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BAYESIAN ESTIMATION FOR THE β-BIRNBAUM-SAUNDERS DISTRIBUTION

WALAA A. SAYED
Department of Mathematics, Faculty of Science, Cairo University, Egypt
Email: shwalaa@sci.cu.edu.eg

EL-SAYED A. ELSHERPIENY
Department of Mathematical Statistics, Institute of Statistical Studies & Research, Cairo University, Egypt
Email: ahmedc55@hotmail.com

MOSTAFA M. EL-FAHHAM
Department of Mathematics, Faculty of Science, Cairo University, Egypt
Email: fahham@sci.cu.edu.eg

SUMMARY

In this article, we present Bayes estimators for the parameters and reliability function of the β-Birnbaum-Saunders distribution under both the symmetric (squared error, SE) loss function and asymmetric (LINEX and general entropy, GE) loss functions. The Bayes estimators can not be obtained in closed form. Approximate Bayes estimators are computed using Lindley's approximation technique. Posterior variance estimates are compared with the variance of the maximum likelihood estimators (MLEs) of the parameters. The different loss functions are compared through posterior risk. A real data set is analyzed for illustrative purpose.

Keywords and phrases: Bayesian estimation, Beta Birnbaum-Saunders distribution, Lindley's approximation, Symmetric and asymmetric loss functions.

1 Introduction

Motivated by the problem of damage caused by vibrations in commercial aircraft, Birnbaum and Saunders [3] introduced a new probabilistic model for modeling the failure time due to fatigue under cyclic loading under the assumption that the failure is due to the development and growth of a dominant crack. This model is so called the fatigue life or two-parameter Birnbaum-Saunders (BS) distribution. It is an attractive alternative distribution to other popular distributions such as the log-normal, Weibull and gamma, since its derivation considers the basic characteristics of the fatigue process.

Cordeiro and Lemonte [7] proposed the new distribution, so called the β-Birnbaum-Saunders (β-BS) distribution, which is an extension of the two-parameter BS distribution. This distribution provides more flexibility to fit various types of lifetime data than the BS distribution.

The cumulative distribution function (cdf) of the β-BS distribution with $\alpha > 0$, $a > 0$, $b > 0$ are shape parameters and $\beta > 0$ is a scale parameter is defined by

$$F(t) = I_{\Phi(\upsilon)}(a, b), \ \ t > 0, \tag{1.1}$$

where $\Phi(.)$ is the standard normal cumulative function, $\upsilon = \alpha^{-1}\rho(t/\beta)$, $\rho(z) = z^{1/2} - z^{-1/2}$, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y \omega^{a-1} (1-\omega)^{b-1} d\omega$ is the incomplete beta function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the beta function and $\Gamma(.)$ is the gamma function. The probability density function (pdf) corresponding to (1.1) is given by

$$f(t) = \frac{k(\alpha, \beta)}{B(a, b)}\, t^{-3/2}\, (t + \beta)\, \exp\left\{-\frac{\tau(t/\beta)}{2\alpha^2}\right\} \Phi(\upsilon)^{a-1}\{1 - \Phi(\upsilon))\}^{b-1}, \tag{1.2}$$

where $k(\alpha, \beta) = \exp\left(\alpha^{-2}\right) / \left(2\alpha\sqrt{2\pi\beta}\right)$ and $\tau(z) = z + z^{-1}$. If T is a random variable with pdf (1.2), we write $T \sim \beta$-BS(α, β, a, b). The β-BS model contains as sub-models the BS and exponentiated Birnbaum-Saunders (EBS) distributions when $a = b = 1$ and $b = 1$, respectively.

Statistical inference for the parameters of the β-BS distribution has been discussed by Cordeiro and Lemonte [7], but their work was concerned with the maximum likelihood estimation and not work has been done in the Bayesian estimation.

Some works have been done on estimating the parameters of the BS distribution using Bayesian techniques. Achcar [1] used Laplace's method for approximation and different non-informative priors to derive simple expressions of the marginal posterior and predictive densities of parameters of interest. Xu and Tang [26] proposed the Bayes estimators for the parameters of the BS distribution under the reference prior using the method of Lindley's approximation and Gibbs sampling procedure. Xu and Tang [27] discussed Bayesian analysis of the BS distribution with partial information based on censored samples using Gibbs sampling procedure.

This article is concerned with the Bayesian inference for the unknown parameters and reliability function of the β-BS distribution under the SE, LINEX and GE loss functions using Lindley's approximation technique. This technique depends on the existence of the MLE of the parameters. The article is organized as follows. Section 2, contains the SE as the symmetric loss function and the LINEX and GE as the asymmetric loss functions. The Lindley's approximation technique is presented in Section 3. Approximate Bayes estimators for the parameters and reliability function of the β-BS distribution under the three different loss functions are obtained in Section 4. A real data set has been analyzed in Section 5. Section 6 ends with concluding remarks.

2 Loss Function

2.1 Symmetric Loss Function

Most of the Bayesian inference procedures have been developed under the squared error (SE) loss function, which is frequently used due to its mathematical simplicity and the relation with classical procedures. The SE loss function under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$ is defined as:

$$L(\hat{\theta}, \theta) \propto (\hat{\theta} - \theta)^2. \tag{2.1}$$

The symmetric nature of this loss function gives equal weight to overestimation as well as underestimation. The posterior expected loss of (2.1) is:

$$E_\theta[L(\hat{\theta}, \theta)] \propto E_\theta(\hat{\theta} - \theta)^2, \tag{2.2}$$

where $E_\theta(.)$ is the posterior expectation with respect to the posterior density of θ. The Bayes estimator $\hat{\theta}_{BS}$ of θ under the SE loss function is the value of $\hat{\theta}$ which minimizes (2.2) and is given as:

$$\hat{\theta}_{BS} = E_\theta(\theta|\mathbf{t}). \tag{2.3}$$

2.2 Asymmetric Loss Function

In real life situations, the real loss function is not always symmetric and hence the use of SE loss function may be inappropriate. So this leads to think that an asymmetrical loss function may be more appropriate.

2.2.1 Linear Exponential Loss Function (LINEX)

Varian [25] introduced a very useful asymmetric loss function known as LINEX loss function and was widely used by several authors, among them Canfield [6], Zellner [28], Rojo [20], Basu and Ebrahimi [2], Pandey and Rai [15], Calabria and Pulcini [5], Pandey et al. [17], Pandey [14], Soliman([21], [22]), Nassar and Eissa [13], Soliman([23], [24]), Pandey and Rao [16] and Guure et al. [10]. The LINEX loss function under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$ can be expressed as:

$$L(\Delta) \propto \exp(p\Delta) - p\Delta - 1; \;\; p \neq 0, \tag{2.4}$$

where $\Delta = \hat{\theta} - \theta$ and $\hat{\theta}$ is an estimate of θ. The sign and magnitude of the shape parameter p represent the direction and degree of symmetry, respectively. (If $p > 0$, means overestimation is more serious than underestimation, and vice-versa). For $p \approxeq 0$, the LINEX loss function is approximately the SE loss function and therefore almost symmetric. The posterior expectation of the LINEX loss function (2.4) is:

$$E_\theta[L(\hat{\theta} - \theta)] \propto \exp(p\hat{\theta})E_\theta[\exp(-p\theta)] - p[\hat{\theta} - E_\theta(\theta)] - 1. \tag{2.5}$$

The Bayes estimator $\hat{\theta}_{BL}$ of θ under the LINEX loss function is the value of $\hat{\theta}$ which minimizes (2.5) and is given as:

$$\hat{\theta}_{BL} = -\frac{1}{p} \ln E_\theta[\exp(-p\theta)], \tag{2.6}$$

provided that $E_\theta[\exp(-p\theta)]$ exists and is finite.

2.2.2 General Entropy Loss Function

Another useful alternative asymmetric loss function to the modified LINEX loss function is the general entropy (GE) loss function was proposed by Calabria and Pulcini [4] and is defined as:

$$L(\hat{\theta},\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^{q} - q\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1 \tag{2.7}$$

which has a minimum at $\hat{\theta} = \theta$. This loss function is a generalization of the entropy loss function used by Dey et al. [8], Dey and Liu [9], Parsian and Nematollahi [18] and Li and Ren [11] by taking the shape parameter $q = 1$. When $q > 0$ a positive error causes more serious consequences than a negative error, and vice-versa. The Bayes estimator $\hat{\theta}_{BG}$ of θ under the GE loss function is:

$$\hat{\theta}_{BG} = \left[E_\theta(\theta^{-q})\right]^{-\frac{1}{q}}, \tag{2.8}$$

provided that $E_\theta(\theta^{-q})$ exists and is finite. It can be shown that (see Calabria and Pulcini [4]):

- When $q = 1$, the Bayes estimate (2.8) coincides with the Bayes estimate under the weighted squared error loss function.
- When $q = -1$, the Bayes estimate (2.8) coincides with the Bayes estimate under the SE loss function.

3 Lindley's Procedure

Lindley [12] developed approximate procedure for evaluation of the ratio of integrals of the form

$$\frac{\int \omega(\theta)\exp\left[\ell(\theta)\right]d\theta}{\int \upsilon(\theta)\exp\left[\ell(\theta)\right]d\theta}, \tag{3.1}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ is a parameter, $\ell(\theta)$ is the logarithm of the likelihood function, and $\omega(\theta)$ and $\upsilon(\theta)$ are arbitrary functions of θ. If $\upsilon(\theta)$ is the prior density of θ and $\omega(\theta) = u(\theta)\upsilon(\theta)$, then (3.1) yields the posterior expectation of $u(\theta)$ and can be written as:

$$E[u(\theta)|\mathbf{t}] = \frac{\int u(\theta)\exp\left[\ell(\theta)+\rho(\theta)\right]d\theta}{\int \exp\left[\ell(\theta)+\rho(\theta)\right]d\theta}, \tag{3.2}$$

where $\rho(\theta) = \ln \upsilon(\theta)$. Eq.(3.2) can be asymptotically approximated by

$$E[u(\theta)|\mathbf{t}] = \left[u + \frac{1}{2}\sum_i\sum_j \left(u_{ij} + 2u_i\rho_j\right)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l \ell_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]_{\hat{\theta}} \tag{3.3}$$
$$+ \text{ terms of order } n^{-2} \text{ or smaller,}$$

where $i,j,k,l = 1,2,\ldots,m$, $\hat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $\ell_{ijk} = \partial^3\ell/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$ and σ_{ij} is the $(i,j)^{th}$ element of the variance-covariance matrix of the parameters.

4 Bayesian Estimation

In this section, we consider the Bayes estimators for the unknown parameters and reliability function of the β-BS distribution under the SE, LINEX and GE loss functions. Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a random sample of size n from β-BS distribution with pdf (1.2). The likelihood function can be written as:

$$L(\mathbf{t}|\alpha,\beta,a,b) = \left[\frac{k(\alpha,\beta)}{B(a,b)}\right]^n \prod_{i=1}^{n} t_i^{-\frac{3}{2}}(t_i+\beta)\exp\Big\{-\frac{\tau(t_i/\beta)}{2\alpha^2}\Big\}\Phi(\upsilon_i)^{a-1}\{1-\Phi(\upsilon_i)\}^{b-1}. \quad (4.1)$$

Xu and Tang [27] considered two scenarios for the joint prior densities of (α,β) as following

$$\pi_1(\alpha,\beta) = \pi(\alpha|\beta)\pi(\beta) \text{ and } \pi_2(\alpha,\beta) = \pi(\beta|\alpha)\pi(\alpha), \quad (4.2)$$

where $\pi(\alpha|\beta) = 1/\alpha$, $\pi(\beta|\alpha) = 1/\beta$, $\pi(\alpha) \propto \alpha^{-2a_1-2}\exp(-b_1/\alpha^2)$, $\pi(\beta) \propto \beta^{-a_2-1}\exp(-b_2/\beta)$ and $a_i, b_i > 0$, for $i = 1, 2$. For more details see Xu and Tang [27]. We next consider an independent non-informative priors for a and b as following

$$\pi(a) = 1/a \quad \text{and} \quad \pi(b) = 1/b. \quad (4.3)$$

Thus, the joint prior densities of (α,β,a,b) can be written as

$$\pi_1(\alpha,\beta,a,b) = \pi(\alpha|\beta)\pi(\beta)\pi(a)\pi(b) \quad \text{and} \quad \pi_2(\alpha,\beta,a,b) = \pi(\beta|\alpha)\pi(\alpha)\pi(a)\pi(b). \quad (4.4)$$

For simplicity, we can define the joint prior density of (α,β,a,b) as

$$\pi(\alpha,\beta,a,b) = [\pi_1(\alpha,\beta,a,b)]^{\delta}[\pi_2(\alpha,\beta,a,b)]^{1-\delta}, \quad (4.5)$$

where

$$\pi(\alpha,\beta,a,b) = \begin{cases} \pi_1(\alpha,\beta,a,b) & \text{for } \delta = 1, \\ \pi_2(\alpha,\beta,a,b) & \text{for } \delta = 0. \end{cases}$$

Combining the likelihood function (4.1) and the joint prior density (4.5), then the joint posterior density of (α,β,a,b) can be written as

$$\pi(\alpha,\beta,a,b|\mathbf{t}) \propto L(\mathbf{t}|\alpha,\beta,a,b)\pi(\alpha,\beta,a,b) = KL(\mathbf{t}|\alpha,\beta,a,b)\pi(\alpha,\beta,a,b), \quad (4.6)$$

where K is the normalizing constant given by

$$K^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{t}|\alpha,\beta,a,b)\pi(\alpha,\beta,a,b)d\alpha\, d\beta\, da\, db.$$

The Bayes estimator of an arbitrary function $u(\theta)$ of the unknown parameters θ under SE loss function is the posterior expectation of that function. If we have $\theta = (\alpha,\beta,a,b)$, then

$$E\left[u(\alpha,\beta,a,b)|\mathbf{t}\right] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u(\alpha,\beta,a,b)L(\mathbf{t}|\alpha,\beta,a,b)\pi(\alpha,\beta,a,b)\, d\alpha\, d\beta\, da\, db}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{t}|\alpha,\beta,a,b)\pi(\alpha,\beta,a,b)\, d\alpha\, d\beta\, da\, db}. \quad (4.7)$$

The ratio of integrals in (4.7) does not seem to take a closed form, therefore an application of approximation techniques is suitable for solving such problems. We suggest Lindley's approximation procedure to evaluate the Bayes estimates of the unknown parameters.

The posterior variance of the function $u(\theta)$ is given by:

$$Var\left[u(\alpha,\beta,a,b)|\mathbf{t}\right] = E\left[u(\alpha,\beta,a,b)^2|\mathbf{t}\right] - (E\left[u(\alpha,\beta,a,b)|\mathbf{t}\right])^2. \tag{4.8}$$

In our case, for $m = 4$, where $\theta \equiv (\theta_1,\theta_2,\theta_3,\theta_4) = (\alpha,\beta,a,b)$, Eq. (3.3) reduces to

$$\begin{aligned}E\left[u(\alpha,\beta,a,b)|\mathbf{t}\right] = u + (u_1a_1 + u_2a_2 + u_3a_3 + u_4a_4 + a_5 + a_6) + \frac{1}{2}\Big[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13} + u_4\sigma_{14}) \\ + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23} + u_4\sigma_{24}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33} + u_4\sigma_{34}) \\ +D(u_1\sigma_{41} + u_2\sigma_{42} + u_3\sigma_{43} + u_4\sigma_{44})\Big],\end{aligned} \tag{4.9}$$

evaluated at $\hat{\theta} = (\hat{\alpha},\hat{\beta},\hat{a},\hat{b})$, where

$$\begin{aligned}
a_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} + \rho_4\sigma_{i4}, \quad i = 1,2,3,4\\
a_5 &= (u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{14}\sigma_{14} + u_{23}\sigma_{23} + u_{24}\sigma_{24} + u_{34}\sigma_{34})\\
a_6 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33} + u_{44}\sigma_{44})\\
A &= \sigma_{11}\ell_{111} + \sigma_{22}\ell_{221} + \sigma_{33}\ell_{331} + \sigma_{44}\ell_{441} + 2\Big(\sigma_{12}\ell_{121} + \sigma_{13}\ell_{131} + \sigma_{14}\ell_{141} + \sigma_{23}\ell_{231} + \sigma_{24}\ell_{241} + \sigma_{34}\ell_{341}\Big)\\
B &= \sigma_{11}\ell_{112} + \sigma_{22}\ell_{222} + \sigma_{33}\ell_{332} + \sigma_{44}\ell_{442} + 2\Big(\sigma_{12}\ell_{122} + \sigma_{13}\ell_{132} + \sigma_{14}\ell_{142} + \sigma_{23}\ell_{232} + \sigma_{24}\ell_{242} + \sigma_{34}\ell_{342}\Big)\\
C &= \sigma_{11}\ell_{113} + \sigma_{22}\ell_{223} + \sigma_{33}\ell_{333} + \sigma_{44}\ell_{443} + 2\Big(\sigma_{12}\ell_{123} + \sigma_{13}\ell_{133} + \sigma_{14}\ell_{143} + \sigma_{23}\ell_{233} + \sigma_{24}\ell_{243} + \sigma_{34}\ell_{343}\Big)\\
D &= \sigma_{11}\ell_{114} + \sigma_{22}\ell_{224} + \sigma_{33}\ell_{334} + \sigma_{44}\ell_{444} + 2\Big(\sigma_{12}\ell_{124} + \sigma_{13}\ell_{134} + \sigma_{14}\ell_{144} + \sigma_{23}\ell_{234} + \sigma_{24}\ell_{244} + \sigma_{34}\ell_{344}\Big)
\end{aligned}$$

and the subscripts $1,2,3,4$ on the right-hand sides refer to α,β,a,b respectively.

For the prior distribution (4.5),
Case $\delta = 1$, we have

$$\rho = \ln\pi_1(\alpha,\beta,a,b) = \text{constant} - \left[\ln\alpha + (a_2+1)\ln\beta + \frac{b_2}{\beta} + \ln a + \ln b\right].$$

Therefore,

$$\rho_1 = \frac{\partial\rho}{\partial\alpha} = -\frac{1}{\alpha}, \;\; \rho_2 = \frac{\partial\rho}{\partial\beta} = -\frac{(a_2+1)}{\beta} + \frac{b_2}{\beta^2}, \;\; \rho_3 = \frac{\partial\rho}{\partial a} = -\frac{1}{a}, \;\; \rho_4 = \frac{\partial\rho}{\partial b} = -\frac{1}{b}.$$

Case $\delta = 0$, we have

$$\rho = \ln\pi_2(\alpha,\beta,a,b) = \text{constant} - \left[2(a_1+1)\ln\alpha + \frac{b_1}{\alpha^2} + \ln\beta + \ln a + \ln b\right].$$

Therefore,

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = -\frac{2(a_1+1)}{\alpha} + \frac{2b_1}{\alpha^3},\ \ \rho_2 = \frac{\partial \rho}{\partial \beta} = -\frac{1}{\beta},\ \ \rho_3 = \frac{\partial \rho}{\partial a} = -\frac{1}{a},\ \ \rho_4 = \frac{\partial \rho}{\partial b} = -\frac{1}{b}.$$

The derived $\ell_{ij},\ i,j = 1,2,3,4$ and $\ell_{ijk},\ i,j,k = 1,2,3,4$ are given in the Appendix A. Now, we obtain the Bayes estimates of α, β, a and b under the following loss functions:

1. SE loss function

   (a) If $u(\alpha,\beta,a,b) = \alpha$, substituting it in (4.9), we get

   $$\hat{\alpha}_{BS} = \alpha + a_1 + \frac{1}{2}\big[A\sigma_{11} + B\sigma_{21} + C\sigma_{31} + D\sigma_{41}\big]. \tag{4.10}$$

   (b) If $u(\alpha,\beta,a,b) = \beta$, substituting it in (4.9), we get

   $$\hat{\beta}_{BS} = \beta + a_2 + \frac{1}{2}\big[A\sigma_{12} + B\sigma_{22} + C\sigma_{32} + D\sigma_{42}\big]. \tag{4.11}$$

   (c) If $u(\alpha,\beta,a,b) = a$, substituting it in (4.9), we get

   $$\hat{a}_{BS} = a + a_3 + \frac{1}{2}\big[A\sigma_{13} + B\sigma_{23} + C\sigma_{33} + D\sigma_{43}\big]. \tag{4.12}$$

   (d) If $u(\alpha,\beta,a,b) = b$, substituting it in (4.9), we get

   $$\hat{b}_{BS} = b + a_4 + \frac{1}{2}\big[A\sigma_{14} + B\sigma_{24} + C\sigma_{34} + D\sigma_{44}\big]. \tag{4.13}$$

   Also, let $u(\alpha,\beta,a,b) = \alpha^2$, substituting it in (4.9), we get

   $$E\left[\hat{\alpha}^2_{BS}|\mathbf{t}\right] = \alpha^2 + 2\alpha a_1 + \sigma_{11} + \alpha\big[A\sigma_{11} + B\sigma_{21} + C\sigma_{31} + D\sigma_{41}\big]. \tag{4.14}$$

   Hence the posterior variance from (4.8) is

   $$\begin{aligned} Var\left[\hat{\alpha}_{BS}|\mathbf{t}\right] &= E\left[\hat{\alpha}^2_{BS}|\mathbf{t}\right] - \left(E\left[\hat{\alpha}_{BS}|\mathbf{t}\right]\right)^2 \\ &= \sigma_{11} - \left(a_1 + \frac{1}{2}\big[A\sigma_{11} + B\sigma_{21} + C\sigma_{31} + D\sigma_{41}\big]\right)^2 \\ &< \hat{\sigma}_{11}. \end{aligned} \tag{4.15}$$

   Similarly,

   $$Var[\hat{\beta}_{BS}|\mathbf{t}] < \hat{\sigma}_{22}, \quad Var\left[\hat{a}_{BS}|\mathbf{t}\right] < \hat{\sigma}_{33}, \quad Var[\hat{b}_{BS}|\mathbf{t}] < \hat{\sigma}_{44}.$$

2. LINEX loss function

(a) If $u(\alpha,\beta,a,b) = \exp(-p\alpha)$, substituting it in (4.9), then the BL estimate of α from (2.6) is

$$\hat{\alpha}_{BL} = \alpha - \frac{1}{p}\ln\left\{1 - p(a_1 - \frac{p\sigma_{11}}{2} + \frac{1}{2}[A\sigma_{11} + B\sigma_{21} + C\sigma_{31} + D\sigma_{41}])\right\}. \quad (4.16)$$

(b) If $u(\alpha,\beta,a,b) = \exp(-p\beta)$, substituting it in (4.9), then the BL estimate of β from (2.6) is

$$\hat{\beta}_{BL} = \beta - \frac{1}{p}\ln\left\{1 - p\left(a_2 - \frac{p\sigma_{22}}{2} + \frac{1}{2}\left[A\sigma_{12} + B\sigma_{22} + C\sigma_{32} + D\sigma_{42}\right]\right)\right\}. \quad (4.17)$$

(c) If $u(\alpha,\beta,a,b) = \exp(-pa)$, substituting it in (4.9), then the BL estimate of a from (2.6) is

$$\hat{a}_{BL} = a - \frac{1}{p}\ln\left\{1 - p\left(a_3 - \frac{p\sigma_{33}}{2} + \frac{1}{2}\left[A\sigma_{13} + B\sigma_{23} + C\sigma_{33} + D\sigma_{43}\right]\right)\right\}. \quad (4.18)$$

(d) If $u(\alpha,\beta,a,b) = \exp(-pb)$, substituting it in (4.9), then the BL estimate of b from (2.6) is

$$\hat{b}_{BL} = b - \frac{1}{p}\ln\left\{1 - p\left(a_4 - \frac{p\sigma_{44}}{2} + \frac{1}{2}\left[A\sigma_{14} + B\sigma_{24} + C\sigma_{34} + D\sigma_{44}\right]\right)\right\}. \quad (4.19)$$

3. GE loss function

(a) If $u(\alpha,\beta,a,b) = \alpha^{-q}$, substituting it in (4.9), then the BG estimate of α from (2.8) is

$$\hat{\alpha}_{BG} = \alpha\{1 - \frac{q}{\alpha}(a_1 - \frac{(q+1)\sigma_{11}}{2\alpha} + \frac{1}{2}[A\sigma_{11} + B\sigma_{21} + C\sigma_{31} + D\sigma_{41}])\}^{-\frac{1}{q}}. \quad (4.20)$$

(b) If $u(\alpha,\beta,a,b) = \beta^{-q}$, substituting it in (4.9), then the BG estimate of β from (2.8) is

$$\hat{\beta}_{BG} = \beta\{1 - \frac{q}{\beta}(a_2 - \frac{(q+1)\sigma_{22}}{2\beta} + \frac{1}{2}[A\sigma_{12} + B\sigma_{22} + C\sigma_{32} + D\sigma_{42}])\}^{-\frac{1}{q}}. \quad (4.21)$$

(c) If $u(\alpha,\beta,a,b) = a^{-q}$, substituting it in (4.9), then the BG estimate of a from (2.8) is

$$\hat{a}_{BG} = a\{1 - \frac{q}{a}(a_3 - \frac{(q+1)\sigma_{33}}{2a} + \frac{1}{2}[A\sigma_{13} + B\sigma_{23} + C\sigma_{33} + D\sigma_{43}])\}^{-\frac{1}{q}}. \quad (4.22)$$

(d) If $u(\alpha,\beta,a,b) = b^{-q}$, substituting it in (4.9), then the BG estimate of b from (2.8) is

$$\hat{b}_{BG} = b\{1 - \frac{q}{b}(a_4 - \frac{(q+1)\sigma_{44}}{2b} + \frac{1}{2}[A\sigma_{14} + B\sigma_{24} + C\sigma_{34} + D\sigma_{44}])\}^{-\frac{1}{q}}. \quad (4.23)$$

To estimate the reliability function $R(t)$ under the SE loss function, we substitute $u(\alpha, \beta, a, b) = R(t) = 1 - F(t)$ in (4.9), where $F(t)$ given in (1.1).

In (4.9), if $u(\alpha, \beta, a, b) = \exp[-pR(t)]$ and $u(\alpha, \beta, a, b) = R(t)^{-q}$, then the BL and BG estimates of the reliability function $R(t)$ are given from (2.6) and (2.8), respectively.

On differentiating $u(\alpha, \beta, a, b)$ with respect to the parameters α, β, a and b under the SE, LINEX and GE loss functions, the derived u_i and u_{ij}, $i, j = 1, 2, 3, 4$ are given in Appendix B.

Keep in mind that the BS estimators given by Eq. (4.10) to (4.13), (4.14) and (4.15), BL estimators given by Eq. (4.16) to (4.19), BG estimators given by Eq. (4.20) to (4.23) and Bayes estimators of the reliability function are evaluated at $(\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})$.

5 Illustrative Example

To illustrate our results given in this paper we use a real data set from Proschan [19] consisting of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. This set has been used by Cordeiro and Lemonte [7] to show that the β-BS distribution fits the lifetime data better than the BS distribution. The data are presented in Table 1.

Table 1: Number of successive failures for the air conditioning system

194	413	90	74	55	23	97	50	359	50	130	487	57
102	15	14	10	57	320	261	51	44	9	254	493	33
18	209	41	58	60	48	56	87	11	102	12	5	14
14	29	37	186	29	104	7	4	72	270	283	7	61
100	61	502	220	120	141	22	603	35	98	54	100	11
181	65	49	12	239	14	18	39	3	12	5	32	9
438	43	134	184	20	386	182	71	80	188	230	152	5
36	79	59	33	246	1	79	3	27	201	84	27	156
21	16	88	130	14	118	44	15	42	106	46	230	26
59	153	104	20	206	5	66	34	29	26	35	5	82
31	118	326	12	54	36	34	18	25	120	31	22	18
216	139	67	310	3	46	210	57	76	14	111	97	62
39	30	7	44	11	63	23	22	23	14	18	13	34
16	18	130	90	163	208	1	24	70	16	101	52	208
95	62	11	191	14	71							

The MLEs for the parameters of the β-BS and BS distributions as obtained by Cordeiro and Lemonte [7] and we obtained the MLEs for the parameters of the EBS distribution using a Newton-

Raphson method. The MLE of the reliability function of the β-BS distribution is given after replacing α, β, a and b by $\hat{\alpha}, \hat{\beta}, \hat{a}$ and $\hat{b}$. For given values of hyperparameters $(a_1 = a_2 = b_1 = b_2 = 0.01)$, the approximate BS, BL and BG estimators for the parameters of the β-BS distribution are computed using (4.10)-(4.13), (4.16)-(4.19) and (4.20)-(4.23) under the two scenarios for the joint prior densities (4.4), respectively. Also we computed the Bayes estimators for the parameters of the EBS and BS distributions as special cases. All the computations were done using the Mathcad software.

The MLEs and Bayes estimates (BS, BL, BG) for the parameters of the β-BS, EBS and BS distributions are listed in Table 2, Table 3 and Table 4, respectively, with the variance of MLEs and the posterior risk of the SE, LINEX and GE loss functions (in parenthesis). Table 5 contains the MLE and Bayes estimates (BS, BL, BG) of the reliability function of the β-BS distribution with the posterior risk of three loss functions (in parenthesis).

From Tables 2, 3, 4 and 5 some features can be summarized as follows:

1. Both priors perform well.

2. To show the effect of the shape parameters p and q, we observed that the asymmetric Bayes estimates (BL, BG) of the parameters of the β-BS distribution and sub-models (EBS and BS distributions): for $(p > 0, q > -1)$ are underestimates, $(p < 0, q < -1)$ are overestimates and $(p \approxeq 0, q = -1)$ are the same as the SE loss Bayes estimates.

3. The performance of the Lindley's procedure decreases as the number of parameters increases.

4. The approximate BS estimates of the reliability function at $t = 5, 10, 30$ are smaller than (larger than) MLE when $\delta = 1$ $(\delta = 0)$ and its (BL, BG) estimates are smaller than or larger than MLE depending on p and q. Increasing the values p and q result in decreasing values of the reliability estimates. Noted that, at $t = 50$ or greater the Bayes estimates of the reliability function are larger than MLE.

5. On comparing the variance of MLEs and the posterior variance estimates of SE loss function, the BS estimates have smaller variance than the MLEs.

6. The posterior risk of the LINEX loss function decreasing when p approach to zero.

7. The posterior risk of the GE loss function increasing when q is positive and its value increase and when q is negative the posterior risk decreasing for $q > -1$ and increasing for $q < -1$.

8. As example for $(p \approxeq 0, q = -1)$ to compare the posterior risk of SE, LINEX and GE loss functions, we observed that the LINEX loss function has less posterior risk than the other loss functions, so we conclude that LINEX loss function is more preferable loss function.

6 Concluding Remarks

In this article, we have studied the Bayesian inference procedure for the parameters and reliability function of the β-BS distribution under the symmetric (SE) and asymmetric (LINEX and GE) loss functions. Because of the Bayes estimates can not be obtained in explicit forms, we have used Lindley's approximation technique which is easy to use and does not require innovative programming and expensive computer time. It is observed that the posterior variance estimates are smaller than the variance of the MLEs. Also we observed the BL and BG estimates of the parameters and reliability function and the posterior risk of LINEX and GE loss functions affected by the shape parameters p and q. One of the useful properties of working with the LINEX and GE loss functions are the BL and BG estimates are the same BS estimates when $(p \approxeq 0, q = -1)$.

Table 2: MLEs, Bayes estimates for the parameters of the β-BS distribution and variance of MLE and posterior risk under three loss functions (in parenthesis).

	$\delta = 1$				$\delta = 0$			
Estimators	α	β	a	b	α	β	a	b
MLE	1.5487	9.5752	1.7053	0.3941	1.5487	9.5752	1.7053	0.3941
	(0.4302)	(45.4545)	(0.6507)	(0.2260)	(0.4302)	(45.4545)	(0.6507)	(0.2260)
BS	1.8907	9.4187	2.2903	0.6814	1.6119	7.9232	1.9670	0.4979
	(0.3132)	(45.4300)	(0.3085)	(0.1434)	(0.4262)	(42.7253)	(0.5822)	(0.2152)
$BL(p = 1 \times 10^{-6})$	1.8907	9.4187	2.2903	0.6814	1.6119	7.9231	1.9670	0.4979
	(1.5654×10^{-13})	(2.2715×10^{-11})	(1.5412×10^{-13})	(7.1742×10^{-14})	(2.1310×10^{-13})	(2.1363×10^{-11})	(2.9126×10^{-13})	(1.0756×10^{-13})
$BL(p = 0.5)$	1.7981	5.7531	2.1797	0.6394	1.5049	5.5433	1.8069	0.4419
	(0.0463)	(1.8328)	(0.0553)	(0.0210)	(0.0535)	(1.1899)	(0.0801)	(0.0280)
$BL(p = 1.5)$	1.5684	6.9363	1.8101	0.5238	1.3296	6.9083	1.5105	0.3314
	(0.4835)	(3.7237)	(0.7203)	(0.2365)	(0.4235)	(1.5222)	(0.6849)	(0.2497)
$BL(p = -0.5)$	1.9543	13.3504	2.3405	0.7114	1.7126	13.1101	2.0902	0.5483
	(0.0318)	(1.9658)	(0.0251)	(0.0150)	(0.0503)	(2.5935)	(0.0616)	(0.0252)
$BL(p = -1.5)$	2.0098	12.2081	2.3447	0.7420	1.8532	12.1786	2.2077	0.6231
	(0.1786)	(4.1840)	(0.0817)	(0.0909)	(0.3618)	(6.3832)	(0.3610)	(0.1879)
$BG(q = 0.5)$	1.6916	6.7155	2.0485	0.2825	1.4132	5.9163	1.6811	0.1971
	(0.0215)	(0.0453)	(0.0239)	(0.1672)	(0.0214)	(0.0305)	(0.0279)	(0.1143)
$BG(q = 1.5)$	1.5435	6.1261	1.8227	0.2066	1.3171	5.6807	1.5192	0.1766
	(0.2018)	(0.2736)	(0.2469)	(0.9711)	(0.1697)	(0.1525)	(0.2358)	(0.5076)
$BG(q = -0.5)$	1.8333	8.2791	2.2300	0.5512	1.5425	6.9468	1.8757	0.3555
	(0.0188)	(0.0594)	(0.0186)	(0.1670)	(0.0224)	(0.0498)	(0.0268)	(0.1806)
$BG(q = -1)$	1.8907	9.4187	2.2903	0.6814	1.6119	7.9232	1.9670	0.4979
	(0.0684)	(0.2478)	(0.0638)	(0.5461)	(0.0889)	(0.2310)	(0.1012)	(0.6980)
$BG(q = -1.5)$	1.9368	10.5796	2.3313	0.7527	1.6787	9.1041	2.0459	0.6131
	(0.1387)	(0.5460)	(0.1223)	(0.9682)	(0.1942)	(0.5549)	(0.2108)	(1.3594)

Table 3: MLEs, Bayes estimates for the parameters of the EBS distribution and variance of MLE and posterior risk under three loss functions (in parenthesis).

	$\delta = 1$			$\delta = 0$		
Estimators	α	β	a	α	β	a
MLE	2.1789	13.5871	2.5382	2.1789	13.5871	2.5382
	(0.0758)	(19.5211)	(0.2727)	(0.0758)	(19.5211)	(0.2727)
BS	2.2375	13.8727	2.5866	2.2014	14.4137	2.5283
	(0.0724)	(19.4395)	(0.2704)	(0.0753)	(18.8378)	(0.2726)
$BL(p = 1 \times 10^{-6})$	2.2375	13.8727	2.5866	2.2014	14.4137	2.5283
	(3.6018×10^{-14})	(9.7199×10^{-12})	(1.3534×10^{-13})	(3.7615×10^{-14})	(9.4190×10^{-12})	(1.3625×10^{-13})
$BL(p = 0.5)$	2.2190	11.2009	2.5185	2.1825	11.3721	2.4616
	(9.2756×10^{-3})	(1.3359)	(0.0340)	(9.4734×10^{-3})	(1.5208)	(0.0333)
$BL(p = 1.5)$	2.1807	11.5105	2.3979	2.1454	11.5349	2.3523
	(0.0853)	(3.5433)	(0.2830)	(0.0840)	(4.3182)	(0.2640)
$BL(p = -0.5)$	2.2550	16.1395	2.6515	2.2199	16.2850	2.5956
	(8.7418×10^{-3})	(1.1334)	(0.0325)	(9.2631×10^{-3})	(0.9357)	(0.0337)
$BL(p = -1.5)$	2.2854	15.6886	2.7526	2.2539	15.7114	2.7089
	(0.0718)	(2.7239)	(0.2490)	(0.0787)	(1.9465)	(0.2710)
$BG(q = 0.5)$	2.2118	12.8285	2.5064	2.1753	13.3396	2.4501
	(1.9677×10^{-3})	(0.0128)	(5.2711×10^{-3})	(1.9954×10^{-3})	(0.01320)	(5.1340×10^{-3})
$BG(q = 1.5)$	2.1942	12.2596	2.4558	2.1582	12.6967	2.4035
	(0.0179)	(0.1064)	(0.0464)	(0.0179)	(0.1136)	(0.0442)
$BG(q = -0.5)$	2.2291	13.5136	2.5598	2.1927	14.0586	2.5016
	(1.9306×10^{-3})	(0.0132)	(5.2821×10^{-3})	(1.9907×10^{-3})	(0.0131)	(5.2647×10^{-3})
$BG(q = -1)$	2.2375	13.8727	2.5866	2.2014	14.4137	2.5283
	(7.6274×10^{-3})	(0.0527)	(0.0210)	(7.9300×10^{-3})	(0.0511)	(0.0212)
$BG(q = -1.5)$	2.2457	14.2245	2.6129	2.2100	14.7484	2.5551
	(0.0169)	(0.1165)	(0.0467)	(0.0177)	(0.1111)	(0.0476)

Table 4: MLEs, Bayes estimates for the parameters of the BS distribution and variance of MLE and posterior risk under three loss functions (in parenthesis).

	$\delta = 1$		$\delta = 0$	
Estimators	α	β	α	β
MLE	1.5147	41.3240	1.5147	41.3240
	(6.1220×10^{-3})	(12.2075)	(6.1220×10^{-3})	(12.2075)
BS	1.5259	41.4389	1.5218	41.4524
	(5.9977×10^{-3})	(12.2075)	(6.0722×10^{-3})	(12.2042)
$BL(p = 1 \times 10^{-6})$	1.5259	41.4389	1.5218	41.4524
	(3.1587×10^{-15})	(6.1036×10^{-12})	(3.0112×10^{-15})	(6.1021×10^{-12})
$BL(p = 0.5)$	1.5243	39.5155	1.5202	39.5209
	(7.5365×10^{-4})	(0.9617)	(7.6142×10^{-4})	(0.9657)
$BL(p = 1.5)$	1.5213	39.5378	1.5172	39.5387
	(6.8385×10^{-3})	(2.8518)	(6.8804×10^{-3})	(2.8705)
$BL(p = -0.5)$	1.5273	43.2235	1.5233	43.2287
	(7.4523×10^{-4})	(0.8923)	(7.5605×10^{-4})	(0.8882)
$BL(p = -1.5)$	1.5303	43.1258	1.5262	43.1267
	(6.6128×10^{-3})	(2.5303)	(6.7363×10^{-3})	(2.5115)
$BG(q = 0.5)$	1.5229	41.2173	1.5187	41.2307
	(3.2994×10^{-4})	(8.9371×10^{-4})	(3.3266×10^{-4})	(8.9390×10^{-4})
$BG(q = 1.5)$	1.5208	41.0712	1.5167	41.0845
	(2.9836×10^{-3})	(8.0084×10^{-3})	(2.99994×10^{-3})	(8.0128×10^{-3})
$BG(q = -0.5)$	1.5249	41.365	1.5208	41.3785
	(3.2795×10^{-4})	(8.9442×10^{-4})	(3.3155×10^{-4})	(8.9433×10^{-4})
$BG(q = -1)$	1.5259	41.4389	1.5218	41.4524
	(1.3072×10^{-3})	(3.5743×10^{-3})	(1.3233×10^{-3})	(3.5734×10^{-3})
$BG(q = -1.5)$	1.5268	41.5127	1.5228	41.5261
	(2.9299×10^{-3})	(8.0275×10^{-3})	(2.9701×10^{-3})	(8.0241×10^{-3})

Table 5: MLE, Bayes estimates of the reliability function of the β-BS distribution and posterior risk under three loss functions (in parenthesis).

	$\delta = 1$				$\delta = 0$			
t	5	10	30	50	5	10	30	50
MLE	0.9462	0.8772	0.6796	0.5422	0.9462	0.8772	0.6796	0.5422
BS	0.8901	0.6981	0.5747	0.5877	1.1381	0.9085	0.6741	0.6084
	(0.4078)	(0.2676)	(0.0599)	(2.4682×10^{-3})	(0.3742)	(0.2988)	(0.0709)	(1.5804×10^{-4})
$BL(p = 1 \times 10^{-6})$	0.8901	0.6981	0.5747	0.5877	1.1381	0.9085	0.6741	0.6084
	(2.0390×10^{-11})	(1.3389×10^{-13})	(2.9976×10^{-14})	(1.1102×10^{-15})	(1.8718×10^{-13})	(1.4944×10^{-13})	(3.5527×10^{-14})	(0.0000)
$BL(p = 0.5)$	0.7934	0.6380	0.5606	0.5871	1.0374	0.8340	0.6565	0.6083
	(0.0484)	(0.0300)	(7.0626×10^{-3})	(3.1757×10^{-4})	(0.0530)	(0.0372)	(8.8014×10^{-3})	(0.6613×10^{-5})
$BL(p = 1.5)$	0.6556	0.5614	0.5378	0.5857	0.8390	0.7073	0.6233	0.6081
	(0.3518)	(0.2050)	(0.0555)	(3.0241×10^{-3})	(0.4486)	(0.3018)	(0.0761)	(3.7207×10^{-4})
$BL(p = -0.5)$	0.9923	0.7702	0.5905	0.5883	1.2210	0.9807	0.6918	0.6084
	(0.0511)	(0.0361)	(7.8908×10^{-3})	(2.9961×10^{-4})	(0.0415)	(0.0361)	(8.8502×10^{-3})	(1.3217×10^{-5})
$BL(p = -1.5)$	1.1600	0.9214	0.6258	0.5894	1.3193	1.0939	0.7257	0.6084
	(0.4049)	(0.3349)	(0.0766)	(2.5399×10^{-3})	(0.2719)	(0.2782)	(0.0774)	(9.5221×10^{-6})
$BG(q = 0.5)$	0.6552	0.5631	0.5278	0.5837	0.8253	0.6891	0.6030	0.6075
	(0.0394)	(0.0222)	(0.0109)	(1.2595×10^{-3})	(0.0550)	(0.0411)	(0.0174)	(3.4524×10^{-4})
$BG(q = 1.5)$	0.6063	0.5460	0.5142	0.5804	0.7045	0.6215	0.5705	0.6062
	(0.2344)	(0.1128)	(0.0717)	(0.0123)	(0.4023)	(0.2782)	(0.1351)	(4.1050×10^{-3})
$BG(q = -0.5)$	0.7887	0.6326	0.5550	0.5865	1.0313	0.8239	0.6484	0.6082
	(0.0533)	(0.0360)	(0.0142)	(1.1494×10^{-3})	(0.0564)	(0.0482)	(0.0189)	(2.4909×10^{-4})
$BG(q = -1)$	0.8901	0.6981	0.5747	0.5877	1.1381	0.9085	0.6741	0.6084
	(0.2276)	(0.1705)	(0.0635)	(4.3834×10^{-3})	(0.2114)	(0.1941)	(0.0768)	(8.2469×10^{-4})
$BG(q = -1.5)$	0.9980	0.7808	0.5984	0.5888	1.2268	0.9903	0.7000	0.6085
	(0.5131)	(0.4238)	(0.1557)	(9.3926×10^{-3})	(0.4296)	(0.4206)	(0.1718)	(1.4982×10^{-3})

Appendix A

From (4.1), we have

$$\ell(\mathbf{t}|\alpha,\beta,a,b) = n\ln\{k(\alpha,\beta)\} - n\ln\{B(a,b)\} - \frac{3}{2}\sum_{i=1}^{n}\ln(t_i) + \sum_{i=1}^{n}\ln(t_i+\beta) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\tau(t_i^*) + (a-1)\sum_{i=1}^{n}\ln\{\Phi(\upsilon_i)\} + (b-1)\sum_{i=1}^{n}\ln\{1-\Phi(\upsilon_i)\}.$$

Therefore, we obtain

$$\ell_1 = -\frac{n}{\alpha}\left(1+\frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3}\sum_{i=1}^{n}\tau(t_i^*) - \frac{1}{\alpha}\sum_{i=1}^{n}\upsilon_i\,\phi(\upsilon_i)\left\{\frac{(a-1)}{\Phi(\upsilon_i)} - \frac{(b-1)}{[1-\Phi(\upsilon_i)]}\right\},$$

$$\ell_2 = -\frac{n}{2\beta} + \sum_{i=1}^{n}\frac{1}{t_i+\beta} + \frac{1}{2\alpha^2\beta}\sum_{i=1}^{n}\rho(t_i^{*2}) - \frac{1}{2\alpha\beta}\sum_{i=1}^{n}\tau\left(\sqrt{t_i^*}\right)\phi(\upsilon_i)\left\{\frac{(a-1)}{\Phi(\upsilon_i)} - \frac{(b-1)}{[1-\Phi(\upsilon_i)]}\right\},$$

$$\ell_3 = -\,n\{\psi(a)-\psi(a+b)\} + \sum_{i=1}^{n}\ln\Big\{\Phi(\upsilon_i)\Big\},$$

$$\ell_4 = -\,n\{\psi(b)-\psi(a+b)\} + \sum_{i=1}^{n}\ln\Big\{1-\Phi(\upsilon_i)\Big\},$$

$$\ell_{11} = \frac{n}{\alpha^2} + \frac{6n}{\alpha^4} - \frac{3}{\alpha^4}\sum_{i=1}^{n}\tau(t_i^*) - \frac{(a-1)}{\alpha^2}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\upsilon_i^2-2\right) + \frac{\upsilon_i^2\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\} + \frac{(b-1)}{\alpha^2}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i^2-2\right) - \frac{\upsilon_i^2\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\},$$

$$\ell_{12} = -\frac{1}{\alpha^3\beta}\sum_{i=1}^{n}\rho(t_i^{*2}) - \frac{(a-1)}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\upsilon_i^2-1\right) + \frac{\upsilon_i\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\} + \frac{(b-1)}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i^2-1\right) - \frac{\upsilon_i\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\},$$

$$\ell_{13} = -\frac{1}{\alpha}\sum_{i=1}^{n}\frac{\upsilon_i\phi(\upsilon_i)}{\Phi(\upsilon_i)}, \qquad \ell_{14} = \frac{1}{\alpha}\sum_{i=1}^{n}\frac{\upsilon_i\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]},$$

$$\ell_{22} = \frac{n}{2\beta^2} - \sum_{i=1}^{n}\frac{1}{(t_i+\beta)^2} - \frac{1}{\alpha^2\beta^3}\sum_{i=1}^{n}t_i + \frac{(a-1)}{4\alpha^2\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(-\upsilon_i\tau(\sqrt{t_i^*})^2 + \alpha^2\upsilon_i + 2\alpha\tau(\sqrt{t_i^*})\right) - \frac{\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\} + \frac{(b-1)}{4\alpha^2\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i\tau(\sqrt{t_i^*})^2 - \alpha^2\upsilon_i - 2\alpha\tau(\sqrt{t_i^*})\right) - \frac{\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\},$$

$$\ell_{23} = -\frac{1}{2\alpha\beta}\sum_{i=1}^{n}\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{\Phi(\upsilon_i)}, \ell_{24} = \frac{1}{2\alpha\beta}\sum_{i=1}^{n}\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}, \ell_{33} = -n\{\psi^{(1)}(a) - \psi^{(1)}(a+b)\},$$

$$\ell_{34} = n\,\psi^{(1)}(a+b), \ell_{44} = -n\{\psi^{(1)}(b) - \psi^{(1)}(a+b)\},$$

$$\ell_{111} = -\frac{2n(\alpha^2+12)}{\alpha^5} + \frac{12}{\alpha^5}\sum_{i=1}^{n}\tau(t_i^*) - \frac{(a-1)}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\upsilon_i^4 - 7\upsilon_i^2 + 6\right) + \frac{3\upsilon_i^2\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\left(\upsilon_i^2 - 2\right) + \frac{2\upsilon_i^3\phi(\upsilon_i)^3}{\Phi(\upsilon_i)^3}\right\}$$
$$+ \frac{(b-1)}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i^4 - 7\upsilon_i^2 + 6\right) - \frac{3\upsilon_i^2\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\left(\upsilon_i^2 - 2\right) + \frac{2\upsilon_i^3\phi(\upsilon_i)^3}{[1-\Phi(\upsilon_i)]^3}\right\},$$

$$\ell_{112} = \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{t_i}{\beta^2} - \frac{1}{t_i}\right) - \frac{(a-1)}{\alpha^3\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\frac{1}{2}\upsilon_i^4 - \frac{5}{2}\upsilon_i^2 + 1\right) + \frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\left(\frac{3}{2}\upsilon_i^3 - 2\upsilon_i\right)\right.$$
$$\left. + \frac{\upsilon_i^2\tau(\sqrt{t_i^*})\phi(\upsilon_i)^3}{\Phi(\upsilon_i)^3}\right\} + \frac{(b-1)}{\alpha^3\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\frac{1}{2}\upsilon_i^4 - \frac{5}{2}\upsilon_i^2 + 1\right) - \frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\left(\frac{3}{2}\upsilon_i^3 - 2\upsilon_i\right)\right.$$
$$\left. + \frac{\upsilon_i^2\tau(\sqrt{t_i^*})\phi(\upsilon_i)^3}{[1-\Phi(\upsilon_i)]^3}\right\},$$

$$\ell_{113} = -\frac{1}{\alpha^2}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\upsilon_i^2 - 2\right) + \frac{\upsilon_i^2\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\}, \ell_{114} = \frac{1}{\alpha^2}\sum_{i=1}^{n}\left\{\frac{\upsilon_i\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i^2 - 2\right) - \frac{\upsilon_i^2\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\},$$

$$\ell_{221} = \frac{2}{\alpha^3\beta^3}\sum_{i=1}^{n}t_i + \frac{(a-1)}{4\alpha^3\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(-\upsilon_i\tau(\sqrt{t_i^*})^2(\upsilon_i^2 - 3) + \alpha(\alpha\upsilon_i + 2\tau(\sqrt{t_i^*}))(\upsilon_i^2 - 1)\right) + \frac{\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right.$$
$$\left. \times\left(-\tau(\sqrt{t_i^*})^2(3\upsilon_i^2 - 2) + \alpha\upsilon_i(\alpha\upsilon_i + 2\tau(\sqrt{t_i^*}))\right) - \frac{2\upsilon_i\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^3}{\Phi(\upsilon_i)^3}\right\}$$
$$+ \frac{(b-1)}{4\alpha^3\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i\tau(\sqrt{t_i^*})^2(\upsilon_i^2 - 3) - \alpha(\alpha\upsilon_i + 2\tau(\sqrt{t_i^*}))(\upsilon_i^2 - 1)\right) + \frac{\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right.$$
$$\left. \times\left(-\tau(\sqrt{t_i^*})^2(3\upsilon_i^2 - 2) + \alpha\upsilon_i(\alpha\upsilon_i + 2\tau(\sqrt{t_i^*}))\right) + \frac{2\upsilon_i\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^3}{[1-\Phi(\upsilon_i)]^3}\right\},$$

$$\ell_{222} = -\frac{n}{\beta^3} + 2\sum_{i=1}^{n}\frac{1}{(t_i+\beta)^3} + \frac{3}{\alpha^2\beta^4}\sum_{i=1}^{n} t_i + \frac{(a-1)}{8\alpha^3\beta^3}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(-\tau(\sqrt{t_i^*})^3(\upsilon_i^2-1) + 3\alpha^2\tau(\sqrt{t_i^*})(\upsilon_i^2-3)\right.\right.$$

$$\left.\left. + 6\alpha\upsilon_i(\tau(\sqrt{t_i^*})^2-\alpha^2)\right) + \frac{3\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\left(-\upsilon_i\tau(\sqrt{t_i^*})^2 + 2\alpha\tau(\sqrt{t_i^*}) + \alpha^2\upsilon_i\right) - \frac{2\tau(\sqrt{t_i^*})^3\phi(\upsilon_i)^3}{\Phi(\upsilon_i)^3}\right\}$$

$$+ \frac{(b-1)}{8\alpha^3\beta^3}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\tau(\sqrt{t_i^*})^3(\upsilon_i^2-1) - 3\alpha^2\tau(\sqrt{t_i^*})(\upsilon_i^2-3) - 6\alpha\upsilon_i(\tau(\sqrt{t_i^*})^2-\alpha^2)\right)\right.$$

$$\left. + \frac{3\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\left(-\upsilon_i\tau(\sqrt{t_i^*})^2 + 2\alpha\tau(\sqrt{t_i^*}) + \alpha^2\upsilon_i\right) + \frac{2\tau(\sqrt{t_i^*})^3\phi(\upsilon_i)^3}{[1-\Phi(\upsilon_i)]^3}\right\},$$

$$\ell_{223} = \frac{1}{4\alpha^2\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(-\upsilon_i\tau(\sqrt{t_i^*})^2 + \alpha^2\upsilon_i + 2\alpha\tau(\sqrt{t_i^*})\right) - \frac{\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\},$$

$$\ell_{224} = \frac{1}{4\alpha^2\beta^2}\sum_{i=1}^{n}\left\{\frac{\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i\tau(\sqrt{t_i^*})^2 - \alpha^2\upsilon_i - 2\alpha\tau(\sqrt{t_i^*})\right) - \frac{\tau(\sqrt{t_i^*})^2\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\},$$

$$\ell_{331} = 0, \ell_{332} = 0, \ell_{333} = -\,n\{\psi^2(a) - \psi^2(a+b)\}, \ell_{334} = n\psi^2(a+b), \ell_{441} = 0, \ell_{442} = 0,$$

$$\ell_{443} = n\psi^2(a+b), \ell_{444} = -\,n\{\psi^2(b) - \psi^2(a+b)\},$$

$$\ell_{123} = -\frac{1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{\Phi(\upsilon_i)}\left(\upsilon_i^2-1\right) + \frac{\upsilon_i\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{\Phi(\upsilon_i)^2}\right\},$$

$$\ell_{124} = \frac{1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\tau(\sqrt{t_i^*})\phi(\upsilon_i)}{[1-\Phi(\upsilon_i)]}\left(\upsilon_i^2-1\right) - \frac{\upsilon_i\tau(\sqrt{t_i^*})\phi(\upsilon_i)^2}{[1-\Phi(\upsilon_i)]^2}\right\}, \ell_{134} = 0, \ell_{234} = 0,$$

where $t_i^* = (t_i/\beta)$, $\phi(\cdot)$ is the standard normal density function, $\psi(.)$ is the digamma function, $\psi^{(1)}(.)$ is the second of the polygamma functions (trigamma function) and $\psi^{(2)}(.)$ is the third of the polygamma functions. The $\hat{\sigma}_{ij}$ are the variance-covariance matrix of $(\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})$ are given as the elements in the inverse of the matrix $\{-\ell_{ij}\}$ all evaluated at $(\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})$. Putting $a = b = 1$ and $b = 1$ in the above equations, we can get the first, second and third derivatives from the BS and EBS distributions, respectively.

Appendix B

- **SE loss function**: If $u(\alpha, \beta, a, b) = R(t) = \left[1 - I_{\Phi(\upsilon)}(a, b)\right]$.

Therefore, we obtain

$$u_1 = M, u_2 = N, u_3 = F_a, u_4 = -F_b, u_{11} = \frac{\upsilon\, \phi(\upsilon) M}{\alpha}\left\{\frac{(b-1)}{[1-\Phi(\upsilon)]} - \frac{(a-1)}{\Phi(\upsilon)}\right\} + \frac{M}{\alpha}\left\{\upsilon^2 - 2\right\},$$

$$u_{12} = \frac{\upsilon\, \phi(\upsilon) N}{\alpha}\left\{\frac{(b-1)}{[1-\Phi(\upsilon)]} - \frac{(a-1)}{\Phi(\upsilon)}\right\} + \frac{N}{\alpha}\left\{\upsilon^2 - 1\right\},$$

$$u_{13} = M\left\{\ln\{\Phi(\upsilon)\} - \psi(a) + \psi(a+b)\right\}, u_{14} = -M\left\{-\ln\{1-\Phi(\upsilon)\} + \psi(b) - \psi(a+b)\right\},$$

$$u_{22} = -\frac{1}{\beta}\left\{\frac{\alpha M}{4\beta} + N\right\} - \frac{\tau(\sqrt{t^*})\phi(\upsilon) N}{2\beta\alpha}\left\{\frac{(a-1)}{\Phi(\upsilon)} - \frac{(b-1)}{[1-\Phi(\upsilon)]}\right\} + \frac{M}{\alpha}\left\{\frac{\tau(\sqrt{t^*})}{2\beta}\right\}^2,$$

$$u_{23} = N\left\{\ln\{\Phi(\upsilon)\} - \psi(a) + \psi(a+b)\right\}, u_{24} = -N\left\{-\ln\{1-\Phi(\upsilon)\} + \psi(b) - \psi(a+b)\right\},$$

$$\begin{aligned}
u_{33} = {} & \frac{2\Phi(\upsilon)^a\, \Gamma(a)^2\, {}_3\tilde{F}_2(\Phi(\upsilon))}{B(a,b)}\left\{\ln\{\Phi(\upsilon)\} + \psi(a+b)\right\} \\
& - I_{\Phi(\upsilon)}(a,b)\left\{\left[\ln\{\Phi(\upsilon)\} - \psi(a) + \psi(a+b)\right]^2 + \left[\psi^{(1)}(a+b) - \psi^{(1)}(a)\right]\right\} \\
& + \frac{\Phi(\upsilon)^a\, \Gamma(a)^2}{B(a,b)}\left\{{}_3\tilde{F}_2^{(\{0,0,0\},\{0,1\},0)}(\Phi(\upsilon)) + {}_3\tilde{F}_2^{(\{0,0,0\},\{1,0\},0)}(\Phi(\upsilon))\right. \\
& \left. + {}_3\tilde{F}_2^{(\{0,1,0\},\{0,0\},0)}(\Phi(\upsilon)) + {}_3\tilde{F}_2^{(\{1,0,0\},\{0,0\},0)}(\Phi(\upsilon))\right\},
\end{aligned}$$

$$\begin{aligned}
u_{34} = {} & -F_b\left\{\ln\{\Phi(\upsilon)\} - \psi(a) + \psi(a+b)\right\} - I_{\Phi(\upsilon)}(a,b)\, \psi^{(1)}(a+b) \\
& - \frac{\Phi(\upsilon)^a\, \Gamma(a)^2}{B(a,b)}\left\{\left[\psi(b) - \psi(a+b)\right]{}_3\tilde{F}_2(\Phi(\upsilon)) + {}_3\tilde{F}_2^{(\{0,0,1\},\{0,0\},0)}(\Phi(\upsilon))\right\},
\end{aligned}$$

$$\begin{aligned}
u_{44} = {} & -\frac{2[1-\Phi(\upsilon)]^b\, \Gamma(b)^2\, {}_3\tilde{F}_2(1-\Phi(\upsilon))}{B(a,b)}\left\{\ln\{1-\Phi(\upsilon)\} + \psi(a+b)\right\} \\
& + I_{1-\Phi(\upsilon)}(b,a)\left\{\left[-\ln\{1-\Phi(\upsilon)\} + \psi(b) - \psi(a+b)\right]^2 - \left[\psi^{(1)}(b) - \psi^{(1)}(a+b)\right]\right\} \\
& - \frac{[1-\Phi(\upsilon)]^b\, \Gamma(b)^2}{B(a,b)}\left\{{}_3\tilde{F}_2^{(\{0,0,0\},\{0,1\},0)}(1-\Phi(\upsilon)) + {}_3\tilde{F}_2^{(\{0,0,0\},\{1,0\},0)}(1-\Phi(\upsilon))\right. \\
& \left. {}_3\tilde{F}_2^{(\{0,1,0\},\{0,0\},0)}(1-\Phi(\upsilon)) + {}_3\tilde{F}_2^{(\{1,0,0\},\{0,0\},0)}(1-\Phi(\upsilon))\right\},
\end{aligned}$$

where

$$M = \left\{\frac{\upsilon_r\, \phi(\upsilon_r)\, \Phi(\upsilon_r)^{a-1}\, [1-\Phi(\upsilon_r)]^{b-1}}{\alpha B(a,b)}\right\}, \quad N = \left\{\frac{\tau\left(\sqrt{t_r^*}\right)\phi(\upsilon_r)\Phi(\upsilon_r)^{a-1}\, [1-\Phi(\upsilon_r)]^{b-1}}{2\beta\alpha B(a,b)}\right\},$$

$$F_a = \left\{ \frac{\Phi(\upsilon_r)^a \, \Gamma(a)^2 \, {}_3\tilde{F}_2(\Phi(\upsilon_r))}{B(a,b)} - I_{\Phi(\upsilon_r)}(a,b) \Big[\ln \{\Phi(\upsilon_r)\} - \psi(a) + \psi(a+b) \Big] \right\}$$

$$F_b = \frac{[1-\Phi(\upsilon_r)]^b \, \Gamma(b)^2 \, {}_3\tilde{F}_2(1-\Phi(\upsilon_r)) + B_{1-\Phi(\upsilon_r)}(b,a) \Big[-\ln \{1-\Phi(\upsilon_r)\} + \psi(b) - \psi(a+b) \Big]}{B(a,b)}$$

$${}_p\tilde{F}_q^{(\{1,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \frac{\partial \, {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\partial a_1}$$

$${}_p\tilde{F}_q^{(\{0,\ldots,0\},\{1,0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \frac{\partial \, {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\partial b_1}.$$

The differentiation of regularized hypergeometric function can be found in the Wolfram website. [1]

- **LINEX loss function**: If $u(\alpha,\beta,a,b) = \exp[-pR(t)]$. Therefore, we obtain

$$u_1 = -p\left(\frac{\partial R(t)}{\partial \alpha}\right)\exp[-pR(t)], \text{ similarly } u_i, \quad i = 2,3,4$$

$$u_{11} = p\exp[-pR(t)]\left\{p\left(\frac{\partial R(t)}{\partial \alpha}\right)^2 - \left(\frac{\partial^2 R(t)}{\partial \alpha^2}\right)\right\}, \text{ similarly } u_{ij}, \quad i,j = 1,2,3,4.$$

- **GE loss function**: If $u(\alpha,\beta,a,b) = R(t)^{-q}$.

  Therefore, we obtain

$$u_1 = -qR(t)^{-(q+1)}\left(\frac{\partial R(t)}{\partial \alpha}\right), \text{ similarly } u_i, \quad i = 2,3,4$$

$$u_{11} = qR(t)^{-(q+1)}\left\{\frac{(q+1)}{R(t)}\left(\frac{\partial R(t)}{\partial \alpha}\right)^2 - \left(\frac{\partial^2 R(t)}{\partial \alpha^2}\right)\right\}, \text{ similarly } u_{ij}, \quad i,j = 1,2,3,4.$$

and the subscripts $1,2,3,4$ on the left-hand sides refer to α,β,a,b respectively.

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