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RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF GENERALIZED ORDER STATISTICS FROM DOUBLY TRUNCATED MAKEHAM DISTRIBUTION AND ITS CHARACTERIZATION

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SUMMARY

The concept of generalized order statistics (gos) was introduced by Kamps (1995). Since generalized order statistics is an unified approach of other ordered random scheme, therefore, recurrence relations between moments of generalized order statistics is of special interest. In this paper some recurrence relations for moments of generalized order statistics from doubly truncated Makeham distribution are obtained. Further, the results are deduced for order statistics and record values. In the last section a characterization theorem is presented.

Keywords and phrases: Makeham distribution, generalized order statistics, order statistics, record values, recurrence relations and characterization.

AMS Classification: 62G30, 62E10.

1 Introduction

Let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed random variables (rvs) with absolutely continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, \ldots, n$ are called gos if their joint pdf is given by

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i)\Big)[1 - F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) \leq x_1 \cdots \leq x_n \leq F^{-1}(1)$ of $\mathbb{R}^n$.

If $m_i = 0$, $i = 1, \ldots, n-1$ and $k = 1$, we obtain the joint pdf of the order statistics and for $m_i = -1,\ k \in N$, we get the joint pdf of kth record values.

In view of (1.1) with $m_i = m, i = 1, \dots n-1$, the *pdf* of $rth\ gos$, $X(r,n,m,k)$ is

$$f_{X(r,n,m,k)}(x) \;=\; \frac{C_{r-1}}{(r-1)!}\,[\bar{F}(x)]^{\gamma_r - 1}\, f(x)\, g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$\begin{aligned} f_{X(r,s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}&[\bar{F}(x)]^{m}\, g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\times [\bar{F}(y)]^{\gamma_s - 1}\, f(x)\, f(y), \;\; \alpha \le x < y \le \beta, \end{aligned} \tag{1.3}$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i, \;\; h_m(x) = \begin{cases} -\dfrac{1}{m+1}\,(1-x)^{m+1} & , \;\; m \ne -1 \\ \log\left(\dfrac{1}{1-x}\right) & , \;\; m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \;\; x \in [0,1).$

Makeham distribution is an important life distribution and has been widely used to fit actuarial data (see, Marshall and Olkin, 2007). For a description on the genesis and applications of Makeham distribution one may refer to Makeham (1860).

A random variable X is said to have Makeham distribution if its *pdf* is of the form

$$f_1(x) = [1 + \theta(1 - e^{-x})]e^{-x - \theta(x + e^{-x} - 1)}, \;\; x \ge 0, \;\; \theta \ge 0 \tag{1.4}$$

with the corresponding *df*

$$F_1(x) = 1 - e^{-x - \theta(x + e^{-x} - 1)}, \;\; x \ge 0, \;\; \theta \ge 0. \tag{1.5}$$

Now if for given P_1 and Q_1

$$\int_0^{Q_1} f_1(x)dx = Q \;\; \text{and} \;\; \int_0^{P_1} f_1(x)dx = P,$$

then the truncated *pdf* is given by

$$f(x) = \frac{1}{P-Q}[1 + \theta(1 - e^{-x})]e^{-x - \theta(x + e^{-x} - 1)}, \;\; Q_1 \le x \le P_1 \tag{1.6}$$

and corresponding truncated *df* $F(x)$ is

$$\bar{F}(x) = \frac{f(x)}{1 + \theta(1 - e^{-x})} - P_2, \tag{1.7}$$

where

$$1 - P = e^{-P_1 - \theta(P_1 + e^{-P_1} - 1)}, \; 1 - Q = e^{-Q_1 - \theta(Q_1 + e^{-Q_1} - 1)}, Q_2 = \frac{1-Q}{P-Q} \;\; \text{and} \;\; P_2 = \frac{1-P}{P-Q}.$$

Some recurrence relations for moments of generalized order statistics are obtained by Kamps (1995), Cramer and Kamps (2000), Kamps and Cramer (2001), Pawlas and Szynal (2001), Ahmad and Fawzy (2003), Athar and Islam (2004), Ahmad (2007), Khan $et\ al.$ (2007) among others. Here in this paper, we have obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated Makeham distribution and its various deductions and particular cases are discussed. At the end characterization of this distribution through conditional expectation is presented.

2 Single Moments

Lemma 2.1. *For* $2 \le r \le n$, $n \ge 2$ *and* $k = 1, 2, \ldots,$

$$(i)\ E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n,m,k)\big] = \frac{C_{r-2}}{(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1}[\bar F(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big)dx. \tag{2.1}$$

$$(ii)\ E\big[X^j(r-1,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big] = -\frac{(m+1)C_{r-2}}{\gamma_1(r-2)!}\, j \int_{Q_1}^{P_1} x^{j-1}[\bar F(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big)dx. \tag{2.2}$$

$$(iii)\ E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big] = \frac{C_{r-1}}{\gamma_1(r-1)!}\, j \int_{Q_1}^{P_1} x^{j-1}[\bar F(x)]^{\gamma_r} g_m^{r-1}\big(F(x)\big)dx. \tag{2.3}$$

Proof. We have by Athar and Islam (2004),

$$E\big[\xi\{X(r,n,m,k)\}\big] - E\big[\xi\{X(r-1,n,m,k)\}\big] = \frac{C_{r-2}}{(r-1)!} \int_{\alpha}^{\beta} \xi'(x)\big[\bar F(x)\big]^{\gamma_r} g_m^{r-1}\big(F(x)\big)dx, \tag{2.4}$$

where $\xi(x)$ is a Borel measurable function of $x \in (\alpha, \beta)$. The equation (2.1) can be established by letting $\xi(x) = x^j$ in the (2.4). Relations (2.2) and (2.3) can be seen on the lines of (2.1).

Theorem 1. *For the given Makeham distribution and* $n \in N,\ m \in \mathbb{R},\ 2 \le r \le n,$

$$\begin{aligned} E\big[X^j(r,n,m,k)\big] - E\big[X^j(r-1,n-1,m,k)\big] = -P_2\, K \Big\{ & E\big[X^j(r,n-1,m,k+m)\big] \\ & - E\big[X^j(r-1,n-1,m,k+m)\big]\Big\} \\ & + \frac{j}{\gamma_1} E\big[\psi\{X(r,n,m,k)\}\big], \end{aligned} \tag{2.5}$$

where

$$K = \frac{C_{r-2}^{(n-1)}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \frac{\gamma_i^{(n-1)}}{\gamma_i^{(n-1)} + m},\ \gamma_i^{(n-1)} = k + (n-i-1)(m+1), \psi(x) = \frac{x^{j-1}}{1+\theta(1-e^{-x})}.$$

Proof. In view of (1.7) and (2.3), we have

$$\begin{aligned}
&E\big[X^j(r,n,m,k)\big]-E\big[X^j(r-1,n-1,m,k)\big]\\
&=\frac{C_{r-1}}{\gamma_1(r-1)!}\,j\int_{Q_1}^{P_1}x^{j-1}\big[\bar F(x)\big]^{\gamma_r-1}\Big\{-P_2+\frac{f(x)}{1+\theta(1-e^{-x})}\Big\}g_m^{r-1}\big(F(x)\big)dx\\
&=-P_2\frac{C_{r-2}^{(n-1)}}{(r-1)!}\,j\int_{Q_1}^{P_1}x^{j-1}\big[\bar F(x)\big]^{\gamma_r^{(n-1,k+m)}}g_m^{r-1}\big(F(x)\big)dx\\
&\quad+\frac{j}{\gamma_1}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1}\psi(x)\big[\bar F(x)\big]^{\gamma_r-1}g_m^{r-1}\big(F(x)\big)f(x)dx
\end{aligned}$$

as $\gamma_r-1=\gamma_r^{(n-1,k+m)}=(k+m)+(n-1-r)(m+1),\ C_{r-1}=\gamma_1C_{r-2}^{(n-1)}$ and hence the required result.

Remark 1. At $P=1,\ Q=0$, we get the relation for non-truncated case

$$E\big[X^j(r,n,m,k)\big]-E\big[X^j(r-1,n-1,m,k)\big]=\frac{j}{\gamma_1}E\big[\psi\{X(r,n,m,k)\}\big].$$

Remark 2. Recurrence relation for single moments of order statistics $(m=0,\ k=1)$ is

$$\begin{aligned}
E(X_{r:n}^j)-E(X_{r-1:n-1}^j)&=-P_2\big\{E(X_{r:n-1}^j)-E(X_{r-1:n-1}^j)\big\}+\frac{j}{n}E[\psi(X_{r:n})]\\
\text{or }\ E(X_{r:n}^j)&=-P_2E(X_{r:n-1}^j)+Q_2E(X_{r-1:n-1}^j)+\frac{j}{n}E[\psi(X_{r:n})].
\end{aligned}$$

At $j=1$, result is obtained by Aboutahoun and Al-Otaibi (2009).

By convention, we use $X_{n:n-1}=P_1$ and $X_{0:n-1}=Q_1$.

Remark 3. For kth record statistic $(m=-1)$, recurrence relation for single moments reduces as

$$E(X_r^{(k)})^j-E(X_{r-1}^{(k)})^j=-P_2\Big(\frac{k}{k-1}\Big)^{r-1}\Big\{E(X_r^{(k-1)})^j-E(X_{r-1}^{(k-1)})^j\Big\}+\frac{j}{k}E[\psi(X_r^{(k)})].$$

Similarly, the recurrence relation for single moments of order statistics with non-integral sample size for $m=0,\ k=\alpha-n+1,\ \alpha\in\mathbb{R}_+$ and for sequential order statistics for $m=\alpha-1,\ k=\alpha$ may be obtained.

Theorem 2. *For the given Makeham distribution and* $n\in N,\ m\in\mathbb{R},\ 2\le r\le n$,

$$E\big[X^j(r,n,m,k)-X^j(r-1,n-1,m,k)\big]=\frac{(P-Q)}{\gamma_1}K^*\,j\,E[\phi\{X(r,n,m,k+1)\}] \tag{2.6}$$

$$=\frac{j}{\gamma_1}\Big[-(1-P)E[\psi\{X(r,n,m,k)\}]+E[\phi\{X(r,n,m,k)\}]\Big], \tag{2.7}$$

where

$$\phi(x)=\frac{x^{j-1}e^{\{x+\theta(x+e^{-x}-1)\}}}{1+\theta(1-e^{-x})},\ \ K^*=\frac{C_{r-1}}{C_{r-1}^{k+1}}=\prod_{i=1}^{r}\Big(\frac{\gamma_i}{\gamma_i+1}\Big).$$

Proof. On using (1.6) in (2.3), we get

$$
\begin{aligned}
&E\big[X^j(r,n,m,k)\big]-E\big[X^j(r-1,n-1,m,k)\big]\\
&=\frac{C_{r-1}}{\gamma_1(r-1)!}\,j\int_{Q_1}^{P_1}x^{j-1}[\bar F(x)]^{\gamma_r}\Big\{\frac{(P-Q)e^{\{x+\theta(x+e^{-x}-1)\}}}{1+\theta(1-e^{-x})}f(x)\Big\}g_m^{r-1}\big(F(x)\big)dx\\
&=\frac{(P-Q)}{\gamma_1}\frac{C_{r-1}}{C_{r-1}^{(k+1)}}\,j\,\Big\{\frac{C_{r-1}^{(k+1)}}{(r-1)!}\int_{Q_1}^{P_1}\phi(x)[\bar F(x)]^{\gamma_r^{(k+1)}-1}g_m^{r-1}\big(F(x)\big)f(x)dx\Big\},
\end{aligned}
$$

where $\gamma_r^{(k+1)}=(k+1)+(n-r)(m+1)$ and hence the theorem.

To prove (2.7), note that

$$
\frac{\bar F(x)}{f(x)}=\frac{1}{1+\theta(1-e^{-x})}-(1-P)\frac{e^{\{x+\theta(x+e^{-x}-1)\}}}{1+\theta(1-e^{-x})}
$$

and the result follows from (2.3).

Theorem 3. *For the given Makeham distribution and* $n\in N,\ m\in\mathbb{R},\ 2\le r\le n,$

$$
\begin{aligned}
&E\big[X^j(r,n,m,k)\big]-E\big[X^j(r-1,n,m,k)\big]\\
&=-P_2\,K^{**}\,E[X^j(r,n-1,m,k+m)]-E[X^j(r-1,n-1,m,k+m)]\\
&\quad+\frac{j}{\gamma_r}E[\psi\{X(r,n,m,k)\}],
\end{aligned}
\tag{2.8}
$$

where

$$
K^{**}=\frac{C_{r-2}}{C_{r-2}^{n-1,k+m}}=\prod_{i=1}^{r-1}\Big(\frac{\gamma_i}{\gamma_i^{(n-1)}+m}\Big)=\prod_{i=1}^{r-1}\Big(\frac{\gamma_i}{\gamma_i-1}\Big).
$$

Proof. Proof follows on the lines of Theorem 2.1 using (1.7) in (2.1).

3 Product Moments

Theorem 4. *For the given Makeham distribution and* $1\le r<s\le n-1,\ m\in\mathbb{R},\ n\ge 2$ *and* $k=1,2,\ldots$

$$
\begin{aligned}
&E[X^i(r,n,m,k).X^j(s,n,m,k)]-E[X^i(r,n,m,k).X^j(s-1,n,m,k)]\\
&=-P_2K_1\Big\{E[X^i(r,n-1,m,k+m).X^j(s,n-1,m,k+m)]\\
&\quad-E[X^i(r,n-1,m,k+m).X^j(s-1,n-1,m,k+m)]\Big\}\\
&\quad+\frac{j}{\gamma_s}E\big[\psi\{X(r,n,m,k).X(s,n,m,k)\}\big],
\end{aligned}
\tag{3.1}
$$

where

$$\psi\{X(r,n,m,k).X(s,n,m,k)\} = \frac{x^i.y^{j-1}}{1+\theta(1-e^{-y})}$$
$$K_1 = \frac{C_{s-2}}{C_{s-2}^{(n-1,k+m)}} = \prod_{i=1}^{s-1}\Big(\frac{\gamma_i}{\gamma_i^{(n-1)}+m}\Big) = \prod_{i=1}^{s-1}\Big(\frac{\gamma_i}{\gamma_i-1}\Big).$$

Proof. In view of Athar and Islam (2004), we have

$$\begin{aligned} &E[X^i(r,n,m,k).X^j(s,n,m,k)] - E[X^i(r,n,m,k).X^j(s-1,n,m,k)] \\ &= \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} j \int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}\big(F(x)\big) \\ &\quad \times [h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1}[\bar{F}(y)]^{\gamma_s} dy\, dx. \end{aligned} \tag{3.2}$$

Now using (1.7) in (3.2), we get

$$\begin{aligned} &= \frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} j \int_{Q_1}^{P_1}\int_{x}^{P_1} x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}\big(F(x)\big) \\ &\times [h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}\Big\{ -P_2 + \frac{f(y)}{1+\theta(1-e^{-y})}\Big\} dy\, dx. \end{aligned}$$

Hence (3.1) can be established after noting that $\gamma_s - 1 = \gamma_s^{(n-1,k+m)},\ C_{s-1} = \gamma_s C_{s-2}$.

Remark 4. Recurrence relation between product moments of order statistics $(m=0,\ k=1)$ is

$$\begin{aligned} E\big(X_{r,s:n}^{(i,j)}\big) &= E\big(X_{r,s-1:n}^{(i,j)}\big) - P_2\, \frac{n}{n-s+1}\Big\{E\big(X_{r,s:n-1}^{(i,j)}\big) - E\big(X_{r,s-1:n-1}^{(i,j)}\big)\Big\} \\ &\quad + \frac{j}{n-s+1} E\big[\psi(X_{r,s:n})\big]. \end{aligned} \tag{3.3}$$

At $i=j=1$, (3.3) reduces to result obtained by Aboutahoun and Al-Otaibi (2009).

Remark 5. Recurrence relation for product moments of $k-th$ record value (at $m=-1$) is given as

$$E(X_{r,s}^{(k)})^{(i,j)} - E(X_{r,s-1}^{(k)})^{(i,j)} = -P_2\Big(\frac{k}{k-1}\Big)^{s-1}\Big\{E(X_{r,s}^{(k-1)})^{(i,j)} - E(X_{r,s-1}^{(k-1)})^{(i,j)}\Big\} + \frac{j}{k} E\big[\psi(X_{r,s}^{(k)})\big].$$

4 Characterization

Let $X(r,n,m,k),\ r=1,2,\ldots$ be *gos*, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, in view of (1.2) and (1.3) is

$$f_{s|r}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[\bar{F}(x)]^{m-\gamma_r+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(y). \tag{4.1}$$

Theorem 5. *Let X be an absolutely continuous rv with df $F(x)$ and pdf $f(x)$ with $F(x) < 1$, for all $x \in (0, \infty)$. Then for two consecutive values r and $r+1$, $2 \le r+1 \le s \le n$,*

$$E[X(s,n,m,k) + \theta e^{-X(s,n,m,k)}|X(l,n,m,k) = x]$$
$$= \Big(x + \frac{\theta}{\theta+1}\, e^{-x}\Big) + \frac{1}{\theta+1} \sum_{j=l+1}^{s} \frac{1}{\gamma_j}, \;\; l = r, r+1 \tag{4.2}$$

if and only if X has the df

$$F(x) = 1 - e^{-x-\theta(x+e^{-x}-1)}, \;\; x \ge 0, \;\; \theta \ge 0. \tag{4.3}$$

Proof. We have for $s \ge r+1$,

$$\begin{aligned} g_{s|r} &= E[X(s,n,m,k) + \theta e^{-X(s,n,m,k)}|X(r,n,m,k) = x] \\ &= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \\ &\quad \times \int_x^\infty (y + \theta\, e^{-y}) \Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{\gamma_s - 1} \Big[1 - \frac{(\bar{F}(y))^{m+1}}{(\bar{F}(x))^{m+1}}\Big]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \end{aligned} \tag{4.4}$$

Let $u = \bar{F}(y)/\bar{F}(x) = e^{-\{(y-x)(1+\theta)+\theta(e^{-y}-e^{-x})\}}$, then $y + \theta\, e^{-y} = x + \frac{\theta}{\theta+1}\, e^{-x} - \frac{1}{\theta+1} \log u$.

Thus (4.4) becomes

$$\begin{aligned} &E[X(s,n,m,k) + \theta e^{-X(s,n,m,k)}|X(r,n,m,k) = x] \\ &= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r-1}} \int_0^1 \{x + \frac{\theta}{\theta+1}\, e^{-x} - \frac{\log u}{\theta+1}\}\, u^{\gamma_s - 1} [1 - u^{m+1}]^{s-r-1} du. \end{aligned}$$

Set $u^{m+1} = t$, to get

$$\begin{aligned} &E[X(s,n,m,k) + \theta e^{-X(s,n,m,k)}|X(r,n,m,k) = x] \\ &= x + \frac{\theta e^{-x}}{\theta+1} - \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r+1}(\theta+1)} \int_0^1 \log t\; t^{\frac{\gamma_s}{m+1}-1} (1-t)^{s-r-1} dt \\ &= x + \frac{\theta e^{-x}}{\theta+1} - \frac{C_{s-1}}{C_{r-1}(s-r-1)!(\theta+1)(m+1)^{s-r+1}} B\Big(\frac{\gamma_s}{m+1}, s-r\Big) \Big[\psi\big(\frac{\gamma_s}{m+1}\big) - \psi\big(\frac{\gamma_r}{m+1}\big)\Big], \end{aligned}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ [Gradshteyn and Ryzhik, 2007, pp-540]. Since,

$$\psi(x-n) - \psi(x) = -\sum_{k=1}^{n} \frac{1}{x-k}. \quad [c.f. \text{ Gradshteyn and Ryzhik, 2007, pp-905}]$$

Therefore,

$$E[X(s,n,m,k) + \theta e^{-X(s,n,m,k)}|X(r,n,m,k) = x] = \Big(x + \frac{\theta}{\theta+1}\, e^{-x}\Big) + \frac{1}{\theta+1} \sum_{j=r+1}^{s} \frac{1}{\gamma_j}.$$

To show that (4.2) implies (4.3), we have

$$g_{s|r+1}(x) - g_{s|r}(x) = -\frac{1}{(\theta+1)\gamma_{r+1}}.$$

Therefore,

$$-\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = 1 + \theta(1 - e^{-x}) \qquad \text{[Beg and Ahsanullah, 2006]}$$

and hence

$$\frac{f(x)}{\bar{F}(x)} = 1 + \theta(1 - e^{-x}).$$

Implying that

$$\bar{F}(x) = e^{-x-\theta(x+e^{-x}-1)}, \;\; x \geq 0, \;\; \theta \geq 0.$$

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