Dhaka Univ. J. Sci. 60(2): 191-195, 2012 (July)

Connections on Bundles

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Received on 25. 05. 2011.Accepted for Publication on 15. 12. 2011

Abstract

This paper is a survey of the basic theory of connection on bundles. A connection on tangent bundle TM, is called an affine connection on an m-dimensional smooth manifold M. By the general discussion of affine connection on vector bundles that necessarily exists on M which is compatible with tensors.

I. Introduction

In order to differentiate sections of a vector bundle [5] or vector fields on a manifold we need to introduce a structure called the connection on a vector bundle. For example, an affine connection is a structure attached to a differentiable manifold so that we can differentiate its tensor fields. We first introduce the general theorem of connections on vector bundles. Then we study the tangent bundle. TM is a m-dimensional vector bundle determine intrinsically by the differentiable structure [8] of an m-dimensional smooth manifold M.

II. Connections on Vector Bundles

A connection on a fiber bundle [7] is a device that defines a notion of parallel transport on the bundle, that is, a way to connect or identify fibers over nearby points. If the fiber bundle is a vector bundle, then the notion of parallel transport is required to be linear. Such a connection is equivalently specified by a covariant derivative, which is an operator that can differentiate sections of that bundle along tangent directions in the base manifold [3]. Connections in this sense generalize, to arbitrary vector bundles, the concept of a linear connection on the tangent bundle of a smooth manifold, and are sometimes known as linear connections. Nonlinear connections are connections that are not necessarily linear in this sense.

Definition 1. A connection on a vector bundle E is a map

$$D : \Gamma(E) \to \Gamma(T^*M \otimes E) \tag{1}$$

which satisfies the following conditions:

(i) For any $s_1, s_2 \in \Gamma(E)$,

$$D(s_1 + s_2) = Ds_1 + Ds_2$$

(ii) For $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$,

$$D(\alpha s) = d\alpha \otimes s + \alpha Ds$$

Suppose X is a smooth tangent vector fields on M and $s \in \Gamma(E)$. Let

$$D_X s = < X, Ds > \tag{2}$$

where $<,>$ represents the pairing between TM and T^*M. Then $D_X s$ is a section of E, called the absolute differential quotient or the covariant derivative of the section s along X.

Theorem 1. A connection always exists on a vector bundle.

Proof. Choose a coordinate covering $\{U_\alpha\}_{\alpha\in A}$ of M. Since vector bundles are trivial locally, we may assume that there is local frame field S_α for any U_α. By the local structure of connections, we need only construct a $q \times q$ matrix w_α on each U_α such that the matrices satisfy

$$w' = dA.A^{-1} + A.w.A^{-1} \tag{3}$$

under a change of the local frame field, which is the transformation formula for a connection, a most important formula in differential geometry.

We may assume that $\{U_\alpha\}$ is locally finite, and $\{g_\alpha\}$ is a corresponding sub-ordinate partition of unity such that supp $g_\alpha \subset U_\alpha$. When $U_\alpha \cap U_\beta \neq \emptyset$, there naturally exists a non-degenerate matrix $A_{\alpha\beta}$ of smooth functions on $U_\alpha \cap U_\beta$ such that

$$S_\alpha = A_{\alpha\beta}.S_\beta\ ,\quad det\ A_{\alpha\beta} \neq 0 \tag{4}$$

For every $\alpha \in A$, choose an arbitrary $q \times q$ matrix ϕ_α of differential 1-forms on U_α. Let

$$w_\alpha = \sum_{\beta\in A} g_\beta.\big(dA_{\alpha\beta}.A_{\alpha\beta}^{-1} + A_{\alpha\beta}.\ \phi_\beta\ .A_{\alpha\beta}^{-1}\big) \tag{5}$$

where the terms in the sums over β with $U_\alpha \cap U_\beta = \emptyset$ are zero. Then w_α is a matrix of differential 1-forms on U_α. We need only demonstrate the following transformation formula for $U_\alpha \cap U_\beta \neq \emptyset$:

$$w_\alpha = dA_{\alpha\beta}.A_{\alpha\beta}^{-1} + A_{\alpha\beta}.w_\beta\ .A_{\alpha\beta}^{-1}. \tag{6}$$

This can be done by a direct calculation. First observe that when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, the following is true in the intersection:

$$A_{\alpha\beta}\ .A_{\beta\gamma} = A_{\alpha\gamma}.$$

Thus on $U_\alpha \cap U_\beta \neq \emptyset$ we have

$$A_{\alpha\beta}.w_\beta.A_{\alpha\beta}^{-1} = \sum_{\substack{\gamma \\ U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset}} g_\gamma.A_{\alpha\beta}.\left(dA_{\beta\alpha}.A_{\beta\alpha}^{-1} + A_{\beta\gamma}.\phi_\gamma.A_{\beta\gamma}^{-1}\right).A_{\alpha\beta}^{-1}$$
$$= w_\alpha - dA_{\alpha\beta}.A_{\alpha\beta}^{-1}$$

This is precisely (6). We see from the above that there is much freedom in the choice of a connection. This completes the proof of the theorem. □

Remark 1. In particular, if we let $\phi_\beta = 0$ in (6), then we obtain a connection D on E whose connection matrix on U_α is

$$w_\alpha = \sum_\beta g_\beta.\left(dA_{\alpha\beta}.A_{\alpha\beta}^{-1}\right)$$

By the transformation formula (3) for connection matrices, the vanishing of a connection matrix is not an invariant property. In fact, for an arbitrary connection, we can always find a local frame field with respect to which the connection matrix is zero at some point. This fact is useful in calculations involving connections.

Theorem 2. Suppose D is a connection on a vector bundle E, and $p \in M$. Then there exists a local frame field S in a coordinate neighborhood of p such that the corresponding connection matrix w is zero at p.

Proof. Choose a coordinate neighborhood $(U;u^i)$ of p such that $u^i(p) = 0, 1 \le i \le m$. Suppose S' is a local frame field on U with corresponding connection matrix $w^i = (w_\alpha'^\beta)$,

where

$$w_\alpha'^\beta = \sum_{i=1}^m \Gamma_{\alpha i}'^\beta\, u^i \tag{7}$$

and the $\Gamma_{\alpha i}'^\beta$ are smooth functions on U. Let

$$a_\alpha^\beta = \delta_\alpha^\beta - \sum_{i=1}^m \Gamma_{\alpha i}'^\beta(p).u^i$$

Then $A = (a_\alpha^\beta)$ is the identity matrix at p. Hence there exists a neighborhood $V \subset U$ of p such that A is non-degenerate in V. Thus

$$S = A.S' \tag{8}$$

is a local frame field on V. Since

$$dA(p) = -w'(p),$$

we can obtain from (3),

$$w(p) = (dA.A^{-1} + A.w'.A^{-1})(p)$$
$$= -w'(p) + w'(p)$$
$$= 0$$

Thus S is the desired local frame field. □

Theorem 3. Suppose X, Y are two arbitrary smooth tangent vector fields on the manifold M. Then

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} \tag{9}$$

Proof. Because the absolute differential quotient and the curvature operator are local operators, we need only consider the operations of both sides of (9) on a local section. Suppose $s \in \Gamma(E)$ has the local expression

$$s = \sum_{\alpha=1}^q \lambda^\alpha s_\alpha$$

Then

$$D_X s = \textstyle\sum_{\alpha=1}^q (X\lambda^\alpha + \sum_{\beta=1}^q \lambda^\beta < X, w_\beta^\alpha >)\, s_\alpha, \tag{10}$$

and $D_Y D_X s = \sum_{\alpha=1}^q\{ Y(X\lambda^\alpha) + \sum_{\beta=1}^q (X\lambda^\beta < Y, w_\beta^\alpha > + Y\lambda^\beta < X, w_\beta^\alpha >)$
$\sum_{\beta=1}^q \lambda^\beta (Y < X, w_\beta^\alpha > + \sum_{\gamma=1}^q < X, w_\beta^\gamma > < Y, w_\gamma^\alpha >)\} s_\alpha$.

Hence $D_X D_Y s - D_Y D_X s = \sum_{\alpha=1}^q\{ [X,Y]\lambda^\alpha + \sum_{\beta=1}^q \lambda^\beta(< [X,Y], w_\beta^\alpha > + < X \Lambda Y, dw_\beta^\alpha > - \sum_{\gamma=1}^q w_\beta^\gamma \Lambda w_\gamma^\alpha >)\} s_\alpha = D_{[X,Y]}s + \sum_{\alpha,\beta=1}^q \lambda^\beta < X \Lambda Y, \Omega_\beta^\alpha > s_\alpha$ (11)

That is,

$$R(X,Y)s = D_X D_Y s - D_Y D_X s - D_{[X,Y]}s$$

This completes the proof of the theorem. □

Theorem 4. The curvature matrix Ω satisfies the Bianchi identity

$$d\Omega = w \Lambda \Omega - \Omega \Lambda w.$$

Proof: Apply exterior differentiation [9] to both sides of $\Omega = dw - w \Lambda w$ $d\Omega = -dw \Lambda w + w \Lambda dw$

$$= -(\Omega + w\Lambda w)\Lambda w + w \Lambda (\Omega + w \Lambda w)$$
$$= w \Lambda \Omega - \Omega \Lambda w$$

This completes the proof of the theorem. □

Remark 2. If a section s of a vector bundle E satisfies the condition $Ds = 0$, then s is called a parallel section.

III. Affine Connections

Definition 2. Let M be a smooth n-dimensional manifold, O_M be the set of smooth functions and $\Gamma(TM)$ be the vector space of smooth vector fields. An affine connection on M is a map (denoted by ∇)

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_X Y$$

such that

(i) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

(ii) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

$(iii)\ \nabla_X (f\,Y) = X(f)\,Y + f\ \nabla_X Y$

$(iv)\ \ \nabla_{f\,X} Y = f\,\nabla_X Y\ \ ;\forall f \in O_M$ and $X, Y \in \Gamma(TM)$

IV. Affine Connection in Two Coordinates Charts

Let (U, φ) be a coordinate chart on a manifold M, with coordinates $(x^1, x^2, \ldots, x^n)$. Then the vector fields X and Y can be expressed as

$$X = \sum_{i=1}^{n} X^i(x)\ \frac{\partial}{\partial x^i}$$

$$Y = \sum_{j=1}^{n} Y^j(x)\ \frac{\partial}{\partial x^j}$$

For some smooth functions $X^i(x)$ and $Y^j(x)$. In U, $\frac{\partial}{\partial x^i}$ are smooth vector fields. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ is again a smooth vector field. Thus

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k\ \frac{\partial}{\partial x^k}$$

For some smooth functions $\Gamma_{ij}^k(x)$. Here $\Gamma_{ij}^k(x)$ is a n^3 function.

$\Rightarrow\ \nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^k\ e_k$; where $e_i = \frac{\partial}{\partial x^i}$, $e_j = \frac{\partial}{\partial x^j}$ and $e_k = \frac{\partial}{\partial x^k}$

Let us compute $\nabla_X Y$

$\nabla_X Y = \nabla_{\sum_{i=1}^{n} X^i e_i} \sum_{j=1}^{n} Y^j e_j$

$= \sum_{j=1}^{n} \left(\nabla_{\sum_{i=1}^{n} X^i e_i} Y^j e_j \right)$ [By axiom (i)]

$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\nabla_{X^i e_i} Y^j e_j \right)$ [By axiom (ii)]

$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(X^i\ \nabla_{e_i} Y^j e_j \right)$ [By axiom (iv)]

$= \sum_{i=1}^{n}\sum_{j=1}^{n} X^i \left(e_i(Y^j) e_j + Y^j\,\nabla_{e_i} e_j \right)$ [By axiom (iii)]

$\nabla_X Y = \sum_{i=1}^{n}\sum_{j=1}^{n} X^i \left(\frac{\partial}{\partial x^i}(Y^j) e_j + \sum_{k=1}^{n} \Gamma_{ij}^k\ e_k\,Y^j \right)$

The functions $\Gamma_{ij}^k(x)$ are called coordinate symbols of the affine connection ∇. The vector field $\nabla_X Y$ is often called covariant derivative of vector field Y along the vector field X.

Definition 3. If the torsion tensor of an affine connection ∇ is zero, then the connection is said to be torsion free.

A torsion-free affine connection always exists. In fact, if the coefficients of a connection ∇ are Γ_{jk}^i , then the set

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2}\left(\Gamma_{ik}^j + \Gamma_{ki}^j\right).$$

Obviously, $\tilde{\Gamma}_{jk}^i$ is symmetric with respect to the lower indices and satisfies

$$\Gamma'^j_{ik} = {}^q_{pr}\ \frac{\partial w^j}{\partial u^q}\frac{\partial u^p}{\partial w^i}\frac{\partial u^r}{\partial w^k} + \frac{\partial^2 u^p}{\partial w^i \partial w^k}\cdot\frac{\partial w^j}{\partial u^p} \qquad (12)$$

under a local change of coordinates. Therefore the $\tilde{\Gamma}_{ik}^j$ are the coefficients of some connection $\tilde{\nabla}$ and $\tilde{\nabla}$ is torsion-free.

Theorem 5. Suppose ∇ is a torsion-free affine connection on M. Then for any point $p \in M$ there exists a local coordinate system u^i such that the corresponding connection coefficients Γ_{ik}^j vanish at p.

Proof. Suppose $(W; w^i)$ is a local coordinating system at p with connection coefficients $\tilde{\Gamma}'^j_{ik}$. Let

$$u^i = w^i + \frac{1}{2}\,\Gamma'^i_{jk}(p)(w^j - w^j(p))\,(w^k - w^k(p)) \qquad (13)$$

Then, $$\left.\frac{\partial u^i}{\partial w^j}\right|_p = \delta_j^i\ ,\quad \left.\frac{\partial^2 u^i}{\partial w^i \partial w^k}\right|_p = \Gamma'^i_{jk}(p) \qquad (14)$$

Thus the matrix $\left(\frac{\partial u^i}{\partial w^j}\right)$ is non-degenerate near p, and (13) provides for a change of local coordinates in a neighborhood of p. From (12) we see that the connection coefficients Γ_{ik}^j in the new coordinate system u^i satisfy

$$\Gamma_{ik}^j(p) = 0 \quad ; \quad 1 \le i, j, k \le m$$

This completes the proof of the theorem. □

Theorem 6. Suppose ∇ is a torsion-free affine connection on M. Then we have the Bianchi identity:

$$R_{ikl,h}^j + R_{ilh,k}^j + R_{ihk,l}^j = 0\,.$$

Proof. From Theorem 4, we have

$$d\Omega_i^j = w_i^k \wedge \Omega_k^j - \Omega_i^k \wedge w_k^j,$$

that is,

$$\frac{\partial R_{ikl}^j}{\partial u^h}\ du^h \wedge du^k \wedge du^l = \left(\Gamma_{ih}^p\ R_{pkl}^j - \Gamma_{ph}^j\,R_{ikl}^p \right) du^h \wedge du^k \wedge du^l.$$

Therefore

$$R_{ikl,h}^j\ du^h \wedge du^k \wedge du^l = -\left(\Gamma_{kh}^p\ R_{ipl}^j - \Gamma_{lh}^p\,R_{ikp}^j \right) du^h \wedge du^k \wedge du^l = 0,$$

where in the last equality we have used the torsion-free property of the connection. Hence

$$\left(R_{ikl,h}^j + R_{ilh,k}^j + R_{ihk,l}^j\right) du^h \wedge du^k \wedge du^l = 0 \qquad (15)$$

Now since the coefficients of (15) are skew-symmetric with respect to k, l, h , we have

$$R_{ikl,h}^j + R_{ilh,k}^j + R_{ihk,l}^j = 0$$

This completes the proof of the theorem. □

V. Connection Compatible with Tensors

Let M be a smooth manifold and τ be any tensor in M. Mostly this can be interested in the case when $\tau = g$ is a semi-Riemannian metric tensor on M, i.e., τ is a non--degenerate [1] symmetric (2, 0)- tensor, or when $\tau = \omega$ is symplectic form on M, i.e., τ is a non-degenerate closed 2-form [7], [9]. If ∇ is a connection in M, i.e., a connection on the tangent bundle TM, then we have naturally induced connections on all tensor bundles on M, all of which is denoted by the same symbol ∇.

Definition 4. The torsion of ∇ is the anti-symmetric tensor

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[X,Y]$ denotes the Lie brackets of the vector fields X and Y; ∇ is called symmetric if $T = 0$.

The connection ∇ is said to be compatible with τ is ∇ - -parallel, i.e., when $\nabla\tau = 0$.

Establishing whether a given tensor τ admits compatible connections is a local problem. Namely, one can use partition of unity to extend locally defined connections and observe that a convex combination of compatible connections is a compatible connection. In local coordinates, finding a connection compatible with a given tensor reduces to determining the existence of solutions for a non homogeneous linear system for the Christoffel symbols of the connection.

It is well known that semi-Riemannian metric tensors admit a unique compatible symmetric connection, called the Levi-Civita connection of the metric tensor, which can be given explicitly in [4]. Uniqueness of the Levi-Civita connection can be obtained by a curious combinatorial argument, as follows.

Suppose that ∇ and $\tilde{\nabla}$ are connections on M; their difference $\nabla - \tilde{\nabla}$ is a tensor, that is denoted by t

$$t(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

where X and Y are smooth vector fields on M. If both ∇ and $\tilde{\nabla}$ are symmetric connection, then t is symmetric

$$t(X,Y) - t(Y,X) = \tilde{\nabla}_X Y - \nabla_X Y - \tilde{\nabla}_Y X + \nabla_Y X$$
$$= [X,Y] + [Y,X] = 0.$$

Lemma 1. Let U be a set and $\rho : U \times U \times U \to \nabla$ be a map that is symmetric in its first two variables and anti--symmetric in its last two variables. Then ρ is identically zero.

Proof. Let $u_1, u_2, u_3 \in U$ be fixed. We have

$$\rho(u_1,u_2,u_3) = \rho(u_2,u_1,u_3) = -\rho(u_2,u_3,u_1) =$$
$$-\rho(u_3,u_2,u_1),$$

so that ρ is anti-symmetric in the first and the third variables. On the other hand

$$\rho(u_1,u_2,u_3) = -\rho(u_3,u_2,u_1) = -\rho(u_2,u_3,u_1)$$
$$= \rho(u_1,u_3,u_2),$$

so that ρ is symmetric in the second and the third variables. This concludes the proof. □

Theorem 7. There exists at most one symmetric connection which is compatible with a semi Riemannian metric.

Proof. Assume that g is a semi-Riemannian metric on M, and let ∇ and $\tilde{\nabla}$ are two symmetric connections such that $\nabla g = \tilde{\nabla} g = 0$; for all $p \in M$, consider the map $\rho : T_pM \times T_pM \times T_pM \to \nabla$ given by

$$\rho(X,Y,Z) = g(t(X,Y),Z),$$

where t is the difference $\nabla - \tilde{\nabla}$. Since t is symmetric, then ρ is symmetric in the first two variables. On the other hand, ρ is anti-symmetric in the last two variables

$$\rho(X,Y,Z) + \rho(X,Z,Y) = g(\tilde{\nabla}_X Y,Z) -$$
$$g(\nabla_X Y,Z) + g(\tilde{\nabla}_X Z,Y) - g(\nabla_X Z,Y)$$
$$= \tilde{\nabla}_g(X,Y,Z) - \nabla_g(X,Y,Z) = 0.$$

By Lemma 1, $\rho = 0$, hence t =0, and thus $\tilde{\nabla} = \nabla$. Hence completes the proof. □

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