(\boldsymbol{X}) + 6\kappa_q^{(1,0,2)} g_i^2(\boldsymbol{X}) + 12\kappa_q^{(2,1,1)} + 4\kappa_q^{(1,1,3)} + 6\kappa_q^{(2,2,2)} + 4\kappa_q^{(3,3,1)}\right].$$

Proof. First consider that

$$\begin{aligned}
\Delta_\theta &= R(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X})) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) \\
&= E_\theta\left[L(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X})) - L(\boldsymbol{\theta}, \boldsymbol{\delta}^0)\right] \\
&= E_\theta\Big\{\sum_{i=1}^p \left[g_i^4(\boldsymbol{X}) + (4(X_i-\theta_i)g_i^3(\boldsymbol{X}) + 6(X_i-\theta_i)^2 g_i^2(\boldsymbol{X}) + 4(X_i-\theta_i)^3 g_i(\boldsymbol{X}))\right]\Big\}.
\end{aligned}$$

Then by making use of Stein's identity similar to Eq. (2.1) we get

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left[(X_i-\theta_i)g_i^3(\boldsymbol{X})\right] &= \int_0^\infty E_{\boldsymbol{\theta}}\Big[(X_i-\theta_i)g_i^3(\boldsymbol{X})\Big| V=t\Big] dG(t) \\
&= \int_0^\infty t^{-q}\, E_{\boldsymbol{\theta}}\Big[\frac{\partial}{\partial X_i} g_i^3(\boldsymbol{X})\Big| V=t\Big] dG(t) \\
&= E_\theta \kappa_q^{(1,1,3)}
\end{aligned}$$

Applying extended Stein's identity (2.1) repeatedly, we find

$$\begin{aligned}
E_\theta\left[(X_i-\theta_i)^2 g_i^2(\boldsymbol{X})\right] &= \int_0^\infty E_{\boldsymbol{\theta}}\Big[(X_i-\theta_i)^2 g_i^2(\boldsymbol{X})\Big| V=t\Big] dG(t) \\
&= \int_0^\infty t^{-q}\, E_{\boldsymbol{\theta}}\Big[\frac{\partial}{\partial X_i}(X_i-\theta_i) g_i^2(\boldsymbol{X})\Big| V=t\Big] dG(t) \\
&= \int_0^\infty t^{-q}\, E_{\boldsymbol{\theta}}\Big[g_i^2(\boldsymbol{X}) + t^{-q}\frac{\partial^2}{\partial X_i^2} g_i^2(\boldsymbol{X})\Big| V=t\Big] dG(t) \\
&= E_\theta \kappa_q^{(1,0,2)} + E_\theta \kappa_q^{(2,2,2)}.
\end{aligned}$$

In a similar fashion, it yields

$$\begin{aligned}
E_\theta\big[(X_i-\theta_i)^3 g_i(\boldsymbol{X})\big] &= \int_0^\infty E_{\boldsymbol{\theta}}\big[(X_i-\theta_i)^3 g_i(\boldsymbol{X})\Big| V=t\big] dG(t)\\
&= \int_0^\infty t^{-q}\ E_{\boldsymbol{\theta}}\Big[\frac{\partial}{\partial X_i}(X_i-\theta_i)^2 g_i(\boldsymbol{X})\Big| V=t\Big] dG(t)\\
&= \int_0^\infty t^{-q}\ E_{\boldsymbol{\theta}}\Big[2(X_i-\theta_i)g_i(\boldsymbol{X})+t^{-q}(X_i-\theta_i)^2\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})\Big| V=t\Big] dG(t)\\
&= \int_0^\infty t^{-q}\ E_{\boldsymbol{\theta}}\Big[2t^{-q}\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})+t^{-q}\frac{\partial}{\partial X_i}\Big((X_i-\theta_i)\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})\Big)\Big| V=t\Big] dG(t)\\
&= \int_0^\infty t^{-q} E_{\boldsymbol{\theta}}\Big[3t^{-q}\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})+t^{-2q}\frac{\partial^3}{\partial X_i^3}g_i(\boldsymbol{X})\Big| V=t\Big] dG(t)\\
&=3E_\theta\kappa_q^{(2,1,1)}+E_\theta\kappa_q^{(3,3,1)}.
\end{aligned}$$

Gathering all the above relevant terms, the result follows. □

From lemma 2.1 one can find that any estimator $\boldsymbol{\delta}(\boldsymbol{X})=\boldsymbol{X}+\boldsymbol{g}(\boldsymbol{X})$ dominates $\boldsymbol{\delta}^0(\boldsymbol{X})=\boldsymbol{X}$ under the quartic loss (1.2) as soon as $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right]<\infty$, for every $1\le i\le p$, is satisfied and

$$\eth g(\boldsymbol{x})\le 0 \quad \forall \boldsymbol{x}\in\mathbb{R}^p$$

with strict inequality on a set of positive Lebesgue measure.

3 Class of James-Stein Estimators

Two typical classes of Shrinkage Stein-type estimators include well-known classes of James-Stein estimators. We basically involve with two types of shrinkage functions. Item one includes the one in (2.3) resulting on the following James-Stein (JS) estimator

$$\boldsymbol{\delta}_a^{JS}(\boldsymbol{X})=\boldsymbol{X}-\frac{a}{\|\boldsymbol{X}\|^2}\boldsymbol{X}. \tag{3.1}$$

For better understanding of the finiteness condition $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right]<\infty$, we turn our attention to the shrinkage function

$$\boldsymbol{g}(\boldsymbol{X})=\boldsymbol{g}_a(\boldsymbol{X})=-\frac{a}{\|\boldsymbol{X}\|^2}\boldsymbol{X},\ a>0.$$

For the specified class in (3.1), applying Theorem 1 of Bock et al. (1983) we have

$$\begin{aligned}
E_{\boldsymbol{\theta}}\Big(\frac{g_i^4(\boldsymbol{x})}{a^4}\Big) &\le E_{\boldsymbol{\theta}}\Big(\frac{1}{\|\boldsymbol{x}\|^4}\Big)\\
&= E_t E_{\boldsymbol{\theta}}\left(\frac{1}{\|\boldsymbol{x}\|^4}\middle| V=t\right)\\
&= \int_0^\infty t^{2q}E_{\boldsymbol{\theta}}\big[\chi_p^{-2}\left(t\|\boldsymbol{\theta}\|^2\right)\Big| V=t\big]^2 dG(t)<\infty
\end{aligned} \tag{3.2}$$

for $p > 4$ and such distribution in which $\kappa_q^{(-2)} < \infty$.

Item two deals with the shrinkage function $\boldsymbol{g}_{a,b}(\boldsymbol{X}) = -\frac{a}{\|\boldsymbol{X}\|^2+b}\boldsymbol{X}$ (with $a > 0$ and $b > 0$) resulting on the following JS estimator

$$\boldsymbol{\delta}_{a,b}^{JS}(\boldsymbol{X}) = \boldsymbol{X} - \frac{a}{\|\boldsymbol{X}\|^2 + b}\boldsymbol{X}.$$

Now consider the JS class of estimators defined by (3.2). For the finiteness condition $E_{\boldsymbol{\theta}}[g_i^4(\boldsymbol{X})] < \infty$ consider that

$$\begin{aligned} E_{\boldsymbol{\theta}}\left(\frac{g_i^4(\boldsymbol{x})}{a^4}\right) &\leq E_{\boldsymbol{\theta}}\left(\frac{\|\boldsymbol{x}\|^4}{b^4}\right) \\ &= \frac{1}{b^4}\int_0^\infty t^{-2q} E_{\boldsymbol{\theta}}\left[\chi_p^2\left(t\|\boldsymbol{\theta}\|^2\right)\middle| V = t\right]^2 dG(t) \\ &= \frac{1}{b^4}\int_0^\infty t^{-2q}\left[p(p+2) + 2(p+2)t^q\|\boldsymbol{\theta}\|^2 + t^{2q}\|\boldsymbol{\theta}\|^4\right] dG(t) \\ &= \frac{1}{b^4}\left[p(p+2)\kappa_q^{(2)} + 2(p+2)\|\boldsymbol{\theta}\|^2\kappa_q^{(1)} + \|\boldsymbol{\theta}\|^4\right] < \infty, \end{aligned} \tag{3.3}$$

provided that $\kappa_q^{(i)} < \infty$, $i = 1, 2$ and $\|\boldsymbol{\theta}\|^2 < \infty$.

Lemma 3.1. Assume that $p \geq 5$. Under quartic loss (1.2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}_a^{JS}(\boldsymbol{X})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is expressed as

$$\begin{aligned} \eth\boldsymbol{g}_a(\boldsymbol{X}) = a\Bigg[&\left(a^3 + 24\kappa_q^{(1)}a^2 + 144\kappa_q^{(2)}a + 192\kappa_q^{(3)}\right)\frac{\sum_{i=1}^p X_i^4}{\|\boldsymbol{X}\|^8} + 6\kappa_q^{(1)}\left(a - 2t(p-2)\right)\frac{1}{\|\boldsymbol{X}\|^2} \\ &- 12\left(a^2\kappa_q^{(1)} - (p-10)a\kappa_q^{(2)} - 2\kappa_q^{(3)}(p-8)\right)\frac{1}{\|\boldsymbol{X}\|^4}\Bigg]. \end{aligned}$$

Proof. The result follows from Lemma 2.2 of Fourdrinier et al. (2008) and Lemma 2.1. □

Theorem 1. *Under the model* (1.1)*, the JS estimator given by* (3.1) *dominates* $\boldsymbol{\delta}^0(\boldsymbol{X})$ *under quartic loss* (1.2)*, for all* $\boldsymbol{\theta} \in \mathbb{R}^p$*, provided* $p \geq 7$*, and*

$$0 < a < min\left\{2\kappa_q^{(1)}(p-2), sup\{s \geq 0/h(s) = 0\}\right\},$$

where

$$h(s) = s^3 + 12\kappa_q^{(1)}s^2 + 18sp\kappa_q^{(2)} - 12\kappa_q^{(3)}(p - 4 - 2\sqrt{2})(p - 4 + 2\sqrt{2}).$$

Proof. Under conditioning, the risk difference Δ_θ is bounded above by

$$\begin{aligned} & aE_t\Bigg\{E_\theta\Bigg[(a^3 + 24t^{-q}a^2 + 144at^{-2q} + 192t^{-3q})\frac{1}{\|\boldsymbol{X}\|^4} + 6t^{-q}(a - 2t^{-q}(p-2))\frac{1}{\|\boldsymbol{X}\|^2} \\ & -12(a^2t^{-q} - (p-10)at^{-2q} - 2t^{-3q}(p-8))\frac{1}{\|\boldsymbol{x}\|^4}\Bigg]\Bigg\} \\ = \;& aE_t\Bigg\{E_\theta\Bigg[(a^3 + 12t^{-q}\,a^2 + 12a\,t^{-2q}(p+2) + 24pt^{-3q})\frac{1}{\|\boldsymbol{X}\|^4} \\ & +6\,t^{-q}(a - 2t^{-q}(p-2))\frac{1}{\|\boldsymbol{X}\|^2}\Bigg]\Bigg\}. \end{aligned} \tag{3.4}$$

Thus applying Lemma A.3 of Fourdrinier et al. (2008), by the assumption $a < 2\kappa_q^{(1)}(p-2)$, the bound in (3.4) reduces to

$$a\left[(a^3+12\kappa_q^{(1)}a^2+12a\kappa_q^{(2)}(p+2)+24p\kappa_q^{(3)}+6\kappa_q^{(2)}(a-2\kappa_q^{(1)}(p-2))(p-4))\right]E_\theta\left[\frac{1}{\|\boldsymbol{X}\|^4}\right]$$
$$=a\left[(a^3+12\kappa_q^{(1)}a^2+18ap\kappa_q^{(2)}-12\kappa_q^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2}))\right]E_\theta\left[\frac{1}{\|\boldsymbol{X}\|^4}\right].$$

setting

$$h(a)=a^3+12\kappa_q^{(1)}a^2+18ap\kappa_q^{(2)}-12\kappa_q^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2})$$

it is clear that $h(0)=-12\kappa_q^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2})<0$ since $p\geq 7$ and $\kappa_q^{(1)}>0$. Furthermore this cubic polynomial is increasing in a and hence negative on the interval $[0,a_0]$ where a_0 is its smallest root $a_0>0$. Finally, for $0<a<min\{2\kappa_q^{(1)}(p-2),a_0\}$, we have $\Delta_\theta<0$, which is the desired domination result. □

Theorem 2. *Under the model* (1.1)*, the JS type estimator in* (3.1) *dominates* $\boldsymbol{\delta}^0(\boldsymbol{X})$ *under quartic loss* (1.2)*, for all* $\boldsymbol{\theta}\in\mathbb{R}^p$*, provided* $p\geq 5$ *and* $0<a\leq 2\kappa_q^{(1)}(p-4)$.

Proof. Applying Lemma A.2 of Fourdrinier et al. (2008) and continuing in the same way as Theorem 3.1, the result follows. □

Theorem 3. *Under the model* (1.1)*, the risk* $R(0,\boldsymbol{\delta}_a^{JS})$ *of the JS estimator at* $\boldsymbol{\theta}=\mathbf{0}$ *under quartic loss* (1.2) *is*

$$\frac{3}{(p+2)}\left(\kappa_q^{(2)}p\,(p+2)-4ap\,\kappa_q^{(1)}+6a^2-4\frac{a^3}{\kappa_q^{(1)}(p-2)}+\frac{a^4}{\kappa_q^{(2)}(p-2)(p-4)}\right)$$

and it is finite provided $p-4>0$.

Proof. Similar to the computations in the proof of Proposition 2.2 of Fourdrinier et al. (2008), the risk of JS estimators at **0** under quartic loss (1.2) is given by

$$R(\mathbf{0},\boldsymbol{\delta}_a^{JS})=E_t\left\{E_\theta\left[\sum_{i=1}^p\left(1-\frac{a}{\|\boldsymbol{X}\|^2}\right)^4X_i^4\right]\right\}$$
$$=pE_t\left\{E_\theta\left[\frac{X_i^4}{\|\boldsymbol{X}\|^4}\right]E_\theta\left[\|\boldsymbol{X}\|^4\left(1-\frac{a}{\|\boldsymbol{X}\|^2}\right)^4\right]\right\},$$

since $Y_i|t=\left.\frac{X_i^2}{\|\boldsymbol{X}\|^2}\right|t$ is independent of $\|\boldsymbol{X}\|^2|t$ for $i=1,\ldots,p$. As the distribution of $Y_i|t$ is

$Beta\,(1/2,(p-1)/2)$ and the distribution of $\|\boldsymbol{X}\|^2|t$ is $t^{-1}\chi^2_p$,

$$\begin{aligned} E_\theta\left[\frac{X_i^4}{\|\boldsymbol{X}\|^4}\right] &= E_t E_\theta[Y_i^2] \\ &= E_t\left[\frac{B(\frac{1}{2}+2,(p-1)/2)}{B(\frac{1}{2},(p-1)/2)}\right] \\ &= \frac{3}{p\,(p+2)}, \end{aligned}$$

since $\int_0^\infty dG(t) = 1$ and

$$\begin{aligned} E_\theta\left[\|\boldsymbol{X}\|^4\left(1-\frac{a}{\|\boldsymbol{X}\|^2}\right)^4\right] &= E_t E_\theta\left[\|\boldsymbol{X}\|^4 - 4a\|\boldsymbol{X}\|^2 + 6a^2 - 4\frac{a^3}{\|\boldsymbol{X}\|^2} + \frac{a^4}{\|\boldsymbol{X}\|^4}\right] \\ &= \int_0^\infty \Bigg[t^{-2q}\;p\;(p+2) - 4\;a\;p\;t^{-q} + 6a^2 - 4\frac{a^3}{t^{-q}(p-2)} \\ &\quad + \frac{a^4}{t^{-2q}(p-2)(p-4)}\Bigg] dG(t). \end{aligned}$$

Simplifying the above result, completes the proof. □

Similar to the Lemma 3.1, we have the following parallel result to the Lemma 3.1 of Fourdrinier et al. (2008) under the model (1.1).

Lemma 3.2. Assume that $p \geq 3$. Under quartic loss (1.2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}^{JS}_{a,b}(\boldsymbol{X})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is expressed as

$$\begin{aligned} \eth g_{a,b}(\boldsymbol{X}) = a\Bigg[&\left(a^3 + 24\kappa_q^{(1)}a^2 + 144\kappa_q^{(2)}a + 192t^3\right)\frac{\sum_{i=1}^p X_i^4}{(\|\boldsymbol{X}\|^2+b)^4} - 12p\kappa_q^{(2)}\frac{1}{\|\boldsymbol{X}\|^2+b} \\ &+ 6\kappa_q^{(1)}(a+4\kappa_q^{(1)})\frac{\|\boldsymbol{X}\|^2}{(\|\boldsymbol{X}\|^2+b)^2} - 12\kappa_q^{(1)}(a+2\kappa_q^{(1)})(a+8\kappa_q^{(1)})\frac{\|\boldsymbol{X}\|^2}{(\|\boldsymbol{X}\|^2+b)^3} \\ &+ 12p\kappa_q^{(2)}(a+2\kappa_q^{(1)})\frac{1}{(\|\boldsymbol{X}\|^2+b)^2}\Bigg]. \end{aligned}$$

Theorem 4. *Under the model* (1.1)*, the JS estimator* $\boldsymbol{\delta}^{JS}_{a,b}(\boldsymbol{X})$ *given by* (3.2) *dominates* $\boldsymbol{\delta}^0(\boldsymbol{X})$ *under quartic loss* (1.2)*, for all* $\boldsymbol{\theta} \in \mathbb{R}^p$*, provided* $p \geq 3$ *and*

$$0 < a \leq 2\kappa_q^{(1)}(p-2) \quad \textit{and} \quad b \geq max\left\{\frac{a^3 + 24\kappa_q^{(1)}a^2 + 12\kappa_q^{(2)}a(p+12) + 24p\kappa_q^{(3)}}{12\kappa_q^{(1)}(3p\kappa_q^{(1)} - a - 4\kappa_q^{(1)})}, a + 2\kappa_q^{(1)}\right\}.$$

The proof is similar to that of given in Theorem 3.1 with some utilities provided in the proof of Theorem 3.1 of Fourdrinier et al. (2008).

4 Examples

In this section we provide some examples of scale mixture of normal distributions to determine the superiority conditions proposed in the previous section precisely. Examples consist of multivariate

Student's t (MT), multivariate Slash (MS) and multivariate exponential power (MEP) distributions. For this purpose, we need to compute the expression $\kappa_q^{(i)}$ given by (2.2) for each distribution.

4.1 MT distribution

Suppose that $\boldsymbol{X}$ is distributed according to a MT distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\boldsymbol{I}_p$ and $\nu > 0$ degrees of freedom, denoted by $\boldsymbol{X} \sim t_p(\boldsymbol{\theta}, \boldsymbol{I}_p, \nu)$, with the following pdf

$$f(\boldsymbol{x}) = \frac{\Gamma\left(\frac{p+\nu}{2}\right)}{(\pi\nu)^{\frac{p}{2}}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{\boldsymbol{x}'\boldsymbol{x}}{\nu}\right)^{-\frac{p+\nu}{2}}.$$

The distribution is the mixture of multivariate normal distributions with the inverse gamma distribution as the weight function given by

$$G(t) = \frac{\nu^{\frac{\nu}{2}} t^{\frac{\nu}{2}-1} e^{-\frac{\nu t}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}. \tag{4.1}$$

Note that here, $q = 1$. Then using equations (2.2) and (4.1) we obtain

$$\kappa_1^{(i)} = \int_0^\infty \frac{\nu^{\frac{\nu}{2}} t^{\frac{\nu}{2}-i-1} e^{-\frac{\nu t}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}\, dt = \left(\frac{\nu}{2}\right)^i \frac{\Gamma\left(\frac{\nu}{2} - i\right)}{\Gamma\left(\frac{\nu}{2}\right)}.$$

Consequently since

$$\kappa_1^{(1)} = \frac{\nu}{\nu - 2} < \infty, \quad \kappa_1^{(2)} = \frac{\nu^2}{(\nu-2)(\nu-4)} < \infty, \quad \kappa_1^{(-2)} = \frac{(\nu-2)(\nu-4)}{\nu^2} < \infty,$$

the finiteness conditions in (3.2) and (3.3) are satisfied.

4.2 MS distribution

Suppose that $\boldsymbol{X}$ is distributed according to a MS distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\boldsymbol{I}_p$ and shape $q > 0$, denoted by $\boldsymbol{X} \sim S_p(\boldsymbol{\theta}, \boldsymbol{I}_p, q)$, with the following pdf

$$\begin{aligned} f(\boldsymbol{x}) &= q \int_0^1 u^{q+p-1} \phi_p(u\boldsymbol{x}; u\boldsymbol{\theta}, \boldsymbol{I}_p) du \\ &= \begin{cases} \frac{q 2^{\frac{p+q}{2}-1} \gamma\left(\frac{p+q}{2}; \frac{\|\boldsymbol{x}-\boldsymbol{\theta}\|^2}{2}\right)}{(2\pi)^{\frac{p}{2}} \|\boldsymbol{x}-\boldsymbol{\theta}\|^{p+q}} & \boldsymbol{x} \neq \boldsymbol{0} \\ \frac{q}{p+q}\left(\frac{1}{2\pi}\right)^{\frac{p}{2}} & \boldsymbol{x} = \boldsymbol{0} \end{cases}, \end{aligned}$$

where $\gamma(a; z) = \int_0^z t^{a-1} e^{-t} dt = \sum_{k=0}^\infty \frac{(-1)^k z^{a+k}}{k!(a+k)}$. See Wang and Genton (2006) for more details.

Note that the MS distribution is a scale mixture of the normal distributions (see e.g. Fang et al., 1990) and so it can be represented as:

$$\boldsymbol{X}|V = t \sim N_p(\boldsymbol{\theta}, t^{-\frac{1}{q}} \boldsymbol{I}_p), \quad V \sim U(0, 1). \tag{4.2}$$

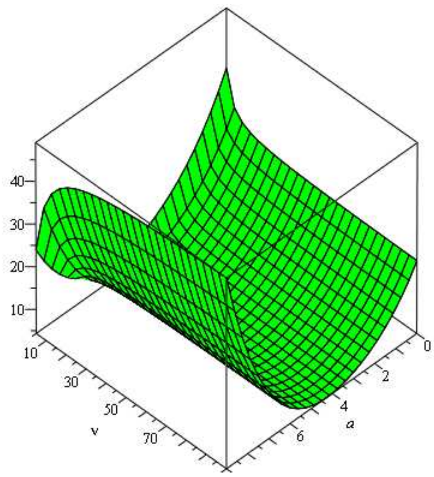

Figure 1: $R(\mathbf{0}, \boldsymbol{\delta}_a^{JS})$ **for MT model when** $p = 7$

Note that here, we have $\frac{1}{q}$ rather than q in (2.2). Thus by making use of the equations (2.2) and (4.2) we have

$$\kappa_{\frac{1}{q}}^{(i)} = \int_0^1 t^{-\frac{i}{q}} dt = \frac{q}{q-i}.$$

Here we face an unusual situation since for $q = i = 1$, or $q = i = 2$ the required finiteness conditions do not satisfy. However for $q \geq 3$ one can immediately find that all the required finiteness conditions are satisfied.

4.3 MEP

Suppose that $\boldsymbol{X}$ is distributed according to a MEP distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\boldsymbol{I}_p$ and kurtosis parameter $\beta \in (0,1)$, denoted by $\boldsymbol{X} \sim EP_p(\boldsymbol{\theta}, \boldsymbol{I}_p, \beta)$, with the following pdf (see Gomez et al., 1998)

$$f(\boldsymbol{x}) = \frac{p\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}\Gamma\left(1+\frac{p}{2\beta}\right)2^{1+\frac{p}{2\beta}}} \exp\left\{-\frac{1}{2}\left[(\boldsymbol{x}-\boldsymbol{\theta})'(\boldsymbol{x}-\boldsymbol{\theta})\right]^{\beta}\right\}.$$

First consider that for $\alpha \in (0,1)$ and $\sigma > 0$; we denote by $S_\alpha(.;\sigma)$ the density of the (positive) stable distribution having characteristic function (see Samorodnitsky and Taqqu, 2000, p.8)

$$\phi(z) = \exp\left[-\sigma^\alpha |z|^\alpha e^{-i\frac{\pi}{2}\ \alpha sign(z)}\right].$$

For the index of stability or characteristic exponent $\alpha \in (0,1)$ the Laplace transform of the distribution function F of the density $S_\alpha(.;\sigma)$ is (see Samorodnitsky and Taqqu, 2000, p.15) $\mathcal{L}_F(z) =$

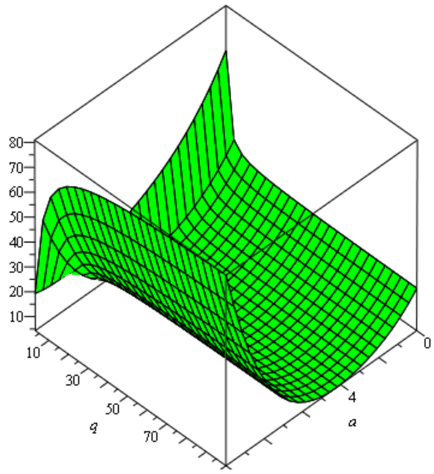

Figure 2: $R(\mathbf{0}, \boldsymbol{\delta}_a^{JS})$ **for MS model when** $p = 7$

$\exp\left(-\sigma^\alpha z^\alpha\right)$. In particular, for $\sigma = 2^{-\frac{1}{\alpha}}$ we have the density $S_\alpha(.; 2^{-\frac{1}{\alpha}})$ and then $\mathcal{L}_F(z) = \exp\left(-\frac{1}{2}t^\alpha\right)$.

Afterward, note that the MEP distribution is a scale mixture of the normal distributions (see Theorem 1 of Gómez-Sánchez-Manzano et al., 2006) and so it can be represented as:

$$\boldsymbol{X}|V = t \sim N_p(\boldsymbol{\theta}, t^2\boldsymbol{I}_p), \quad V \sim f_V(.) \quad \text{where}$$
$$f_V(t) = \frac{2^{1+\frac{p}{2}-\frac{p}{2\beta}}\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(1+\frac{p}{2\beta}\right)} t^{p-3} S_\beta\left(t^{-2}; 2^{1-\frac{1}{\beta}}\right) \tag{4.3}$$

Similarly, we take $q = -2$ in (2.2) and thus using the equations (2.2) and (4.3) we obtain

$$\begin{aligned}\kappa_{-2}^{(i)} &= \int_0^\infty t^{2i} \frac{2^{1+\frac{p}{2}-\frac{p}{2\beta}}\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(1+\frac{p}{2\beta}\right)} t^{p-3} S_\beta\left(t^{-2}; 2^{1-\frac{1}{\beta}}\right) dt \\ &= \frac{2^{1+\frac{p}{2}-\frac{p}{2\beta}}\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(1+\frac{p}{2\beta}\right)} \int_0^\infty t^{2i+p-3} S_\beta\left(t^{-2}; 2^{1-\frac{1}{\beta}}\right) dt. \end{aligned} \tag{4.4}$$

Note that if we take the mixing variable in (4.3) to be $W = 2^{(1/\beta)^{-1}} V^{-2}$ then $F_W(t)$ is proportional to $t^{-p/2} S_\beta(t, 1)$. Thus from equation (2.2.18) of Zolotarev (1986) or Theorem 1 of Nolan (1997), (4.4) reduces to

$$\begin{aligned}\kappa_{-2}^{(i)} &\propto \int_0^\infty t^{2i-\frac{p}{2}} S_\beta\left(t; 1\right) dt \\ &\propto \int_0^\infty x^{2i} \int_0^{\frac{\pi}{2}} g(\xi, \beta, x) e^{-g(\xi,\beta,x)} d\xi dx, \quad g(\xi, \beta, x) = \left(\frac{x\cos\xi}{\sin\beta\xi}\right)^{\frac{\beta}{\beta-1}} \frac{\cos(\beta-1)\xi}{\cos\xi} \end{aligned} \tag{4.5}$$

the second integral in (4.5) does not converge. See Matsui and Takemura (2004) for more details. Thus we conclude that the finiteness condition in (2.2) does not satisfy and our method do not work in this example.

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