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ON CHARACTERIZATIONS OF DISTRIBUTIONS BY TRUNCATED MOMENTS

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SUMMARY

In this paper we have shown that the conditional expectation of the n-th power of the variable can be expressed as a function of hazard rate. Based on these properties some characterizations of exponential, normal, Pareto, power function and uniform distributions are given.

Keywords and phrases: Characterization, Conditional expectation, Exponential distribution, Hazard rate, Normal distribution, Pareto distribution, Power function distribution and uniform distribution

1 Introduction

It is well known that characterizations of statistical distributions play a vital role in statistical theory. Mean residual lives or conditional expectations have been frequently used in reliability theory and in financial modeling. Actuaries use tail conditional expectations and tail conditional variances as measures of risk. Ahmed [1] characterized beta, binomial, and Poisson distributions by connecting conditional expectation with hazard rate. Osaki [6] presented characterization of gamma and negative binomial distributions. Nassar [5] characterized mixture of exponential distributions. Further discussions on characterization of various distributions can be found in Ahsanullah and Hamedani [2], Galambos and Kotz [3], Kotz [4], and Shanbhag [7].

This paper characterizes normal, uniform, Pareto, and exponential distribution. The article is organized as follows. In Section 2, we present the necessary and sufficient condition for identifying each distribution from their higher order conditional mean expressed in terms of hazard rate.

Notation	
X	a random variable having an absolutely continuous distribution
$f(x)$	a probability density function (pdf)
$F(x)$	a cumulative distribution function (cdf)
$h(t) = \frac{f(t)}{1-F(t)}, 0 < F(t) < 1,$	a hazard function

Exponential distribution, Exp(λ) with pdf

$$f(x,\lambda) = \begin{cases} \lambda\, e^{-\lambda x} & x > 0, \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Normal distribution, N(μ, σ) with pdf

$$f(x,\mu,\sigma) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} & -\infty < x, \mu < \infty,\ \sigma > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Pareto distribution, PA(x, α, δ) with pdf

$$f(x,\alpha,\delta) = \begin{cases} \frac{\delta\alpha^{\delta}}{x^{\delta+1}} & 0 < \alpha < x < \infty, \delta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Power function distribution, POW(x, a, b, m) with pdf

$$f(x,a,b,m) = \begin{cases} \frac{m x^{m-1}}{b^m - a^m} & -\infty < a < b < \infty, m > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Uniform distribution, U(x, a, b) with pdf

$$f(x,a,b) = \begin{cases} \frac{1}{b-a} & -\infty < a < b < \infty \\ 0 & \text{otherwise.} \end{cases}$$

2 Main Results

We present below a lemma which will be used in the characterizations of the theorems.

Lemma 2.1. *Let X be an absolutely continuous (with respect to Lebesgue measure) random variable with cdf $F(x)$ and pdf $f(x)$. We assume $\alpha = inf\{x|F(x) > 0\}$ and $\beta = sup\{x|F(x) < 1\}$. If for any positive integer n, $E(X^n)$ exists, then*

$$E(X^n|X > t) = A(n) + g(t)h(t),\ \ \alpha < t < \beta,$$

where $A(n)$ is independent of t but may depend on n and $h(t) = \frac{f(t)}{1-F(t)}$ iff

$$f(x) = ce^{\int \frac{A(n)-g(t)-t^n}{g(t)} dt}$$

and c is determined by the condition

$$\int_\alpha^\beta f(x)dx = 1.$$

Proof. We have

$$\frac{\int_t^\infty x^n f(x)dx}{\bar{F}(t)} = A(n) + g(t)h(t), \ -\alpha < t < \beta$$

that is,

$$\int_t^\infty x^n f(x)dx = A(n)\bar{F}(t) + g(t)f(t).$$

Differentiating the above equation with respect to t, we obtain

$$-t^n f(t) = -A(n)f(t) + g'(t)f(t) + g(t)f'(t).$$

On simplification, we get

$$\frac{f'(t)}{f(t)} = \frac{A(n) - g'(t) - t^n}{g(t)}.$$

Integrating the above equation with respect to t, we have

$$f(x) = ce^{\int \frac{A(n)-g'(x)-x^n}{g(x)} dx},$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

The following theorem gives a characterization of $Exp(\lambda)$.

Theorem 1. *Let X be an absolutely continuous random variable with pdf $f(x)$ and finite $E(X^n)$ for any positive integer n, $0 < x < \infty$. Then*

$$E(X^n|X > t) = \sum_{r=0}^{n} n^{(r)} \frac{t^{n-r}}{\lambda^{r+1}} h(t), \tag{2.1}$$

where $n^{(r)} = n(n-1)(n-2)\cdots(n-r+1)$, $n^{(0)} = 1$. iff X has the exponential distribution Exp(λ), $\lambda > 0$.

Proof. If X has exponential distribution, then (2.1) is true. Suppose that

$$E(X^n|X > t) = \sum_{r=0}^{n} n^{(r)} \frac{t^{n-r}}{\lambda^{r+1}} h(t),$$

then $A(n) = 0$ and

$$g(t) = \sum_{r=0}^{n} n^{(r)} \frac{t^{n-r}}{\lambda^{r+1}}.$$

We have

$$t^n + g'(t) = \lambda g(t).$$

Then

$$\frac{f'(t)}{f(t)} = \frac{A(n) - (t^n + g'(t))}{g(t)} = -\lambda.$$

Hence

$$f(x) = e^{-\int \lambda dx} = ce^{-\lambda x}$$

Using the boundary conditions $F(0) = 0$ and $F(\infty) = 1$, we have

$$F(x) = 1 - \lambda e^{-\lambda x},\ x > 0,\ \lambda > 0.$$

The following theorem gives a characterization of $N(0,1)$ distribution.

Theorem 2. *Let X be an absolutely continuous random variable with pdf $f(x)$ for $-\infty < x < \infty$. Suppose $E(X^n)$ exists for some positive integer n, then X is $N(0,1)$ iff*

$$E(X^n|X > t) = A(n) + g(t)h(t),\ -\infty < t < \infty,$$

where $A(n) = 0$,

$$g(t) = t^{n-1} + (n-1)t^{n-3} + \cdots + (n-1)(n-3)\cdots 4t^2 + (n-1)(n-3)\cdots 2, \tag{2.2}$$

for odd n and $A(n) = (n-1)(n-3)\cdots 3$,

$$g(t) = t^{n-1} + (n-1)t^{n-3} + \cdots + (n-1)(n-3)\cdots 3t, \tag{2.3}$$

for even n.

Proof. It is easy to show that if X is distributed as $N(0,1)$, then (2.2) and (2.3) are true for odd and even integers respectively. We will prove only if condition. If n is odd integer, then using (2.2), we obtain

$$A(n) - (g'(t) + t^n) = -tg(t).$$

If n is even integer, then from (2.3), we have

$$A(n) - (g'(t) + t^n) = -tg(t).$$

Thus for all integer n,

$$\frac{f'(t)}{f(t)} = -t.$$

On integrating with respect to t and using the condition $\int_{-\infty}^{\infty} f(x)dx = 1$, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},\ -\infty < x < \infty.$$

The following theorem gives the characterization of Pareto distribution.

Theorem 3. *Let X be a continuous random variable with pdf $f(x),\ 0<\alpha<x<\infty$. Then X has a Pareto distribution, PA(δ,α) iff,*

$$E(X^n|X>t)=\frac{t^{n+1}}{\delta-n}h(t),\ \textit{for any fixed } n<\delta \textit{ and } t>\alpha. \tag{2.4}$$

Proof. It is easy to show that if X has the distribution $PA(\alpha,\delta)$, then (2.4) is true. We will prove here the only if condition. If

$$E(X^n|X>t)=\frac{t^{n+1}}{\delta-n}h(t)$$

then $A(n)=0$ and $g(t)=\frac{t^{n+1}}{\delta-n}$. We have

$$g'(t)=\frac{n+1}{\delta-n}t^n \Rightarrow \frac{g'(t)+t^n}{g(t)}=\frac{\delta+1}{t}$$

Thus

$$\frac{f'(t)}{f(t)}=\frac{\delta+1}{T}$$

Using the boundary conditions $F(\alpha)=0$ and $F(\infty)=1$, we obtain

$$f(x)=\begin{cases}\frac{\delta\alpha^\delta}{x^{\delta+1}} & 0<\alpha<x<\infty\\ 0 & \text{otherwise.}\end{cases}$$

The following theorem gives a characterization of the power function distribution.

Theorem 4. *Let X be an absolutely continuous random variable with pdf $f(x),\ -\infty<a<x<b<\infty$. Then for $-\infty<a<t<b<\infty$ and $m>0$*

$$E(X^n|X>t)=\begin{cases}\frac{h(t)}{m+n}(b^{m+n}t^{-m+1}-t^{n+1}),\ m+n\neq 0\\ (\ln b-\ln t)t^{1-m},\ m+n=0,\end{cases} \tag{2.5}$$

iff X has the power function distribution with

$$f(x)=\frac{mx^{m-1}}{b^m-a^m},\ m\neq 0.$$

Proof. If $f(x)=\frac{mx^{m-1}}{b^m-a^m}$, then (2.5) is true. Suppose that $m>0$ and $m+n\neq 0$, then

$$\begin{aligned} g(t)&=\frac{1}{m+n}(b^{m+n}t^{-(m-1)}-t^{n+1})\\ g'(t)&=\frac{1}{m+n}(-(m-1)b^{m+n}t^{-m}-(n-1)t^n)\\ t^n+g'(t)&=t^n+\frac{1}{m+n}(-(m-1)b^{m+n}t^{-m}-(n+1)t^n)\\ &=\frac{m-1}{m+n}(t^n-b^{m+n}t^{-m})\\ \frac{t^n+g'(t)}{g(t)}&=\frac{m-1}{t}\end{aligned}$$

Therefore,

$$f(t) = c\,e^{\int \frac{m-1}{t}dt} = ct^{m-1}.$$

Using the boundary conditions $F(a) = 0$ and $F(b) = 1$, we have then

$$f(x) = \frac{mx^{m-1}}{b^m - a^m},\ m > 0,\ -\infty < a < x < b < \infty.$$

Suppose $m > 0$ and $m + n = 0$, then $A(n) = 0$ and $g(t) = (\ln b - \ln t)t^{1-m}$. We have

$$t^n + g'(t) = (1-m)(\ln b - \ln t)t^{-m}$$

Thus

$$\frac{f'(t)}{f(t)} = \frac{A(n) - t^n - g'(t)}{g(t)} = \frac{m-1}{t}$$

$$f(x) = \frac{mx^{m-1}}{b^m - a^m}$$

Therefore,

$$f(t) = ce^{\int \frac{m-1}{t}dt} = ct^{m-1}$$

Using the boundary conditions $F(a) = 0$ and $F(b) = 1$, we have then

$$f(x) = \frac{mx^{m-1}}{b^m - a^m},\ m > 0\ -\infty < a < x < b < \infty.$$

The following theorem gives a characterization of the uniform distribution.

Theorem 5. *Suppose that X is an absolutely continuous random variable with pdf $f(x)$ and $-\infty < a < x < b < \infty$. Then X has a uniform distribution $U(a,b)$ iff*

$$E(X^n|X > t) = \begin{cases} \frac{b^{n+1} - t^{n+1}}{n+1}h(t) & \text{for any } n,\ n \neq -1 \\ (\ln b - \ln t)h(t) & \text{for } n = -1, \end{cases}$$

where $-\infty < a < t < b < \infty$

Proof. The proof follows from the Theorem 4 with $m = 1$.

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References

[1] Ahmed, A. N. (1991). Characterization of Beta, Binomial, and Poisson Distributions. *IEEE Transactions on Reliability* **40***(3), 290-295.*

[2] Ahsanullah, M. and Hamedani, G.G. (2005). Characterizations of univariate continuous distributions based on hazard function II. *Journal of Statistical Theory and Applications* **4***(3), 218-238.*

[3] Galambos, J. and Kotz, S. (1978). *Characterization of Probability Distributions* Springer-Verlage.

[4] Kotz, S. (1974). Characterization of statistical distribution: A supplement to recent surveys. *Rev. Int. Statist.* **42***(1), 39-65.*

[5] Nassar, M. M. and Mahmoud, M. R. (1985). On characterization of a mixture of exponential distributions. *IEEE Transactions on Reliability* **R-34***, 484-488.*

[6] Osaki, S. and Li, X. (1988). Characterization of gamma and negative binomial distributions. *IEEE Transactions on Reliability* **R-37***, 379-382.*

[7] Shanbhag, D. N. (1970). The characterizations for exponential and geometric distributions. *Journal of American Statistical Association* **65***(331), 1256-259.*