Algorithm to Perform a Complete RHS Parametric Analysis for LPP with Bounded Variables

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Abstract

Linear Programming problem (LPP)s with upper bounded variables can be solved using the Bounded Simplex method, without the explicit consideration of the upper bounded constraints. One can consider the upper bounded constraints explicitly and perform the regular right-hand-side parametric analysis of LPPs with bounded variables. This paper develops a method to perform the parametric analysis where the upper bounded constraints are considered implicitly, thus reduce the size of the basis matrix.

I. Introduction

Consider the LP problem with upper bounded variables

$$\text{Max} \quad z = c^T x$$

$$\text{s.t.} \quad Ax \le b \qquad (1)$$

$$0 \le x \le U$$

These problems have been solved by using the upper bound simplex technique without increasing the size of the problem (see: Murty[1], Dantzig[2]). Wagner[3] developed the dual simplex method for LP with bounded variables, which is further studied by Maros[4, 5].

However the data obtained to estimate the parameter values (A, c, b, U) used in the above model often are rather crude or nonexistent, so that the parameters in the original formulation may represent deliberate overestimates or underestimates to protect the interests of the estimators. For these reason it is important to perform a parametric analysis to investigate the effects on the optimal solution provided by the simplex method (SM) if the parameters take on other possible values. The traditional way to evaluate any imprecision in the parameters of an LP model is through a post-optimization analysis, with the help of sensitivity analysis, shadow prices and parametric programming. The method of sensitivity analysis in SM is well developed on the foundation of optimal basis, it requires little computational effort, has been introduced in numerous papers and text books so far (see, for example: Dantzig[2, 6], Gal[7]) and used in many LP codes.

In this paper we introduced a parameter θ in the right hand side vector b of (1). This parametric demand has practical significance since the uncertainty in demand requirements is commonly seen in locational decision making environments. This paper presents an algorithm to perform the parametric analysis of the above problem.

Panwalkar[8] established a method for the parametric analysis of LP problems with upper bounded variables, where the upper bounds in (1) have been parametrized as $0 \le x \le U + \theta V$, that provides an approximation to the maximum value of the parameter θ maintaining the feasibility of the problem. Later Dahiya and Verma[9] discussed the positive sensitivity analysis (PSA) in LP with bounded variables within which the components of a given optimal solution which are strictly between their bounds remain strictly between bounds and which are at their lower and upper bounds remain at lower and upper bounds respectively.

In this paper, we consider the parametric programming where the demand is allowed to vary as a linear function of the parameter θ within the given bounds of the variables. An algorithm has been established that will find all possible optimal solutions as a linear function of the parameter. Finally a numerical example is provided.

II. RHS Parametric Analysis

Consider the LP problem with bounded variables

$$\text{Max} \quad z = c^T x$$

$$\text{Subject to} \quad Ax = b + \theta b' \qquad (2)$$

$$0 \le x \le U$$

where $c, x, U \in \Re^n$, $b \in \Re^m$, $A = [a_{ij}]_{m \times n}$ is a matrix of rank m. U is the vector of the upper bounds of the decision variables and θ is a real valued parameter. Let c_j be the jth component of the vector c, A_j be the jth activity vector i.e. the column of the coefficient matrix in the simplex tableau corresponding to the variable x_j, B and N denote the submatrix of A in the simplex tableau corresponding to the basic (x_B) and non-basic (x_N) variables respectively. Let $(U_B)_i$ be the upper bound of the ith basic variable $(x_B)_i$. Let

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$J = \{1, 2, \cdots, n\}$,

$I = \{i \in J \mid x_i \text{ is a basic variable}\}$. It is assumed that problem (2) has an optimal solution for $\theta = 0$. Then the optimum simplex tableau of (2) for $\theta = 0$ has the form,

$$z = c_B^T x_B + c_N^T x_N, \tag{3}$$

$$B x_B + N x_N = b \tag{4}$$

Any variable that reaches at its upper bound, is replaced by $x_i' = u_i - x_i$, thus all the m basic variables are nonnegative and the remaining ($n - m$) non-basic variables are at zero level. To establish the method for the parametric analysis we need the following result.

Theorem. Consider the problem described in (2). If θ is any real valued parameter then the solution obtained for $\theta = 0$ will remain optimal for $\theta_L \le \theta \le \theta_U$, where

$$\theta_L = \max\{\theta_{L_1}, \theta_{L_2}\}, \qquad \theta_U = \min\{\theta_{U_1}, \theta_{U_2}\},$$

$$\theta_{L_1} = \max_i \{\frac{-(B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i > 0\},$$

$$\theta_{L_2} = \max_i \{\frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i < 0\},$$

$$\theta_{U_1} = \min_i \{\frac{-(B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i < 0\},$$

$$\theta_{U_2} = \min_i \{\frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i > 0\}.$$

Proof. For any real valued parameter θ the ith basic variable $(x_B)_i$ can be written using equation (4) as,

$$(x_B)_i = (B^{-1}b)_i + \theta(B^{-1}b')_i + \sum_{j \in J \mid I} P_j x_j \tag{5}$$

where $P_j = B^{-1}N$.

Since the dual solution of (2) does not depend on θ, therefore it remains dual feasible for any θ. Thus if x_j in (5) remain non-basic, then the feasibility of the problem remains unaffected i.e. the solution (5) will remain optimum only if,

$$0 \le (x_B)_i = (B^{-1}b)_i + \theta(B^{-1}b')_i \le (U_B)_i \tag{6}$$

The non-negativity condition of $(x_B)_i$ is satisfied if,

$$\theta \ge \frac{-(B^{-1}b)_i}{(B^{-1}b')_i}, \quad \text{for } (B^{-1}b')_i > 0 \tag{7}$$

and $\quad \theta \le \frac{-(B^{-1}b)_i}{(B^{-1}b')_i}, \quad \text{for } (B^{-1}b')_i < 0 \tag{8}$

The condition $(x_B)_i \le (U_B)_i$ is satisfied if,

$$\theta \le \frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i}, \quad \text{for } (B^{-1}b')_i > 0 \tag{9}$$

and

$$\theta \ge \frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i}, \quad \text{for } (B^{-1}b')_i < 0 \tag{10}$$

Now combining (7) and (10) let us define

$$\theta_{L_1} = \max_i \{\frac{-(B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i > 0\},$$

$$\theta_{L_2} = \max_i \{\frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i < 0\} \text{ and}$$

$$\theta_L = \max\{\theta_{L_1}, \theta_{L_2}\}.$$

Similarly using (8) and (9) let us define

$$\theta_{U_1} = \min_i \{\frac{-(B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i < 0\},$$

$$\theta_{U_2} = \min_i \{\frac{(U_B)_i - (B^{-1}b)_i}{(B^{-1}b')_i} \mid (B^{-1}b')_i > 0\} \qquad \text{and}$$

$\theta_U = \min\{\theta_{U_1}, \theta_{U_2}\}$. Thus θ_L and θ_U become the lower and upper bound for θ respectively, for which the solution remain optimum, hence the result.

Remark1. If all $(B^{-1}b')_i \le 0$, then let $\theta_{L_1} = -\infty$. Also if corresponding to every $(B^{-1}b')_i < 0$, the basic variable $(x_B)_i$ does not have a finite upper bound, then $\theta_{L_2} = -\infty$.

Remark2. If all $(B^{-1}b')_i \geq 0$, then let $\theta_{U_1} = \infty$. Also if corresponding to every $(B^{-1}b')_i > 0$, the basic variable $(x_B)_i$ does not have a finite upper bound, then $\theta_{U_2} = \infty$ Gass and Saaty[10, 11] showed that there are a finite number of such characteristic values.

In general if θ_L^k represent the kth characteristic value of θ, then the lower bound at the kth stage is obtained as,

$$\theta_{L_1}^k = \max_i \{ \frac{-(B^{-1}b)_i^k}{(B^{-1}b')_i^k} \mid (B^{-1}b')_i^k > 0 \} \qquad (11)$$

$$\theta_{L_2}^k = \max_i \{ \frac{(U_B)_i^k - (B^{-1}b)_i^k}{(B^{-1}b')_i^k} \mid (B^{-1}b')_i^k < 0 \} \qquad (12)$$

and $\theta_L^k = \max\{\theta_{L_1}^k, \theta_{L_2}^k\}$.

In (11) and (12) the notation $(\ )_i^k$ denotes the ith term in the column vector $(\ )$ at the kth iteration and the optimum solution at k^{th} stage is given by

$$x_B^k = (B^k)^{-1}[b^k + \theta b'^k], \ \theta_L^k \leq \theta \leq \theta_L^{k-1} \qquad (13)$$

Similarly if θ_U^m represent the mth characteristic value of θ, then the upper bound at the mth stage is obtained as,

$$\theta_{U_1}^m = \min_i \{ \frac{-(B^{-1}b)_i^m}{(B^{-1}b')_i^m} \mid (B^{-1}b')_i^m < 0 \} \qquad (14)$$

$$\theta_{U_2}^m = \min_i \{ \frac{(U_B)_i^m - (B^{-1}b)_i^m}{(B^{-1}b')_i^m} \mid (B^{-1}b')_i^m > 0 \} \qquad (15)$$

and $\theta_U^m = \min\{\theta_{U_1}^m, \theta_{U_2}^m\}$.

The optimum solution at m^{th} stage is given by (13) for $\theta_U^{m-1} \leq \theta \leq \theta_U^m$.

For problems without upper bounds the new basis can be obtained by carrying out a simplex iteration. For problems with upper bounds the algorithm for performing the parametric analysis is described next.

III. Algorithm

Assume that the LP problem (2) with bounded variables provides an optimum solution for $\theta = 0$.

Step1. Solve problem (2) for $\theta = 0$ using bounded simplex or simplex method and obtain the optimum table.

Step2. Using theorem 1, determine the characteristic values for θ i.e. θ_L^k and θ_U^k for $k = 1$, such that the optimum solution obtained in step1 remains optimum.

Step3. For $\theta < 0$, let $\theta_L^k = \theta_L$ denote the characteristic value for the lower bound of θ i.e. the value of θ at which the basis for optimum solution changes.

IF $\theta_L^k = -\infty$, there is no characteristic value for the problem. The basis at $\theta = 0$ will remain optimum for all $\theta \leq \theta_U^k$ and **GOTO** Step 5.

ELSE If θ_L^k is finite valued, then we have the following two cases.

Case I. IF $\theta_L^k = \theta_{L_1}^k$, for $i = r$ (say) $\in I$.

THEN x_r becomes non-basic and a dual simplex iteration is carried out to obtain the new basis as well as the new optimum solution since for $\theta < \theta_L^k$ the basic variable x_r becomes negative and **GOTO** Step4.

IF the dual simplex iteration cannot be carried out **THEN** there is no feasible solution for $\theta < \theta_L^k$ and **GOTO** Step 5.

Case II. IF $\theta_L^k = \theta_{L_2}^k$, for $i = m$ (say) $\in I$.

THEN x_m becomes non-basic at its upper bound as for $\theta < \theta_L^k$ the value of x_m violates the upper bound constraint. Substitute $x_m = u_m - x_m'$ and obtain the new basis presenting a new optimum solution by carrying out a simplex iteration **GOTO** Step4.

Step4. For $k = k + 1$, determine the new characteristic values of θ and the corresponding optimum solutions using (11-13) and repeat Step 3-4.

Step5. For $\theta > \theta_U^m > 0$, where $\theta_U^m = \theta_U^1$ is obtained in Step 2, the parametric analysis is performed as follows.

Step6. **IF** $\theta_U^m = \infty$, **THEN** there is no characteristic value for the upper bound of θ. The basis at $\theta = 0$ will remain optimum for all $\theta \geq \theta_L^m$ and the analysis stopped.

ELSEIF θ_U^m is finite valued, then we have the following two cases.

Case I.

IF $\theta_U^m = \theta_{U_1}^m$, for $i = r$ (say) $\in I$ **THEN** x_r becomes non-basic and a dual simplex iteration is carried out to obtain the new basis as well as the new optimum solution **GOTO** Step 7.

IF the dual simplex iteration cannot be carried out then there is no feasible solution for $\theta > \theta_U^m$ and the analysis stopped at this stage.

Case II.

IF $\theta_U^m = \theta_{U_2}^m$, for $i = n$ (say) $\in I$ **THEN** x_n becomes non-basic at its upper bound, substitute $x_n = u_n - x'_n$ and obtain the new basis presenting a new optimum solution by carrying out a simplex iteration and **GOTO** Step 7.

Step7. For $m = m + 1$ evaluate the characteristic values for θ as well as the corresponding optimum solutions using (13-15) and repeat Step 6-7.

Stop.

IV. Numerical Example

Consider the problem

$$\text{Max} \quad z = 3x_1 + 5x_2 + 2x_3$$

Subject to
$$x_1 + x_2 + 2x_3 \leq 10 - \theta$$
$$2x_1 + 4x_2 + 3x_3 \leq 16 + 2\theta$$
$$0 \leq x_1 \leq 4, 0 \leq x_2 \leq 3, 0 \leq x_3 \leq 3$$

Step1. At $\theta = 0$ the optimum solution can be obtained easily and the optimum table is

Table. 1.

	x'_1	x_3	x_5	Solution
x_4	- ½	5/4	-1/4	4
x_2	- ½	¾	1/4	2
	½	7/4	5/4	22

The basic variables in the tableau are $x_4 = 4$, $x_2 = 2$ and all the non-basic variables x'_1, x_3, x_5 are at zero level. The value of x_1 is easily obtained as $x_1 = u_1 - x'_1 = 4$. Thus x_1 is at its upper bound and the value of the objective $z = 22$.

Step2. Now $B = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix}$, $b' = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. So $B^{-1}b' = \begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix}$.

Therefore $\theta_{L_1}^1 = \max\{\frac{-2}{1/2}\} = -4$, for $i = 2$, $\theta_{L_2}^1 = -\infty$

and $\theta_{U_1}^1 = \min\{\frac{-4}{-3/2}\} = 8/3$, for $i = 4$,

$\theta_{U_2}^1 = \min\{\frac{3-2}{1/2}\} = 2$, for $i = 2$. Thus

$\theta_L^1 = \max\{\theta_{L_1}^1, \theta_{L_2}^1\} = -4 = \theta_{L_1}^1$ with $i = 2$ and

$\theta_U^1 = \min\{\theta_{U_1}^1, \theta_{U_2}^1\} = 2 = \theta_{U_2}^1$, with $i = 2$. Hence the optimum solution of the problem is given by

$$x = B^{-1}(b + \theta b') = \begin{pmatrix} 4 \\ 2 + \theta/2 \\ 0 \end{pmatrix}, \quad z = 22 + \frac{5}{2}\theta, \quad \text{for}$$

$\theta_L^1 \leq \theta \leq \theta_U^1$, where $\theta_L^1 = -4$, $\theta_U^1 = 2$ are the first characteristic values of the lower and upper bound of θ respectively for which the solution is optimum.

Step3. $\theta_L^1 = -4 = \theta_{L_1}^1$ with $i = 2$, thus x_2 becomes non-basic and substituted by $x'_1 = u_1 - x_1$ and the optimum table for $\theta < \theta_{L_1}^1$ is obtained by carrying out a dual simplex iteration as

Table. 2.

	x_2	x_3	x_5	Solution
x_4	- 1	½	- ½	2
x_1	2	3/2	½	8
	1	5/2	3/2	24

Step4. Using (12-14) for $k = 2$, we obtained $\theta_L^2 = \max\{\theta_{L_1}^2, \theta_{L_2}^2\} = -8 = \theta_{L_1}^2$ with $i = 1$. The optimum

solution of the problem is $x = \begin{pmatrix} 8+\theta \\ 0 \\ 0 \end{pmatrix}$, $z = 24+3\theta$,

for $\theta_L^2 \le \theta \le \theta_L^1$, where $\theta_L^2 = -8$. Now $\theta_L^2 = \theta_{L_1}$ for $i = 1$, as the dual simplex method cannot be applied therefore the analysis stopped at this stage.

Step5-6. From Table 1, since $\theta_U^1 = \theta_{U_2}^1$ for $i = 2$, therefore x_2 becomes non-basic at its upper bound for $\theta > \theta_U^2$, substituted by $x_2 = u_2 - x'_2$ and the new optimum basis is obtained by carrying out a simplex iteration as

Table. 3.

	x'_1	x'_2	x_5	Solution
x_4	1/3	5/3	- 2/3	17/3
x_3	- 2/3	- 4/3	1/3	- 4/3
	5/3	7/3	2/3	- 8/3

Step7. Using (14 – 16) for $m = 2$, we obtained $\theta_U^2 = \min\{\theta_{U_1}^2, \theta_{U_2}^2\} = 17/7 = \theta_{U_1}^2$, with $i = 4$. The optimum solution of the problem is given by

$x = \begin{pmatrix} 4 \\ 3 \\ -4/3 + 2\theta/3 \end{pmatrix}$, $z = \frac{73}{3} + \frac{4}{3}\theta$, for

$\theta_U^1 \le \theta \le \theta_U^2$, where $\theta_U^2 = 17/7$.

Following the same procedure described in the algorithm we obtained the following optimum solutions.

Table. 4.

	x'_1	x'_2	x_4	Solution
x_5	- ½	- 5/2	- 3/2	-17/2
x_3	- ½	- ½	½	3/2
	2	4	1	3

$x = \begin{pmatrix} 4 \\ 3 \\ 3/2 - \theta/2 \end{pmatrix}$, $z = 30 - \theta$, for $\theta_U^2 \le \theta \le \theta_U^3$,

where $\theta_U^3 = 3$.

Table. 5.

	x'_2	x_3	x_4	Solution
x_5	- 2	- 1	- 2	-10
x_1	- 1	2	1	7
	2	4	3	21

$x = \begin{pmatrix} 7-\theta \\ 3 \\ 0 \end{pmatrix}$, $z = 26 - 3\theta$, for $\theta_U^3 \le \theta \le \theta_U^4$, where

$\theta_U^4 = 7$.

Table. 6.

	x_1	x_3	x_4	Solution
x_5	- 2	- 5	- 4	-24
x_2	1	2	1	10
	2	8	5	21

$x = \begin{pmatrix} 0 \\ 10-\theta \\ 0 \end{pmatrix}$, $z = 50 - 5\theta$, for $\theta_U^4 \le \theta \le \theta_U^5$, where

$\theta_U^5 = 10$.

As the dual simplex method cannot be applied therefore the analysis stopped at this stage.

The optimum solutions thus obtained are given below

$$\max z = \begin{cases} 24 + 3\theta, & -8 \le \theta \le -4 \\ 22 + 5\theta/2, & -4 \le \theta \le 2 \\ 73/3 + 4\theta/3, & 2 \le \theta \le 17/7 \\ 30 - \theta, & 17/7 \le \theta \le 3 \\ 26 - 3\theta, & 3 \le \theta \le 7 \\ 50 - 5\theta, & 7 \le \theta \le 10 \end{cases}$$ and

$$x = \begin{cases} (8+\theta,0,0), & -8 \le \theta \le -4 \\ (4,2+\theta/2,0) & -4 \le \theta \le 2 \\ (4,3,-4/3+2\theta/3), & 2 \le \theta \le 17/7 \\ (4,3,3/2-\theta/2), & 17/7 \le \theta \le 3 \\ (7-\theta,3,0) & 3 \le \theta \le 7 \\ (0,10-\theta,0) & 7 \le \theta \le 10 \end{cases}$$

which is identical to the solutions obtained by the regular parametric analysis.

V. Conclusions

In this paper an algorithm to perform the complete right-hand side parametric analysis for LP problems with bounded variables is developed. The upper bound constraints are considered implicitly thus reduce the size of the basis matrix greatly and the computation become highly efficient than the regular parametric analysis. A numerical example has solved in this regard that clearly showed the efficiency of using the developed algorithm. Besides one can build a computer program to perform the parametric analysis using the same algorithm which is necessary for large scale problems.

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