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Multilinear Algebras and Tensors with Vector Subbundle of Manifolds

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Abstract

In the present paper some aspects of tensor algebra, tensor product, exterior algebra, symmetric algebra, module of section, graded algebra, vector subbundle are studied. A *Theorem* 1.32. is established by using sections and fibrewise orthogonal sections of an application of Gran-Schmidt.

Keywords: Multilinear and tensor algebra, tangent and tensor bundle, subbundle associated frame bundles, graded and Symmetric algebra.

1. Introduction

Multilinear algebra and tensor algebra of R −modules are needed to use higher order tensors. The tangent bundle, various tensor bundle, subbundle and associated frame bundles will play important roles as the theory of manifolds is developed. A theorem related with subbundle is treated with various tensor, graded algebra, tensor product, and trivial bundles.

II. Tensor Algebra

In order to study R −multilinear maps, we build a universal model of multilinear objects called the tensor algebra over R, where R will be the ring $C^\infty(M)$.

Definition 1.1. An R −module V is *free* if there is a subset $B \subset V$ such that every nonzero element $v \in V$ can be written uniquely as a finite R −linear combination of elements of B. The set B will be called a *(free) basis* of R.

Example 1.2. Let $\pi : E \to M$ be a trivial $n-$ plane bundle [1]. Then $\Gamma(E)$ is a free $C^\infty(M)$

−module on a basis of *n* elements. Another example is the integer lattice $\mathbb{Z}^k$, a free $\mathbb{Z}$ −module.

Definition 1.3. If V_1, V_2, V_3 are objects in $\mathcal{M}(R)$, a map $\varphi : V_1 \times V_2 \to V_3$ is $R-linear$ if

$$\varphi(.\ ,V_2) : V_1 \to V_3$$

$$\varphi(V_1,.) : V_2 \to V_3$$

are $R-$ linear, $\forall\ v_i \in V_i,\ i = 1,2.$

Definition 1.4. [2] A *tensor product* of $R-$ modules V_1, V_2 is an R −module $V_1 \otimes V_2$, together with an R −bilinear map

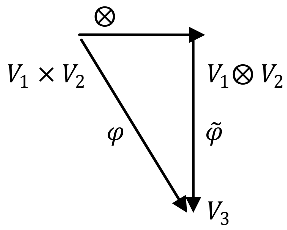
$$\otimes : V_1 \times V_2 \to V_1 \otimes V_2$$

with the following "universal property":

given any R −modules V_3 and any R −bilinear map

$$\varphi : V_1 \times V_2 \to V_3,$$

there is a unique R −linear map $\tilde{\varphi} : V_1 \otimes V_2 \to V_3$ such that the diagram



commutes. Write $\otimes(v, w) = v \otimes w$.

Corollary 1.5. If V_i is an R −module, $i = 1,\ 2,\ 3,$ there are unique R −linear isomorphism

$$V_1 \otimes (V_2 \otimes V_3) = (V_1 \otimes V_2) \otimes V_3$$
$$= V_1 \otimes V_2 \otimes V_3$$

identifying

$$v_1 \otimes (v_2 \otimes v_3) = (v_1 \otimes v_2) \otimes v_3$$
$$= v_1 \otimes v_2 \otimes v_3,$$
$$\forall\ v_i \in V_i,\ i = 1,2,3.$$

Definition 1.6. An element $v \in V_1 \otimes \ldots \otimes V_k$ is said to be *decomposable* if it can be written as a monomial $v = v_1 \otimes \ldots \otimes v_k$, for suitable elements $v_i \in V_i, 1 \le i \le k$. Otherwise, v is said to be *indecomposable.*

Lemma 1.7. If V and W are R −modules with respective bases A and B, then $V \otimes W$ is free with basis $C = \{a \otimes b \mid a \in A, b \in B\}$.

Proof. An arbitrary element $v \in A \otimes B$ can be written as a linear combination of decom-posable. A decomposable element $V \otimes W$ can be expanded the multilinearity of tensor product, to a linear combination of elements of C, proving that C spans $V \otimes W$. It remains to show that, if

$$\sum_{i,j=1}^{p,q} c_{ij}\ a_i \otimes b_j = \sum_{i,j=1}^{p,q} d_{ij}\ a_i \otimes b_j,$$

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where $a_i \in A$ and $b_j \in B,\ 1 \le i \le p,\ 1 \le j \le q$ then all $c_{i,j} = d_{i,j}$. Subtracting one expression from the other, we only need to prove that

$$\sum_{i,j=1}^{p,q} c_{ij}\ a_i \otimes b_j = 0 \ldots \quad (4.1)$$

implies that all $c_{i,j} = 0$. The bilinear functional $\varphi : V \times W \to R$ correspond one to one to any functions $f : A \times B \to R$.The correspondence is $\varphi \leftrightarrow \varphi \mid (A \times B)$. Thus, the linear functionals $\tilde{\varphi} : V \otimes W \to R$ also correspond one to one to these functions $f : A \times B \to R$.

If $(a,b) \in (A \times B)$, let $f_{a,b} : (A \times B) \to R$ be the function taking the value 1on (a,b) and the value 0 on every other element of $(A \times B)$.

The corresponding linear functional will be denoted by $\tilde{\varphi}_{a,b}$. Applying $\tilde{\varphi}_{a_i,b_j}$ to equation (4.1), we see that all $c_{ij} = 0$. This completes the proof.

Proposition 1.8. If $\lambda_i : V_i \to W_i$ is an $R-$

linear map,$1 \le i \le k$, there is a unique $R-$linear map

$$\lambda_1 \otimes \ldots\ldots \otimes \lambda_k : V_1 \otimes \ldots\ldots \otimes V_k \to W_1 \otimes \ldots\ldots \otimes W_k$$

which, on decomposable elements, has the formula

$$(\lambda_1 \otimes \ldots\ldots \otimes \lambda_k)\,(v_1 \otimes \ldots\ldots \otimes v_k) = \lambda_1(v_1) \otimes \ldots\ldots \otimes \lambda_k(v_k).$$

Proof. We know the decomposable span. So, the uniqueness is immediate. For existence, let us define the multilinear map

$$\lambda : V_1 \times \ldots\ldots \times V_k \to W_1 \otimes \ldots\ldots \otimes W_k$$

by

$$\lambda(v_1, \ldots\ldots, v_k) = \lambda_1(v_1) \otimes \ldots\ldots \otimes \lambda_k(v_k).$$

Then $\lambda_1 \otimes \ldots\ldots \otimes \lambda_k$ is defined to be the unique associated linear map. Hence, the proof is complete.

Definition 1.9. For the module of $R-$linear functionals, the *dual* V^* of an $R-$module V is $Hom_R(V,R)$.

Lemma 1.10. If V has a finite free basis $\{v_1, \ldots \ldots\ldots, v_n\}$, then V^* has a finite free basis $\{v_1, \ldots \ldots\ldots, v_n\}$, called the basis and defined by $v_i^*(v_j) = \delta_j^i,\ \ 1 \le i,j \le n$.

Corollary 1.11. If $V_1, \ldots\ldots, V_k$ are free $R-$ modules on bases $B_1, \ldots\ldots, B_k$, respectively, then $V_1 \otimes \ldots\ldots \otimes V_k$ is a free $R-$module with basis

$$B = \{v_1 \otimes \ldots\ldots \otimes v_k \mid v_i \in B_i,\ 1 \le i \le k\}.$$

Proposition 1.12. There is a unique $R-$linear map

$$l : V_1^* \otimes \ldots\ldots\ldots \otimes V_k^* \to (V_1 \otimes \ldots\ldots \otimes V_k)^*$$

which on decomposable elements has the formula

$$l(\eta_1 \otimes \ldots\ldots \otimes \eta_k)\,(v_1 \otimes \ldots\ldots \otimes v_k) = \eta_1(v_1) \otimes \ldots\ldots \otimes \eta_k(v_k).$$

If the $R-$modules V_i are all free on finite bases, then l is a canonical isomorphism.

Proof. Since the decomposable span, uniqueness is immediate. For existence, consider the multi linear functional

$$\theta : V_1^* \times \ldots\ldots\ldots \times V_k^* \times V_1 \times \ldots\ldots \times V_k \to R$$

by

$$\theta(\eta_1, \ldots\ldots, \eta_k, v_1 \ \ldots\ldots, v_k) = \eta_1(v_1) \ldots\ldots\ \eta_k(v_k).$$

by the universal property, this gives the associated linear functional

$$\tilde{\theta} : V_1^* \otimes \ldots\ldots\ldots \otimes V_k^* \otimes V_1 \otimes \ldots\ldots \otimes V_k \to R,$$

and we define

$$l : V_1^* \otimes \ldots\ldots\ldots \otimes V_k^* \to (V_1 \otimes \ldots\ldots \otimes V_k)^*$$

by

$$l(\eta)(v) = \tilde{\theta}\,(\eta \times v).$$

If $\{v_{i,1}, \ldots\ldots, v_{i,m_i}\}$ is a basis of $V_i, 1 \le i \le k$, let $\{v_{i,1}^*, \ldots\ldots, v_{i,m_i}^*\}$ be the dual basis. Let B and B^* be the respective bases of $V_1 \otimes \ldots\ldots \otimes V_k$ and $V_1^* \otimes \ldots\ldots\ldots \otimes V_k^*$ given by the Corollary 1.11. The formula

$$l(v_{1,j_1}^* \otimes \ldots\ldots \otimes v_{k,jk}^*)(v_{1,i_1} \otimes \ldots\ldots \otimes v_{k,i_k}) = \delta_{i_1}^{j_1} \ldots\ldots \delta_{i_k}^{j_k} = \delta_{i_1 \ldots\ldots i_k}^{j_1 \ldots\ldots j_k}$$

shows that l carries the basis B^* one to one onto the basis dual to B, so l is an isomorphism. This completes the proof.

Definition 1.13. [3] A *graded (associated) algebra A* over R is a sequence $\{A^n\}_{n=0}^{\infty}$ of $R-$modules, together with $R-$bilinear maps (multiplication)

$$A^n \times A^m \to A^{n+m},\ \ \forall n,m \ge 0,$$

which is strongly associative in the sense that the compositions

$$(A^n \times A^m) \times A^r \xrightarrow{.\times id} A^{n+m} \times A^r \xrightarrow{.} A^{n+m+r},$$

$$A^n \times (A^m \times A^r) \xrightarrow{id \times .} A^n \times A^{m+r} \xrightarrow{.} A^{n+m+r}$$

are equal, $\forall\ n,m,r \ge 0$.

Definition 1.14. If V is an $R-$module, then $\mathcal{T}(V)$ with multiplication $\otimes$, is called the *tensor algebra* of V. It is clear that the tensor algebra $\mathcal{T}(V)$ is connected.

Theorem 1.15. If $\lambda : V \to W$ is an $R-$linear map, then there is a unique induced homomorphism $\mathcal{T}(\lambda): \mathcal{T}(V) \to \mathcal{T}(W)$ of graded $R-$algebras such that $\mathcal{T}^0(\lambda) = id_R$ and $\mathcal{T}^1(\lambda) = \lambda$.

This homomorphism satisfies

$$\mathcal{T}^n(\lambda)(v_1 \otimes v_2 \otimes \ldots\ldots \otimes v_n) = \lambda\,(v_1) \otimes \lambda\,(v_2) \otimes \ldots\ldots \otimes \lambda(v_n),$$

$\forall\, n \geq 2,\ \forall\, v_i \in V,\ 1 \leq i \leq n.$

Finally, this induced homomorphism makes $\mathcal{T}$ a covariant function from the category of R −modules R −linear maps to the category of graded algebras over R and graded algebra homomorphisms.

Definition 1.16. The space of tensors on V of type (r, s) is the *tensor product*

$\mathcal{T}_s^r(V) = \mathcal{T}_0^r(V) \otimes \mathcal{T}_s^0(V).$

III. Exterior Algebra

The R −module $\Lambda^k(V)$ is called the k th exterior power of V. The connected graded R −algebra

$$\Lambda(\mathrm{V}) = \{\Lambda^k(V)\}_{\mathrm{k}=0}^{\infty}$$

with multiplication

$\Lambda^p(V) \times \Lambda^q(V) \xrightarrow{\Lambda} \Lambda^{p+q}(V)$

is called the exterior algebra of V [4].

Lemma 1.17. Let V be an R −module, $v \in V$. Then $v = -v \Leftrightarrow v = 0$.

Proof. Let V be an R −module where $v \in V$. Then

$v = 0 \Rightarrow v = -v.$

For the converse

$v = -v \Rightarrow 2v = 0$

$\Rightarrow v = 1/2(2v)$

$\Rightarrow v = 1/2(0)$

$\therefore v = 0.$

This completes the proof.

Definition 1.18. Let V and W be R- modules.

An antisymmetric $K - linear\ map\ \varphi : V^k \to W$ is a $K -$ linear map such that

$$\varphi\big(v_{\sigma(1)}, \dots\dots\dots, v_{\sigma(k)}\big) = (-1)^{\sigma}\, \varphi\,(v_1, v_2, \dots\dots, v_k), \quad \forall v_1, v_2, \dots\dots, v_k \in V, \forall\, \sigma \in \textstyle\sum k$$

where $(-1)^{\sigma} = \begin{cases} 1, & \sigma \text{ an even permutation,} \\ -1, & \sigma \text{ an odd permutation.} \end{cases}$

Lemma 1.19. If $\varphi : V^k \to W$ is antisymmetric, then $\tilde{\varphi}\,(\mathfrak{A}^k(V)) = \{0\}$.

Proof. It will be enough to show that $\tilde{\varphi}$ vanishes on a set spanning $\mathfrak{A}^k(V)$. Thus, if $w \in \mathcal{T}^p(V)$,

$u \in \mathcal{T}^q(V),\ p + q = k - 2,$ and $v_1, v_2 \in V$, we

will show that

$\tilde{\varphi}\,(w \otimes (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes u) = 0.$

But the antisymmetry of φ implies that

$\tilde{\varphi}\,(w \otimes v_1 \otimes v_2 \otimes\ u) = -\tilde{\varphi}\,(w \otimes v_2 \otimes v_1 \otimes u),$

and the assertion follows the linearity.

Definition 1.20. An element $w \in \Lambda^k(V)$ that can be expressed in the form $v_1 \wedge v_2 \wedge \dots\dots \wedge v_k$, where $v_i \in V$, $1 \leq i \leq k$, is said to be *decomposable*. Otherwise, w is *indecomposable*.

Definition 1.21. A graded algebra A is *anti*

commutative if $\alpha \in A^k$ and $\beta \in A^r \Rightarrow \alpha\beta =$

$(-1)^{kr}\beta\alpha.$

Corollary 1.22. [3] The graded algebra $\Lambda(\mathrm{V})$ is anticommutative.

Proof. It is enough to verify the Definition 1.20. for decomposable elements of $\Lambda^k(V)$and $\Lambda^r(V)$. But that case is an elementary consequence of the case $k = r = 1$, and this latter case is given by

$$\begin{aligned} v \wedge w &= v \otimes w + \mathfrak{A}^2(V) \\ &= w \otimes v + \mathfrak{A}^2(V) \\ &= -w \wedge v, \end{aligned}$$

$\forall\, v, w \in V$. Thus the graded algebra $\Lambda(\mathrm{V})$ is ticommutative.

Corollary 1.23. If $w \in \Lambda^{2r+1}(V)$, then $w \wedge w = 0$.

Proof. Let $w \in \Lambda^{2r+1}(V)$. Then

$$\begin{aligned} w \wedge w &= (-1)^{(2r+1)(2r+1)}(w \wedge w) \\ &= w \wedge w \end{aligned}$$

Now, by using Lemma 1.17., we have

$w \wedge w = 0.$

This completes the proof

Lemma 1.24. If $\lambda : V \to V$ is linear, then $\Lambda^m(\lambda) : \Lambda^m(V) \to \Lambda^m(V)$ is multiplication by $\det(\lambda)$.

Proof. Relative to a basis $\{e_1, \dots\dots, e_m\}$ of V, write

$$\lambda(e_i) = \sum_{j=1}^{m} a_i^j\, e_j, \quad 1 \leq i \leq m$$

then,

$$\begin{aligned} \Lambda^m(\lambda)(e_1 \wedge \dots\dots \wedge e_m) &= \lambda\,(e_1) \wedge \dots\dots \wedge \lambda\,(e_m) \\ &= \left(\sum_{j=1}^{m} a_1^j e_j\right) \wedge \dots\dots \wedge \left(\sum_{j=1}^{m} a_m^j e_j\right) \\ &= \sum_{1 \leq j_1, \dots\dots, j_m \leq m} a_1^{j_1} \dots\dots a_m^{j_m}\ e_{j_1} \wedge \dots\dots \wedge e_{j_m}. \end{aligned}$$

Any term with a repeated j *index* vanishes. If $J = (j_1, j_2, \dots\dots, j_m)$ contains no repetitions, there is a unique permutation $\sigma\, j \in \sum m$ such that

$j_{\sigma j}(r) = r, \qquad 1 \leq r \leq m.$

Thus,

$\Lambda^m(\lambda)(e_1 \wedge \dots \dots \wedge e_m)$

$$= \left(\sum_{\sigma \in \sum m} (-1)^\sigma a^1_{\sigma(1)} \dots \dots a^m_{\sigma(m)} \right) e_1 \wedge \dots \dots \wedge e_m$$

$= \det(\lambda)(e_1 \wedge \dots \dots \wedge e_m)$.

Hence, the proof is complete.

Lemma 1.25. If R is a field, a set of vectors $w_1, w_2, \dots \dots, w_k \in V, k \geq 2$, is linearly indepen-

dent if and only if $w_1 \wedge w_2 \wedge \dots \dots \wedge w_k \neq 0$.

Proof. If R is a field then consider the set of vectors $w_1, w_2, \dots \dots, w_k \in V,\ k \geq 2$. Again if the set is dependent, the existence of universe in R allows us to assume, without loss of generality, that

$$w_1 = \sum_{i=2}^{k} a_i w_i.$$

Then

$w_1 \wedge w_2 \wedge \dots \dots \wedge w_k$

$$= \sum_{i=2}^{k} a_i w_i \wedge w_2 \wedge \dots \dots \wedge w_k = 0.$$

Conversely, if the set is linearly independent, extend it to a basis by suitable choices of $w_{k+1}, \dots \dots, w_m \in V$. Then, we have

$w_1 \wedge w_2 \wedge \dots \dots \wedge w_k \wedge \dots \dots \wedge w_m$

is a basis of the one-dimensional space $\Lambda^m(V)$, hence is not 0.

This completes the proof.

Lemma 1.26. If V is a free R −module on a finite basis, then each A^k is one to one, hence $A : \Lambda(\mathrm{V}) \hookrightarrow \mathcal{T}(V)$ is a canonical graded linear imbedding.

Proof. Let $\{e_1, \dots \dots, e_m\} \subset V$ be a basis and consider the basis

$$\{e_{i_1} \wedge \dots \dots \wedge e_{i_k}\}_{1 \leq i_1 \leq \dots \dots < i_k \leq i_m}$$

of $\Lambda^k(V)$. Let $\{e_1^*, \dots \dots, e_k^*\} \subset V^*$ be the dual basis. Since $\mathcal{T}^k(V^*) = \mathcal{T}^k(V)^*$, we obtain a subset

$$\{e_{j_1}^* \otimes \dots \dots \otimes e_{j_k}^*\}_{1 \leq j_1 < \dots \dots < j_k \leq j_m} \subset \mathcal{T}^k(V)^*,$$

which is a part of a free basis. Then, since $j_1 < \cdots < j_k$ and $i_1 < \cdots < i_k$,

$(e_{j_1}^* \otimes \dots \dots \otimes e_{j_k}^*)(A^k(e_{i_1} \wedge \dots \dots \wedge e_{i_k}))$

$$= (e_{j_1}^* \otimes \dots \dots \otimes e_{j_k}^*) \left(\sum_{\sigma \in \sum k} (-1)^\sigma e_{i_{\sigma(1)}} \otimes \dots \dots \otimes e_{i_{\sigma(k)}} \right)$$

$= (e_{j_1}^* \otimes \dots \dots \otimes e_{j_k}^*)(e_{i_1} \otimes \dots \dots \otimes e_{i_k})$

$= \delta^{j_1 \dots j_k}_{i_1 \dots i_k}$

and the assertion follows.

IV. Symmetric Algebra

A K −linear map $\varphi : V^k \to W$ is symmetric if, for each $\sigma \in \sum k$,

$\varphi\left(v_{\sigma(1)}, \dots \dots, v_{\sigma(k)}\right) = \varphi(v_1, v_2, \dots \dots, v_k)$,

$$\forall\, v_1, v_2, \dots \dots, v_k \in V.$$

In the usual way, we build a universal, symmetric, K −linear map

$V^k \dashrightarrow \mathfrak{A}^k(V)$,

Usually written with the dots

$(v_1, v_2, \dots \dots, v_k) \longmapsto v_1 v_2 \dots \dots v_k$.

Definition 1.27. [5] The space $\mathfrak{A}^k(V)$ is called the *k th symmetric power* of V, where, as usual, $\mathfrak{A}^0(V) = R$ and $\mathfrak{A}^1(V) = V$. The connected, graded algebra $\mathfrak{A}(V) = \{\mathfrak{A}^k(V)\}_{k=0}^{\infty}$, with multiplication ".", is called the *symmetric algebra* of V.

Definition 1.28. Let V be a finite dimensional vector space over a field $\mathbb{F}$. A function $f : V \to \mathbb{F}$ is a *homogeneous polynomial* of degree k on V if, related to some basis $\{e_1, \dots \dots, e_m\}$ of V,

$$f\left(\sum_{i=1}^{m} x_i e_i\right) = P(x_1, \dots \dots, x_m)$$

is a homogeneous polynomial of degree k in the variables $x_1, \dots \dots, x_m$. The vector space of all homogeneous polynomials of degree k on V will be denoted by $P^k(V)$.

V. The Module of Sections

We are going to view the set of all vector bundles over a fixed manifold M [5] as the objects of a category V_M. Let

$\pi : E \to M$

$\rho : F \to M$

be vector bundles (of possibly differing fibers dimensions). A homomorphism of the n −plane bundle E to the m −plane bundle F is denoted by HOM (E, F) is naturally called $C^\infty(M)$ − module.

Theorem 1.29. [5] The $C^\infty(M)$ −linear map α is a canonical isomorphism of $C^\infty(M)$ − modules.

$$\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) = \Gamma(E \otimes F).$$

Corollary 1.30. [5] There are canonical iso- morphisms $C^{\infty}(M) -$ modules

$\Gamma(\mathcal{T}^k(E)) = \mathcal{T}^k(\Gamma(E))$

$\Gamma(\Lambda^k(E)) = \Lambda^k(\Gamma(E))$

$\Gamma(S^k(E)) = S^k(\Gamma(E))$.

Proof. The first part of these identities is an immediate consequence of theorem 1.29. There is canonical inclusion

$A^k : \Lambda^k(\Gamma(E)) \hookrightarrow \mathcal{T}^k(\Gamma(E))$

$A^k : \Gamma(\Lambda^k(E)) \hookrightarrow \Gamma(\mathcal{T}^k(E))$.

The second part comes from the bundle inclusions. The images of these inclusions correspond perfectly under the identification $\mathcal{T}^k(\Gamma(E)) = \Gamma(\mathcal{T}^k(E))$, proving the second identity. Similarly the third part can be proof which is same as proof of second part.

Lemma 1.31. If F and E are trivial bundles, then α is an isomorphism of $C^{\infty}(M) -$ modules.

Proof. In this case we choose the global sections $\{\sigma_1, \dots\dots, \sigma_n\}$ of E and $\{\mathcal{T}_1, \dots\dots, \mathcal{T}_m\}$ of F which trivialize these bundles. These are free bases of the respective $C^{\infty}(M) -$ modules $\Gamma(E)$ and $\Gamma(F)$, so

$\{\sigma_i \otimes_{C^{\infty}(M)} \mathcal{T}_j\}_{i,j=1}^{n,m}$

is a free basis of $\Gamma(E) \otimes_{C^{\infty}(M)} \Gamma(F)$. The set

$\{\sigma_i \otimes \mathcal{T}_j\}_{i,j=1}^{n,m}$

of point wise tensor products of sections trivializes the bundle $E \otimes F$, hence this is also a free basis of $\Gamma(E \otimes F)$. Since

$\alpha\left(\sigma_i \otimes_{C^{\infty}(M)} \mathcal{T}_j\right) = \sigma_i \otimes \mathcal{T}_j$,

for all relevant indices, we see that α is an isomorphism of $C^{\infty}(M) -$ modules. This completes the proof.

Theorem 1.32. If $F \subseteq E$ is a vector subbundle and if there is given Riemannian metric on E, then the subset $\tilde{F} \subseteq E$, fiber wise perpendicular to F, is a subbundle.

Proof. Here the local triviality all that needs to be proven. There are sections $\sigma_1, \dots\dots, \sigma_r, \sigma_{r+1}$, $\dots\dots, \sigma_n$ of $E|U$, trivializing that bundle, where U is a neighborhood of an arbitrary point of M. These can be chosen so that $\sigma_1, \dots\dots, \sigma_r$ are sections of $F|U$ which trivialize that bundle an application of Gran-Schmidt turns these into fiberwise orthonormal sections $S_1, \dots\dots, S_r, S_{r+1}$ $, \dots\dots, S_n$ with the same properties. It follows that $S_{r+1}, \dots\dots, S_n$ are trivializing sections of $\tilde{F}|U$, proving that $\tilde{F}$ is a subbundle of E. Hence the proof is complete.

VI. Conclusion

A theorem 1.32 is established which is related with a Riemannian metric on the bundle $M \times V$. For each $x \in M$, let $\tilde{E}_x \subset \{x\} \times V$ be the subspace orthogonal to $E_x^{\perp}$. Consequently the set $\tilde{E} = \bigcup_{x \in M} \tilde{E}_x$ is a subbundle of $M \times V$. Also this theorem will follow form a theorem in dimension theory.

References

1. Boothby, W., 1975. An Introduction to Differentiable Manifolds and Differential Geometry, *Academic Press*, *NewYork.*.
2. Donson, C. T. J. and T. Poston, 1997.Tensor Geometry, *Pitman,* London.
3. Ahmed, K. M., 2007. A study of Graded manifolds, *Dhaka Uni. J. Sci.* **55** (**1**): 35-39.
4. Chevally, C. 1956. Fundamental Concepts of Algebra, *Academic Press*, New York.
5. Myers, S. B. and N. E., 1939.Steenrod, the group of isometrics of a Riemannian Manifold, *Ann of Math.* **40**: 400-416.
6. Auslander, L. and R. E. Mackenzie, 1963. Introduction to differential Manifolds, *Mc Graw-Hill,* New York.
7. Brickell, F. and R. S. Clark, 1970. Differential Manifolds, *Van Nostrand Reinhold company*, London.