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A BIVARIATE VERSION OF THE HYPER-POISSON DISTRIBUTION AND SOME OF ITS PROPERTIES

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SUMMARY

A bivariate version of the hyper-Poisson distribution is introduced here through its probability generating function (*pgf*). we study some of its important aspects by deriving its probability mass function, factorial moments, marginal and conditional distributions and obtain certain recurrence relations for its probabilities, raw moments and factorial moments. Further, the method of maximum likelihood is discussed and the procedures are illustrated using a real life data set.

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1 Introduction

Bivariate discrete distributions have received a great deal of attention in the literature. For details see Kumar (2008), Kocherlakota and Kocherlakota (1992) and references therein. Bardwell and Crow (1964) studied the hyper-Poisson distribution (HP distribution), which they defined as follows. A random variable X is said to follow an HP distribution if it has the following probability mass function (*pmf*) , for $x = 0, 1, \ldots$

$$g(x) = P(X = x) = \frac{\theta^x \, \Gamma(\lambda)}{\phi(1; \lambda; \theta) \, \Gamma(\lambda + x)}, \tag{1.1}$$

in which λ, θ are positive real numbers and $\phi(1; \lambda; \theta)$ is the confluent hypergeometric series (for details see Mathai and Saxena, 1973 or Slater, 1960). The probability generating function (*pgf*) of the HP distribution with *pmf* (1.1) is the following

$$G(t) = \frac{\phi(1; \lambda; \theta t)}{\phi(1; \lambda; \theta)}, \tag{1.2}$$

which reduces to Poisson distribution when $\lambda = 1$ and when λ is a positive integer, the distribution is known as the displaced Poisson distribution studied by Staff (1964). Bardwell and Crow (1964) termed the distribution as sub-Poisson when $\lambda < 1$ and super-Poisson when $\lambda > 1$. Bardwell and Crow (1964) and Crow and Bardwell (1965) considered various methods of estimation of the parameters of the distribution. Some queuing theory with hyper-Poisson arrivals has been developed by Nisida (1962) and certain results on moments of hyper-Poisson distribution has been studied in Ahmad (1979). Roohi and Ahmad (2003a) discussed the estimation of the parameters of the hyper-Poisson distribution using negative moments. Roohi and Ahmad (2003b) obtained certain recurrence relations for negative moments and ascending factorial moments of the HP distribution. Kemp (2002) developed q-analogue of the HP distribution and Ahmad (2007) introduced and studied Conway-Maxwell hyper-Poisson distribution. Kumar and Nair (2011, 2012a, 2012b) introduced modified versions of the HP distribution and discussed some of their applications.

Ahmad (1981) introduced a bivariate version of the HP distribution through the following *pgf*

$$Q(t_1, t_2) = (\phi_1 \phi_2)^{-1} \exp[\theta(t_1 - 1)(t_2 - 1)]\phi_1[1; \lambda_1; \theta_1 t_1]\phi_2[1; \lambda_2; \theta_2 t_2], \tag{1.3}$$

in which $\phi_i = \phi(1; \lambda_i; \theta_i)$. For $r \geq 0$, $s \geq 0$, the *pmf* $q(r, s) = P(Z_1 = r, Z_2 = s)$ of $Z = (Z_1, Z_2)$ with *pgf* (1.3) is the following

$$q(r, s) = \frac{e^{\theta}\Gamma(\lambda_1)\Gamma(\lambda_2)}{\phi_1\,\phi_2} \sum_{i=0}^{\min(r,s)} \sum_{j=0}^{r-i} \sum_{k=0}^{s-i} \frac{(-1)^{j+k}\theta_1^{r-i-j}\theta_2^{s-i}\theta^{i+j+k}}{\Gamma(\lambda_1 + r - i - j)\Gamma(\lambda_2 + s - i - k)i!j!k!}, \tag{1.4}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$ and $0 < \theta \leq min(\theta_1/\lambda_1, \theta_2/\lambda_2)$.

Through the present paper we introduce another bivariate version of the HP distribution, which we named as 'the bivariate hyper-Poisson distribution ($BHPD$)'and obtain its important properties. In section 2, it is shown that the $BHPD$ possess a random sum structure. Further we obtain its conditional probability distribution, probability mass function and factorial moments in section 2. In section 3, we develop certain recursion formulae for probabilities, raw moments and factorial moments of the $BHPD$ and in section 4 we discuss the estimation of the parameters of the $BHPD$ by the method of maximum likelihood and the distribution has been fitted to a well-known data set and it is observed that the $BHPD$ gives better fit than the bivariate Poisson distribution and the bivariate hyper-Poisson distribution of Ahmad (1981).

Note that the bivariate version of HP distribution introduced in this paper is relatively simple in terms of its *pmf* and *pgf* compared to the bivariate version due to Ahmad(1981), and further this bivariate form possess a bivariate random sum structure as given in section 2. The random sum structure arises in several areas of research such as ecology, biology, genetics, physics, operation research etc. For details, see Johnson et al. (2005).

2 The BHP distribution

Consider a non-negative integer valued random variable X following HP distribution with *pgf* (1.2), in which $\theta = \theta_1 + \theta_2 + \theta_3$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 \geq 0$. Define $\alpha_j = \theta_j/\theta$, for $j = 1, 2, 3$ and let

$\{Y_n = (Y_{1n}, Y_{2n}),\ n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed bivariate Bernoulli random vectors, each with *pgf*

$$P(t_1, t_2) = \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2.$$

Assume that $X, Y_1, Y_2, \ldots$ are independent. Let $T_0 = (T_{10}, T_{20}) = (0, 0)$ and define

$$T_X = (T_{1X}, T_{2X}) = \left(\sum_{x=1}^{X} Y_{1x}, \sum_{x=1}^{X} Y_{2x} \right).$$

Then the *pgf* of T_X is the following, in which $\Lambda = \phi^{-1}(1; \lambda; \theta_1 + \theta_2 + \theta_3)$.

$$H(t_1, t_2) = G\{P(t_1, t_2)\} = \Lambda\phi(1; \lambda; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2) \tag{2.1}$$

We call a distribution with *pgf* as given in (2.1) as 'the bivariate hyper-Poisson distribution'or in short, 'the $BHPD$'. Clearly the $BHPD$ with $\lambda = 1$ is the bivariate Poisson distribution discussed in Kocherlakotta and Kocherlakotta (1992, pp 90) and when λ is a positive integer the $BHPD$ with *pgf* (2.1) reduces to the *pgf* of a bivariate version of the displaced Poisson distribution.

Let (X_1, X_2) be a random variable having the $BHPD$ with *pgf* (2.1). Then the marginal *pgf* of X_1 and X_2 are respectively

$$H_{X_1}(t) = H(t, 1) = \Lambda\phi[1; \lambda; (\theta_1 + \theta_3)t + \theta_2] \text{ and}$$
$$H_{X_2}(t) = H(1, t) = \Lambda\phi[1; \lambda; (\theta_2 + \theta_3)t + \theta_1].$$

The *pgf* of $X_1 + X_2$ is

$$H_{X_1+X_2}(t) = H(t, t) = \Lambda\phi[1; \lambda; (\theta_1 + \theta_2)t + \theta_3 t^2],$$

which is the *pgf* of a modified version of the HP distribution studied in Kumar and Nair (2011).

Let x be a non-negative integer such that $P(X_2 = x) > 0$. On differentiating (2.1) with respect to $t_2 x$ times and putting $t_1 = t$ and $t_2 = 0$, we get

$$H^{(0,x)}(t, 0) = (\theta_2 + \theta_3 t)^x \left(\prod_{j=0}^{x-1} D_j \right) \Lambda \delta_x(\theta_1 t) \tag{2.2}$$

where $D_j = (1 + j)/(\lambda + j)$ and $\delta_j(t) = \phi(1 + j; \lambda + j; t)$ for $j = 0, 1, 2, \ldots$.

Now applying the formula for the *pgf* of the conditional distribution in terms of partial derivatives of the joint *pgf*, developed by Subrahmaniam (1966), we obtain the conditional *pgf* of X_1 given $X_2 = x$ as

$$H_{X_1|X_2=x}(t) = \left(\frac{\theta_2 + \theta_3 t}{\theta_2 + \theta_3} \right)^x \frac{\phi(1 + x; \lambda + x; \theta_1 t)}{\phi(1 + x; \lambda + x; \theta_1)} = H_1(t) H_2(t), \tag{2.3}$$

where $H_1(t)$ is the *pgf* of a binomial random variable with parameters x and $p = \theta_3(\theta_2 + \theta_3)^{-1}$ and $H_2(t)$ is the *pgf* of a random variable following the HPD with parameters $1 + x$, $\lambda + x$ and

θ_1. Note that, when $\theta_3 = 0$ and/or when $x = 0$, $H_1(t)$ reduces to the *pgf* of a random variable degenerate at zero. Thus the conditional distribution X_1 given $X_2 = x$ given in (2.4) can be viewed as the distribution of the sum of independent random variables V_1 with *pgf* $H_1(t)$ and V_2 with *pgf* $H_2(t)$. Consequently from (2.4) we obtain the following

$$E(X_1\,|X_2 = x\,) = \frac{x\theta_3}{(\theta_2+\theta_3)} + \frac{\theta_1 D_x \delta_{x+1}(\theta_1)}{\delta_x(\theta_1)} \tag{2.4}$$

$$\begin{aligned}\text{Var}(X_1\,|X_2 = x) \;&=\; \frac{x\theta_2\theta_3}{(\theta_2+\theta_3)^2} + \frac{\theta_1 D_x}{\delta_x^2(\theta_1)}\{D_{x+1}\delta_x(\theta_1)\delta_{x+2}(\theta_1)\theta_1 \\ &\quad + \delta_x(\theta_1)\delta_{x+1}(\theta_1) - D_x[\delta_{x+1}(\theta_1)]^2\theta_1\}.\end{aligned} \tag{2.5}$$

In a similar approach, for a non-negative integer x with $P(X_1 = x) > 0$, we can obtain the conditional *pgf* of X_2 given $X_1 = x$ by interchanging θ_1 and θ_2 in (2.3). Therefore it is evident that comments similar to those in case of the conditional distribution of X_1 given $X_2 = x$ are valid regarding conditional distribution of X_2 given $X_1 = x$ and explicit expressions for $E(X_2\,|X_1 = x\,)$ and $Var(X_2\,|X_1 = x)$ can be obtained by interchanging θ_1 and θ_2 in the right hand side expressions of (2.5) and (2.6) respectively.

In order to obtain the probability mass function *pmf* of the $BHPD$, we need the following partial derivatives of $H(t_1, t_2)$, in which r is a non-negative integer.

$$H^{(r,0)}(t_1,t_2) = (\prod_{i=0}^{r-1} D_i)(\theta_1+\theta_3 t_2)^r \Lambda\Delta_r(t_1,t_2), \tag{2.6}$$

where

$$\Delta_j(t_1,t_2) = \phi(1+j;\lambda+j;\theta_1 t_1+\theta_2 t_2+\theta_3 t_1 t_2), j = 0,1,2,\ldots$$

The following derivatives are needed in the sequel, in which $0 \le i \le r$ and $j \ge 1$.

$$\frac{\partial^i(\theta_1+\theta_3 t_2)^r}{\partial t_2^i} = \frac{r!\theta_3^i}{(r-i)!}(\theta_1+\theta_3 t_2)^{r-i} \tag{2.7}$$

$$\frac{\partial^j \Delta_r(t_1,t_2)}{\partial t_2^j} = \prod_{i=r}^{r+j-1} D_i(\theta_2+\theta_3 t_1)^j \Delta_{r+j}(t_1,t_2). \tag{2.8}$$

Differentiating both sides of (2.7) s-times with respect to t_2 and applying (2.8) and (2.9), we get the following

$$\begin{aligned}H^{(r,s)}(t_1,t_2) \;&=\; (\prod_{i=0}^{r-1} D_i)\Lambda \sum_{m=0}^{s} \binom{s}{m} \frac{\partial^m(\theta_1+\theta_3 t_2)^r}{\partial t_2^m}\frac{\partial^{s-m}\Delta_r(t_1,t_2)}{\partial t_2^{s-m}} \\ &=\; (\prod_{i=0}^{r-1} D_i)\Lambda \sum_{m=0}^{\min(r,s)} \binom{s}{m}\frac{r!}{(r-m)!}\theta_3^m(\theta_1+\theta_3 t_2)^{r-m} \\ &\quad\times \prod_{i=r}^{r+s-m-1} D_i(\theta_2+\theta_3 t_1)^{s-m}\Delta_{r+s-m}(t_1,t_2)\end{aligned} \tag{2.9}$$

Now, by putting $(t_1, t_2) = (0, 0)$ in (2.10) and by dividing $r!s!$, we get the *pmf* of the $BHPD$ as

$$h(r,s) = \Lambda\theta_1^r\theta_2^s \sum_{m=0}^{\min(r,s)} \frac{D^*}{m!(r-m)!(s-m)!}(\frac{\theta_3}{\theta_1\theta_2})^m, \qquad (2.10)$$

where

$$D^* = \prod_{j=0}^{r+s-m-1} D_j \text{ and } \prod_{j=0}^{k} D_j = 1, \text{for any } k < 0.$$

By putting $(t_1, t_2) = (1, 1)$ in (2.10) we get the $(r, s)^{th}$ factorial moment $\mu_{[r,s]}$ of the $BHPD$ as

$$\mu_{[r,s]} = \Lambda r!s!(\theta_1 + \theta_3)^r(\theta_2 + \theta_3)^s \sum_{m=0}^{\min(r,s)} \frac{D^*\xi_{r+s-m}}{m!(r-m)!(s-m)!}\beta^m \qquad (2.11)$$

where $\xi_j = \phi(1 + j; \lambda + j; \theta_1 + \theta_2 + \theta_3)$, for $j = 0, 1, \dots$ and $\beta = \theta_3(\theta_1 + \theta_3)^{-1}(\theta_2 + \theta_3)^{-1}$.

From (2.12) we have the following, in which $\psi_j = \Lambda\xi_j$, for $j = 1, 2, \dots$

$$\begin{aligned} E(X_1) &= \mu_{[1,0]} = D_0\psi_1(\theta_1 + \theta_3) && (2.12)\\ E(X_2) &= \mu_{[0,1]} = D_0\psi_1(\theta_2 + \theta_3) && (2.13)\\ \text{Cov}(X_1, X_2) &= D_0(D_1\psi_2 - D_0\psi_1^2)(\theta_1 + \theta_3)(\theta_2 + \theta_3) + D_0\psi_1\theta_3 && (2.14) \end{aligned}$$

where D_0 and D_1 are as given in (2.2).

3 Recurrence relations

Let (X_1, X_2) be a random vector following the $BHPD$ with *pgf* (2.1). For j=0, 1, 2, ..., define $\lambda^* + j = (1 + j, \lambda + j)$ and $\lambda^{(j)} = (1 + j)(\lambda + j)^{-1}$ Now, the *pmf* $h(r, s)$ of the $BHPD$ given in (2.11) we denote by $h(r, s; \lambda^*)$. Then we have the following result in the light of relations:

$$H(t_1, t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r, s; \lambda^*)t_1^r t_2^s = \Lambda\phi(1; \lambda; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2) \qquad (3.1)$$

and

$$\xi_1 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r, s; \lambda^* + 1)t_1^r t_2^s = \phi(2; \lambda + 1; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2), \qquad (3.2)$$

in which ξ_1 is as given in (2.12).

Result 3.1. The probability mass function $h(r, s; \lambda^*)$ of the $BHPD$ satisfies the following recurrence relations.

$$h(r+1,0;\lambda^*) = \frac{D_0\psi_1\theta_1}{(r+1)}h(r,0;\lambda^*+1), r \geq 0 \quad (3.3)$$

$$h(r+1,s;\lambda^*) = \frac{D_0\psi_1}{(r+1)}[\theta_1 h(r,s;\lambda^*+1)+\theta_3 h(r,s-1;\lambda^*+1], r \geq 0, s \geq 1 \quad (3.4)$$

$$h(0,s+1;\lambda^*) = \frac{D_0\psi_1\theta_2}{(s+1)}h(0,s;\lambda^*+1), s \geq 0 \quad (3.5)$$

$$h(r,s+1;\lambda^*) = \frac{D_0\psi_1}{(s+1)}[\theta_2 h(r,s;\lambda^*+1)+\theta_3 h(r-1,s;\lambda^*+1], r \geq 1, s \geq 0 \quad (3.6)$$

Proof. Relation (2.7) with r = 1 gives

$$H^{(1,0)}(t_1,t_2) = D_0(\theta_1+\theta_3 t_2)\Lambda\Delta_1(t_1,t_2) \quad (3.7)$$

On differentiating both sides of (3.1) with respect to t_1, we have

$$H^{(1,0)}(t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(r+1)h(r+1,s;\lambda^*)t_1^r t_2^s \quad (3.8)$$

By using (3.2) and (3.8) in (3.7) we get the following, in which ψ_1 is as given in (2.13).

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(r+1)h(r+1,s;\lambda^*)t_1^r t_2^s = D_0\psi_1[\theta_1 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r,s;\lambda^*+1) t_1^r t_2^s + \theta_3 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r,s;\lambda^*+1)t_1^r t_2^{s+1}] \quad (3.9)$$

On equating the coefficient of $t_1^r t_2^0$ on both sides of (3.9) we get the relation (3.3) and on equating the coefficient of $t_1^r t_2^s$ on both sides of (3.9) we get the relation (3.4). We omit the proof of relations (3.5) and (3.6) as it is similar to that of relations (3.3) and (3.4). □

Result 3.2. For $r, s \geq 0$, simple recurrence relations for factorial moments $\mu_{[r,s]}(\lambda^*)$ of order (r, s) of the $BHPD$ are the following.

$$\mu_{[r+1,s]}(\lambda^*) = D_0\psi_1(\theta_1+\theta_3)\mu_{[r,s]}(\lambda^*+1) + D_0\psi_1\theta_3 s\mu_{[r,s-1]}(\lambda^*+1) \quad (3.10)$$

$$\mu_{[r,s+1]}(\lambda^*) = D_0\psi_1(\theta_2+\theta_3)\mu_{[r,s]}(\lambda^*+1) + D_0\psi_1\theta_3 r\mu_{[r-1,s]}(\lambda^*+1), \quad (3.11)$$

in which $\mu_{[0,0]}(\lambda^*) = 1$.

Proof. Let (X_1, X_2) be a random vector having the $BHPD$ with *pgf* $H(t_1, t_2)$ as given in (2.1). Then the factorial moment generating function $F(t_1, t_2)$ of the $BHPD$ is

$$\begin{aligned} F(t_1,t_2) &= H(1+t_1, 1+t_2) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r,s]}(\lambda^*)\frac{t_1^r t_2^s}{r!s!} \\ &= \Lambda\phi[1;\lambda;\theta_1+\theta_2+\theta_3+(\theta_1+\theta_3)t_1+(\theta_2+\theta_3)t_2+\theta_3 t_1 t_2] \quad (3.12) \end{aligned}$$

Differentiate (3.12) with respect to t_1 to get

$$\begin{aligned}\frac{\partial F(t_1,t_2)}{\partial t_1} &= [(\theta_1+\theta_3)+\theta_3 t_2)]D_0\Lambda \\ &\times \phi[2;\lambda+1;\theta_1+\theta_2+\theta_3+(\theta_1+\theta_3)t_1+(\theta_2+\theta_3)t_2+\theta_3 t_1 t_2] \quad (3.13)\end{aligned}$$

Based on the similar argument as in the proof of Result 3.1., by using (3.12) with λ^* replaced by λ^*+1, one can obtain the following from (3.13).

$$\begin{aligned}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r+1,s]}(\lambda^*)\frac{t_1^r t_2^s}{r!s!} &= D_0\psi_1\{(\theta_1+\theta_3)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r,s]}(\lambda^*+1)\frac{t_1^r t_2^s}{r!s!} \\ &+ \theta_3\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r,s]}(\lambda^*+1)\frac{t_1^r t_2^{s+1}}{r!s!}\} \quad (3.14)\end{aligned}$$

Now on equating the coefficients of $(r!s!)^{-1}t_1^r t_2^s$ on both sides of (3.14) we obtain the relation (3.10). A similar procedure implies (3.11). □

Result 3.3. Two recurrence relations for the (r, s)th raw moments $\mu_{r,s}(\lambda^*)$ of the $BHPD$ are:

$$\mu_{r+1,s}(\lambda^*) = D_0\psi_1\theta_1\sum_{j=0}^{r}\binom{r}{j}\mu_{r-j,s}(\lambda^*+1) + D_0\psi_1\theta_3\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\lambda^*+1) \quad (3.15)$$

and

$$\mu_{r,s+1}(\lambda^*) = D_0\psi_1\theta_2\sum_{k=0}^{s}\binom{s}{k}\mu_{r,s-k}(\lambda^*+1) + D_0\psi_1\theta_3\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\lambda^*+1) \quad (3.16)$$

Proof. The characteristic function $A(t_1,t_2)$ of the $BHPD$ with *pgf* (2.1) is the following. For (t_1,t_2) in R^2,

$$A(t_1,t_2) = H(e^{it_1},e^{it_2}) = \Lambda\phi[1;\lambda^*;\lambda(t_1,t_2;\theta)] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{r,s}(\lambda^*)\frac{(it_1)^r(it_2)^s}{r!s!}, \quad (3.17)$$

where $\lambda(t_1,t_2;\theta)=\theta_1 e^{it_1}+\theta_2 e^{it_2}+\theta_3 e^{i(t_1+t_2)}$, $\theta=(\theta_1,\theta_2,\theta_3)$ and $i=\sqrt{-1}$. On differentiating (3.17) with respect to t_1, we obtain

$$D_0\Lambda\phi[2;\lambda^*+1;\lambda(t_1,t_2;\theta)]\{i(\theta_1+\theta_3 e^{it_2})e^{it_1}\} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}i\mu_{r,s}(\lambda^*)\frac{(it_1)^{r-1}(it_2)^s}{(r-1)!s!}.$$

By using (3.17) with λ^* replaced by $\lambda^* + 1$; and on expanding the exponential functions, we obtain the following, in the light of some standard properties of double sum

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\mu_{r+1,s}(\lambda^*)(it_1)^r(it_2)^s}{r!s!}$$
$$= D_0\psi_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(it_1)^r(it_2)^s}{r!s!}\{\theta_1\sum_{j=0}^{r}\binom{r}{j}\mu_{r-j,s}(\lambda^*+1)+\theta_3\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\lambda^*+1)\} \tag{3.18}$$

Now equate the coefficients of $(r!s!)^{-1}(it_1)^r(it_2)^s$ on both sides of (3.18) to get the relation (3.15). A similar procedure gives (3.16). □

4 Estimation of parameters

Here we obtain the estimators of the $BHPD$ by the method of maximum likelihood. Let $a(r,s)$ be the observed frequency of the $(r,s)^{th}$ cell of the bivariate data. Let y be the highest value of r observed and z be the highest value of s observed. Then by using (2.11) the likelihood function of the sample is the following.

$$L = \prod_{r=0}^{y}\prod_{s=0}^{z}[h(r,s)]^{a(r,s)} \quad \Rightarrow \quad \log L = \sum_{r=0}^{y}\sum_{s=0}^{z}a(r,s)\log h(r,s).$$

Let $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\lambda}$ denotes the likelihood estimators of θ_1, θ_2, θ_3 and λ respectively. Now $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\lambda}$ are obtained by solving the likelihood equations (4.1), (4.2), (4.3)and (4.4) given below.

$$\frac{\partial \log L}{\partial \theta_1} = 0$$

Equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z}a(r,s)\left\{\frac{-1}{\lambda}\frac{\phi(2;\lambda+1;\theta_1+\theta_2+\theta_3)}{\phi(1;\lambda;\theta_1+\theta_2+\theta_3)}+\frac{\sum_{m=0}^{\min(r,s)}\frac{D^*\theta_1^{r-m-1}\theta_2^{s-m}\theta_3^m}{(r-m-1)!(s-m)!m!}}{\xi(r,s)}\right\}=0. \tag{4.1}$$

$$\frac{\partial \log L}{\partial \theta_2} = 0$$

Equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z}a(r,s)\left\{\frac{-1}{\lambda}\frac{\phi(2;\lambda+1;\theta_1+\theta_2+\theta_3)}{\phi(1;\lambda;\theta_1+\theta_2+\theta_3)}+\frac{\sum_{m=0}^{\min(r,s)}\frac{D^*\theta_1^{r-m}\theta_2^{s-m-1}\theta_3^m}{(r-m)!(s-m-1)!m!}}{\xi(r,s)}\right\}=0. \tag{4.2}$$

$$\frac{\partial \log L}{\partial \theta_3} = 0$$

Equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z} a(r,s)\left\{\frac{-1}{\lambda}\frac{\phi(2;\lambda+1;\theta_1+\theta_2+\theta_3)}{\phi(1;\lambda;\theta_1+\theta_2+\theta_3)} + \frac{\sum_{m=0}^{\min(r,s)} \frac{D^*\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m-1}}{(r-m)!(s-m)!(m-1)!}}{\xi(r,s)}\right\} = 0. \tag{4.3}$$

$$\frac{\partial \log L}{\partial \lambda} = 0$$

Equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z} a(r,s)\left\{\frac{-1}{\phi(1;\lambda;\theta_1+\theta_2+\theta_3)}\sum_{x=0}^{\infty}(\theta_1+\theta_2+\theta_3)^x\eta(x) + \right.$$

$$\left.\frac{1}{\xi(r,s)}\sum_{m=0}^{\min(r,s)}\eta(r+s-m)\,\frac{(r+s-m)!\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m}}{(r-m)!(s-m)!m!}\right\} = 0, \tag{4.4}$$

in which $\xi(r,s) = \sum_{m=0}^{\min(r,s)} \frac{D^*\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m}}{(r-m)!(s-m)!m!}$ and $\eta(u) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+u)}[\psi(\lambda)-\psi(\lambda+u)]$.

5 An application

Here we illustrate the method of maximum likelihood estimation using a real life data set taken from Patrat (1993). The description of data is as follows: The North Atlantic coastal states in USA can be affected by tropical cyclones. They divided the states into three geographical zones: Zone 1 (Texas, Louisina, The Mississipi, Alabama), Zone 2 (Florida), and Zone 3 (Other states)

Now the interest is in the study of the joint distribution of the pair (X_1, X_2), where X_1 and X_2 are the yearly frequency of hurricanes affecting respectively zone 1 and zone 3. The observed values of (X_1, X_2) during 93 years from 1899 to 1991 are as given in Table 1. We obtain the corresponding expected frequencies by fitting the bivariate Poisson distribution (BPD), the bivariate hyper-Poisson distribution of Ahmad (1981) ($BHPD_A$) and the bivariate hyper-Poisson distribution ($BHPD$) introduced in this paper using method of maximum likelihood in Table 1. The estimated values of the parameters of the BPD, the $BHPD_A$ and the $BHPD$ and the chi-square values in respective cases are listed in Table 2. From Table 2, it can be observed that the $BHPD$ gives a better fit to this data compared to the existing models- the BPD and the $BHPD_A$.

Table 1: Comparison of observed and theoretical frequencies Hurricanes (1899-1991) having affected Zone 1 and Zone 3, using method of maximum likelihood.

Zone 3	Zone 1	0	1	2	3	Total
0	OBS	27	9	3	2	41
	BPD	28.24	12.71	2.86	0.48	44.29
	$BHPD_A$	28.31	12.49	2.95	0.48	44.23
	$BHPD$	25.64	14.31	2.50	0.26	42.71
1	OBS	24	13	1	0	38
	BPD	20.30	9.79	2.35	0.42	32.86
	$BHPD_A$	20.46	9.56	2.37	0.40	32.79
	$BHPD$	23.23	10.88	2.23	0.27	36.61
2	OBS	8	2	1	0	11
	BPD	7.29	3.75	0.96	0.19	12.19
	$BHPD_A$	7.39	3.65	0.95	0.17	12.16
	$BHPD$	6.60	3.62	0.88	0.12	11.22
3	OBS	1	0	2	0	3
	BPD	2.12	1.16	0.32	0.06	3.66
	$BHPD_A$	1.78	0.93	0.25	0.05	3.01
	$BHPD$	1.11	0.72	0.21	0.14	1.07
Total	OBS	60	24	7	2	93
	BPD	57.95	27.41	6.49	1.15	93
	$BHPD_A$	57.94	26.63	6.52	1.1	92
	$BHPD$	56.58	29.53	5.82	0.79	93

Table 2: Estimated values of the parameters of the BPD, the $BHPD_A$ and the $BHPD$ by the method of maximum likelihood estimation and corresponding chi-square values.

Distributions	Estimation of parameters	Chi−square values
BPD	$\hat{\theta}_1 = 0.683$, $\hat{\theta}_2 = 0.450$, $\hat{\theta}_3 = 0.021$	2.524
$BHPD_A$	$\hat{\theta}_1 = 0.780$, $\hat{\theta}_2 = 0.324$, $\hat{\theta}_3 = 0.021$	2.452
	$\hat{\lambda}_1 = 1.075$, $\hat{\lambda}_2 = 0.619$	
$BHPD$	$\hat{\theta}_1 = 0.414$, $\hat{\theta}_2 = 0.255$, $\hat{\theta}_3 = 0.049$	0.463
	$\hat{\lambda} = 0.457$	

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