Dhaka Univ. J. Sci. 62(1): 25-29, 2014 (January)

Generalized Composite Numerical Integration Rule Over a Polygon Using Gaussian Quadrature

M. Alamgir Hossain[1] and Md. Shafiqul Islam*,[2]

[1]*Department of Mathematics, Jagannath University, Dhaka -1100, Bangladesh*

[2]*Department of Mathematics, Dhaka University, Dhaka – 1000, Bangladesh*

(Received: 22 November 2012; Accepted: 17 July 2013)

Abstract

The aim of this paper is to evaluate double integrals over a polygon exploiting coordinate transformation. At first any polygon with *m*-sides is decomposed into $(m-2)$ triangles. Then each triangle is transformed into a standard triangular finite element using the basis functions in local space. Then the standard triangle is decomposed into $4\times n^2$ right isosceles triangles with side lengths $1/n$, and thus composite numerical integration is employed. In addition, the affine transformation over each decomposed triangle and the use of linearity property of integrals are applied. Finally, each isosceles triangle is transformed into a 2-sqare finite element to compute new s^2 sampling points and corresponding weight coefficients, using s point's conventional Gaussian quadrature, which are applied again to evaluate the double integral. We demonstrate some numerical examples through the proposed method.

Key words: Numerical Integration, Quadrilateral and Triangular Finite Elements, Gaussian Quadrature.

I. Introduction

Numerical integrations over triangular regions were first introduced by Hammer *et al*[1–3], and then by Stroud[4]*. In finite element method, the triangular elements are widely used in the area of numerical integration schemes[5]. The works of Hammer *et al*[1-3] have been further developed by Cowper[6] and thus he provided a table of Gaussian quadrature formulae for symmetrically placed integration points. Lethor[7] and Hillion[8] derived formulas for triangles as product of one-dimensional Gauss quadrature rule. Laursen and Gellert[9] also discussed elaborately symmetric integration formulae of precision up to degree ten. One may realize that a lot of works of numerical integration using Gaussian quadrature over triangular[1-9] region, composite numerical integration using Gauss quadrature over triangular[10-12] region and over quadrilateral[13] region have been done, but no generalized work of composite numerical integration has been attempted so far over the polygon.

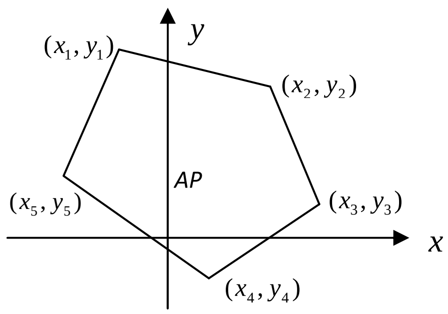
Recently, a rigorous and elaborate survey has been reported by Sarada and Nagaraja[14]. In this paper, they have derived some formulas for limited shapes of triangles and quadrilaterals, and then generalized their process for any arbitrary polygon. This work is based on higher order (e.g., 5, 10, 15 and 20) Gaussian quadrature rule. In contrast to this study, we propose to develop a general composite integration formula over any arbitrary polygon by decomposing the polygon into arbitrary shapes of triangles, described in section II, using lower order (e.g., 3, 5 and 10) Gauss points to get more accuracy. Numerical examples are taken from the recent paper[14], and thus are compared. The subsequent formulations are developed by *Mathematica*.

II. Formulation of Integrals Over an Arbitrary Polygon

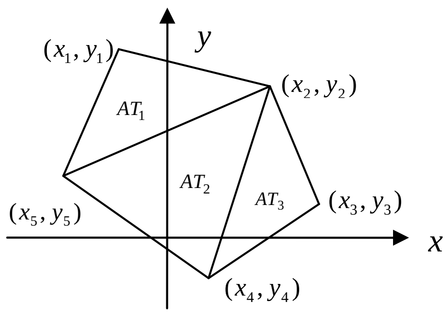
The integral of an arbitrary function, $f(x,y)$ over an arbitrary polygon *AP* with *m* sides is given by

$$I=\iint_{AP} f(x,y)\ dy\,dx=\iint_{AP} f(x,y)\ dx\,dy \qquad (1)$$

At first the polygon *AP* with *m*-sides is decomposed into *m-2* triangles shown in figure 1. The integral *I* of Eq. (1) is the sum of *m-2* integrals whose domains are triangles.



(a) Arbitrary polygon *AP*



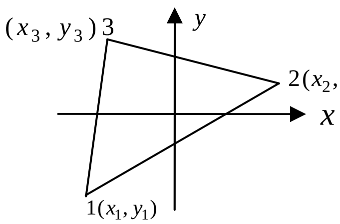
(b) Decomposed triangles AT_t

Fig. 1. Polygon *AP* with *m*-sides is decomposed into *m-2* triangles

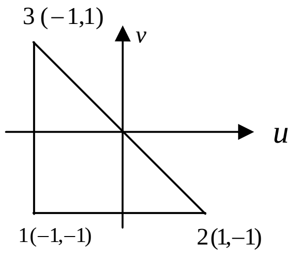
$$I=\sum_{t=1}^{m-2} I_t=\sum_{t=1}^{m-2}\iint_{AT_t} f(x,y)\ dy\,dx=\sum_{t=1}^{m-2}\iint_{AT_t} f(x,y)\ dx\,dy \qquad (2)$$

The each integral I_t of Eq. (2) is then transformed into an integral over the region of the standard triangle $ST=\{(u,v):-1\le v\le 1,-1\le u\le -v\}$ by the liner triangular finite element basis functions $L_i(u,v)$, shown in Fig. 2:

*Author for Correspondence. *e-mail*: mdshafiqul_mat@du.ac.bd



(a) Arbitrary triangle AT_t



(b) Standard triangle *ST*

Fig. 2. Transformation of arbitrary triangle AT_t into equivalent standard triangle *ST*

$L_1(u,v) = -\frac{1}{2}(u+v)$, $L_2(u,v) = \frac{1}{2}(1+u)$ and $L_3(u,v) = \frac{1}{2}(1+v)$.

The coordinates are changed by assuming that

$$x = \sum_{i=1}^{3} x_i L_i \quad \text{and} \quad y = \sum_{i=1}^{3} y_i L_i \tag{3a}$$

and the corresponding *Jacobian*

$$J_1 = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{3b}$$

Therefore using Eqs. (3) and (2) to obtain,

$$I_t = \iint_{AT_t} f(x,y)\; dy\, dx = \iint_{ST} f(u,v)\left|J_1\right|\, du\, dv \tag{4}$$

The integral I_t of Eq. (4) can be further transformed into an integral over the standard 2-square, $\{(\xi,\eta): -1 \le \xi, \eta \le 1\}$ using standard quadrilateral basis functions $Q_i(\xi,\eta)$, as shown in Fig. 3:

$Q_1(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta-1)$,

$Q_2(\xi,\eta) = -\frac{1}{4}(\xi+1)(\eta-1)$, $Q_3(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)$,

$Q_4(\xi,\eta) = -\frac{1}{4}(\xi-1)(\eta+1)$.

Assume that

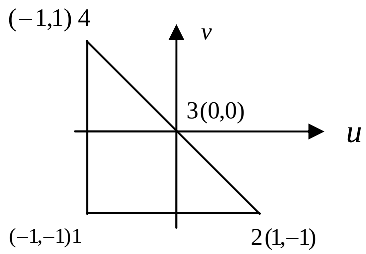
$$u = \sum_{i=1}^{4} u_i\, Q_i = \frac{1}{4}(-1+3\xi-\eta(1+\xi)) = u(\xi,\eta) \tag{5a}$$

$$v = \sum_{i=1}^{4} v_i\, Q_i = \frac{1}{4}(-1+3\eta-\xi(1+\eta)) = v(\xi,\eta) \tag{5b}$$

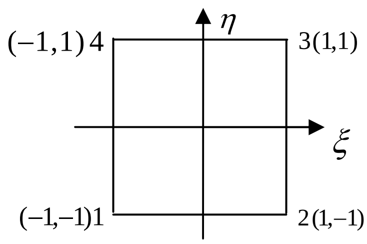
and

$$J_2 = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \frac{1}{4}(2-\eta-\xi) \tag{5c}$$

Note that J_1 depends on the vertices of the given arbitrary triangular region, but J_2 is fixed.



(a) Standard triangle, *ST*



(b) Standard square, *SQ*

Fig. 3. Transformation of standard triangle *ST* into 2-square *SQ*

Let $F(u,v) = f(u,v)\left|J_1\right|$ and using (5), Equ. (4) becomes

$$I_t = \int_{-1}^{1}\int_{-1}^{-u} F(u,v)\; dv\, du = \int_{-1}^{1}\int_{-1}^{1} F(u(\xi,\eta), v(\xi,\eta))\left|J_2\right| d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} F\left(\frac{1}{4}(-1+3\xi-\eta(1+\xi)), \frac{1}{4}(-1+3\eta-\xi(1+\eta))\right)\frac{1}{4}(2-\eta-\xi)\; d\xi\, d\eta \tag{6}$$

Now Eq. (6) represents an integral over the standard 2-square region: $\{(\xi,\eta): -1 \le \xi, \eta \le 1\}$. Hence using Gauss Legendre quadrature rule for the integral I_t of Eq. (6), we have

$$I_t = \sum_{i=1}^{s}\sum_{j=1}^{s} \frac{1}{4}(2-\eta_j-\xi_i)\, w_i\, w_j\, F\left(\frac{1}{4}(-1+3\xi_i-\eta_j(1+\xi_i)), \frac{1}{4}(-1+3\eta_j-\xi_i(1+\eta_j))\right) \tag{7}$$

where (ξ_i, η_j) are Gaussian points in the ξ, η directions of order s and w_i, w_j are the corresponding weight coefficients. We can write Eq. (7) as:

$$I_t = \sum_{k=1}^{N=s\times s} c'_k\; F(x'_k, y'_k) \tag{8}$$

where c'_k, x'_k and y'_k can be written in the form:

$$c'_k = \frac{1}{4}(2-\eta_j-\xi_i)\, w_i\, w_j$$

$$x'_k = \frac{1}{4}(-1+3\xi_i-\eta_j(1+\xi_i))$$

$$y'_k = \frac{1}{4}(-1+3\eta_j-\xi_i(1+\eta_j))$$

$$(k = 1,2,\ldots\ \ldots\ \ldots\ \ldots, N), \quad (i, j = 1,2,\ldots\ \ldots\ \ldots\ \ldots, s) \tag{9}$$

Table 1. Outputs of c'_k, x'_k and y'_k of Eq.s (9) for *s = 5*

k	c'_k	x'_k	y'_k
1	0.028067174431214	-0.950889367642309	0.861470324235019
2	0.046275406309135	-0.758409434945362	0.686239721098985
3	0.036857657161022	-0.476544961484666	0.429634884453998
4	0.015744196426143	-0.194680488023970	0.173030047809011
5	0.002633266629203	-0.002200555327023	-0.002200555327023
6	0.067124593690865	-0.942264702861852	0.502384453182495
7	0.114542702111995	-0.715982010624261	0.360956609587106
8	0.099488780941957	-0.384617327526421	0.153851982579262
9	0.052864972328108	-0.053252644428581	-0.053252644428581
10	0.015744196426143	0.173030047809011	-0.194680488023970
11	0.097927415226499	-0.929634884453998	-0.023455038515334
12	0.172797751608793	-0.653851982579262	-0.115382672473579
13	0.161817283950617	-0.250000000000000	-0.250000000000000
14	0.099488780941957	0.153851982579262	-0.384617327526421
15	0.036857657161022	0.429634884453998	-0.476544961484666
16	0.097655803573857	-0.917005066046144	-0.549294530213163
17	0.176220431895882	-0.591721954534264	-0.591721954534264
18	0.172797751608793	-0.115382672473579	-0.653851982579262
19	0.114542702111995	0.360956609587106	-0.715982010624261
20	0.046275406309135	0.686239721098985	-0.758409434945362
21	0.053501082233226	-0.908380401265687	-0.908380401265687
22	0.097655803573857	-0.549294530213163	-0.917005066046144
23	0.097927415226499	-0.023455038515334	-0.929634884453998
24	0.067124593690865	0.502384453182495	-0.942264702861852
25	0.028067174431214	0.861470324235019	-0.950889367642309

The weighting coefficients c'_k and sampling points (x'_k, y'_k) of various order can be now easily computed from Eq.(9). Using *Mathematica* program, the outputs of c'_k, x'_k and y'_k for $s = 5$ are given in Table 1.

Now we decompose $ST = \{(u,v): -1 \le v \le 1, -1 \le u \le -v\}$ in (u,v)-space of Eq. (4) into $4(n \times n) = 4n^2$ right isosceles triangles, T_i with side lengths $1/n$ [10-13]. Then Eq. (4) reduces to

$$I_t = \sum_{i=1}^{4(n\times n)} \iint_{T_i} F(u,v)\, du\, dv \tag{10}$$

Since each T_i is to be transformed again into a standard triangle and using composite integration rule[10-13] we can obtain the following:

$$I_t = \frac{1}{4n^2} \sum_{k=1}^{N=s\times s} c'_k\, H(x'_k, y'_k) \tag{11}$$

where

$$H(x'_k,y'_k) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1-i} F\left(\frac{x'_k + 2(i-n)+1}{2n}, \frac{y'_k + 2(j-n)+1}{2n}\right)$$
$$+ \sum_{i=0}^{2n-2}\sum_{j=0}^{2n-2-i} F\left(\frac{-x'_k + 2(i-n)+1}{2n}, \frac{-y'_k + 2(j-n)+1}{2n}\right) \tag{12a}$$

and

$$c'_k = \frac{1}{4}(2 - \xi_p - \eta_q)\, w_p\, w_q$$
$$x'_k = \frac{1}{4}(-1 + 3\xi_p - \eta_q(1+\xi_p))$$
$$y'_k = \frac{1}{4}(-1 + 3\eta_q - \xi_p(1+\eta_q))$$
$$(k = 1,2,\ldots\ \ldots\ \ldots\ \ldots, N),\ (i,\ j = 1,2,\ldots\ \ldots\ \ldots\ \ldots, s) \tag{12b}$$

III. Numerical Examples

In this section, all examples have taken from the recent paper[14] which are shown in Tables 2-4.

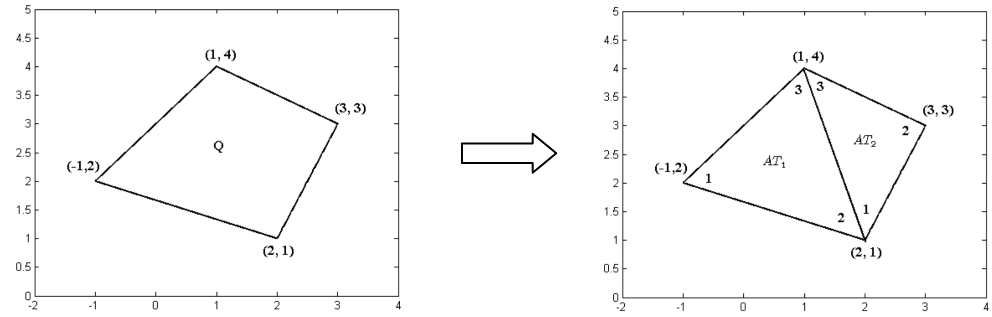


Fig. 4. Quadrilateral region *Q*

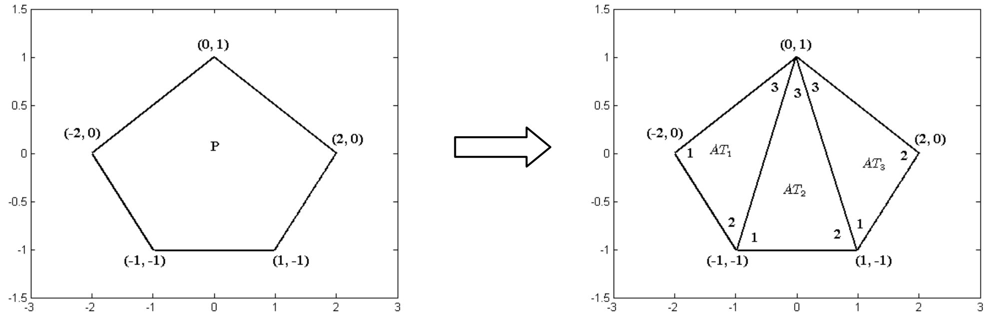


Fig. 5. Pentagon *P*

Table 2. Generalized composite numerical integration over triangular, rectangular and parallelogram regions

Integral with exact value	S	Computed value using present method for (4×5^2) sub triangles	Computed value using present method for (4×10^2) sub triangles	Computed value using the method of Sarada and Nagaraja[14]
$\int_0^1\int_x^{2-x}\sqrt{x+y}\,dy\,dx$	$s=3$	1.131372465703519	1.131371135535597	
$=1.13137084989848$	$s=5$	1.131371010152331	1.131370878227623	1.13137111584312
	$s=10$	1.131370849405025	1.131370849898650	1.13137084930298
$\int_1^3\int_{1-x}^{2x-2}\left[(x+y)^{1/2}(1+x+y)^2\right]dy\,dx$	$s=3$	211.418375413220700	211.418375378138600	
$=211.418375377476$	$s=5$	211.418375377477900	211.418375377475800	211.418375371115
	$s=10$	211.418375377475800	211.418375377475700	211.418375377482
$\int_3^5\int_1^4\sqrt{x+y}\,dx\,dy$	$s=3$	15.247081338038640	15.247081338037520	
$=15.2470813380375$	$s=5$	15.247081338037510	15.247081338037510	15.2470813595701
	$s=10$	15.247081338037510	15.247081338037510	15.2470813380375
$\int_1^2\int_{-1}^{3-x}\sqrt{x^2+y^2}\,dy\,dx$	$s=3$	4.168721788528673	4.168721788576948	
$=4.16872178857775$	$s=5$	4.168721788577755	4.168721788577756	4.16896959209620
	$s=10$	4.168721788577755	4.168721788577755	4.16872193734188
$\int_0^1\int_{1-x}^{2+x}(x+y)^{1/2}(1+x+y)^2\,dy\,dx$	$s=3$	30.631252208539310	30.631252208190520	
$=30.6312522081847$	$s=5$	30.631252208184680	30.631252208184680	30.6312523036460
	$s=10$	30.631252208184680	30.631252208184680	30.6312522081853
$\int_{-1}^0\int_{-y-2}^{y+2}\frac{x^4+y^3}{1+x^2}\,dx\,dy$	$s=3$	1.006013394580186	1.006013395754989	
$=1.00601339577510$	$s=5$	1.006013395775123	1.006013395775098	0.99528697888712
	$s=10$	1.006013395775098	1.006013395775098	1.00637151173877

Table 3. Generalized composite numerical integration over arbitrary quadrilateral region *Q* (Fig.4)

Integral with exact value	S	Computed value using present method for (4×5^2) sub triangles	Computed value using present method for (4×10^2) sub triangles	Computed value using the method of Sarada and Nagaraja[14]
$\iint_Q \left[(x+y)^{1/2}(1+x+y)^2\right]dy\,dx$	$s = 3$	298.234339210737800	298.234339210045600	
$= 298.2343392100\ 34$	$s = 5$	298.234339210034300	298.234339210034300	298.234338347174
	$s = 10$	298.234339210034300	298.234339210034300	298.234339210033
$\iint_Q \frac{1}{\sqrt{x+y}}\,dy\,dx$	$s = 3$	3.549613019979926	3.549613026675375	
	$s = 5$	3.549613026789603	3.549613026789717	3.54960971225221
$= 3.5496130267\ 8971$	$s = 10$	3.549613026789717	3.549613026789717	3.54961302681661

Table 4. Generalized composite numerical integration over any arbitrary pentagon *P* (Fig. 5)

Integral with exact value	S	Computed value using present method for (4×5^2) sub triangles	Computed value using present method for (4×10^2) sub triangles	Computed value using the method of Sarada and Nagaraja[14]
$\iint_P (1-x)\sin(10xy)\,dx\,dy$	$s = 3$	- 0.013100041165331	-0.013103628437347	
$= -0.0131037196\ 69957$	$s = 5$	- 0.013103721381516	-0.013103719671051	- 0.451592814601184
	$s = 10$	- 0.013103719669958	-0.013103719669957	- 0.018237768238161
$\iint_P \frac{x^4+y^3}{1+x^2}\,dx\,dy$	$s = 3$	1.924030542963068	1.924030542637816	
	$s = 5$	1.924030542632652	1.924030542632650	1.90714805151570
$= 1.9240305426\ 3265$	$s = 10$	1.924030542632650	1.924030542632651	1.92457441414507

IV. Conclusions

In this paper, we have discussed the formulation of double integrals over an arbitrary polygon. At first we have decomposed any polygon with *m*-sides into *m-2* triangles. Each triangular region is transformed into a standard triangle $\{(u,v): -1 \le u \le 1, -1 \le v \le -u\}$ by using triangular basis functions. Each of the standard triangles is further decomposed into $4\times n^2$ triangles. Then we map each of the standard triangle into the 2-square using standard quadrilateral basis functions. For each triangle we generate s^2 new Gauss points using the lower order conventional Gauss quadrature of order s, and thus the composite numerical integration over the standard triangular finite elements are applied. We observe that computed results of the given examples converge to the exact solutions correct upto fifteen decimal places. The technique of this paper, using lower order Gauss Legendre quadrature rule, can give high accuracy results compare to the existing formulations in the literatures.

References

1. Hammer, P. C., O. J. Marlowe and A. H. Stroud, 1956. Numerical integration over simplexes and cones. *Math. Tables and other aids to computation*, **10**, 130–136.
2. Hammer, P. C. and A. H. Stroud, 1956. Numerical integration over simplexes. *Math. Tables and other aids to computation,* **10**, 137–139.
3. Hammer, P. C. and A. H. Stroud, 1958. Numerical evaluation of multiple integrals. *Math. Tables and other aids to computation*, **12**, 272–280.
4. Stroud, A. H., 1974. Numerical Quadrature and Solution of Ordinary Differential Equations, Springer-Verlag, New York Berlin Heidelberg.
5. Zienkiewicz, O. C., 1977. The Finite Element Method, 3rd Edn., Mc Graw Hill Inc.
6. Cowper, G. R., 1973. Gaussian quadrature formulas for triangles. *Inter. Jn. Numer. Methods Engrs,* **7** , 405–408.
7. Lethor, F. G., 1976. Computation of double integrals over a triangle. *J. Computational and Applied Mathematics*, **2**, 219–224
8. Hillion, P., 1977. Numerical integration on a triangle, *Inter. Jn. Numer. Methods Engrs*. **11**, 797–815.
9. Lauresn, M. E. and M. Gellert, 1978. Some criteria for numerically integrated matrices and quadrature formulas for triangles, *Inter. Jn. Numer. Methods Engrs*. **12**, 67–76.
10. Rathod, H.T., K.V. Nagaraja and B. Venkatesudu, 2007. Symmetric Gauss Legendre quadrature formulas for composite numerical integration over a triangular surface. *Appl. Math. Comput.*, **188**, 865–876.
11. Rathod, H.T., K.V. Nagaraja and B. Venkatesudu, 2007. On the application of two symmetric Gauss Legendre quadrature rules for composite numerical integration over a triangular surface. *Appl. Math. Comput.*, **190**, 21–39.
12. Islam, M. S. and M. A. Hossain, 2008. Numerical Integration over an Arbitrary Triangular Region. *International e-Journal of Numerical Analysis and Related Topics (IeJNART),* **2**, 25-40.
13. Islam, M. S. and M. A. Hossain, 2009. Numerical Integrations over an Arbitrary Quadrilateral Region. *Appl. Math. Comput.*, **210**, 515 – 524.
14. Sarada, J. and K.V. Nagaraja, 2011. Generalized Gaussian quadrature rules over two-dimensional regions with linear sides. *Appl. Math. Comput.*, **217**, 5612–5621.

Dhaka Univ. J. Sci. 62(1): 25-29, 2014 (January)

*Author for Correspondence. *e-mail*: mdshafiqul_mat@du.ac.bd

Dhaka Univ. J. Sci. 62(1): 25-29, 2014 (January)

*Author for Correspondence. *e-mail*: mdshafiqul_mat@du.ac.bd