

## Numerical Solutions of General Fourth Order Two Point Boundary Value Problems by Galerkin Method with Legendre Polynomials

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### Abstract

In this paper, Galerkin weighted residual method is presented to find the numerical solutions of the general fourth order linear and nonlinear differential equations with *essential* boundary conditions. For this, the given differential equations and the boundary conditions over arbitrary finite domain  $[a, b]$  are converted into its equivalent form over the interval  $[0, 1]$ . Here the Legendre polynomials, over the interval  $[0, 1]$ , are chosen as trial functions satisfying the corresponding homogeneous form of the *Dirichlet* boundary conditions. Details matrix formulations are derived for solving linear and nonlinear boundary value problems (BVPs). Numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation and the results obtained are compared.

**Keywords:** Galerkin method, Fourth order linear and nonlinear BVPs, Legendre polynomials.

### I. Introduction

Many higher order boundary value problems (BVPs) arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plates deflection theory, beam element theory and many more applications of engineering and applied mathematics which are solved either analytically<sup>1</sup> or numerically<sup>2,3</sup>. Only a limited number of such type of BVPs can be solved by using analytical methods leading to closed form solutions. For this, many researchers have attempted for the numerical solutions of fourth order BVPs to obtain high accuracy rapidly by using a numerous method such as, finite difference method, Sinc-Galerkin method and also some other methods using polynomial and nonpolynomial spline functions. Among the various researchers, Usmani<sup>4</sup> and Usmani and Warsi<sup>5</sup> developed and analyzed second order and fourth order convergent methods for the solution of linear fourth order two-point boundary problem using quartic, quintic and sextic polynomial spline functions, respectively. Al-Said and Noor<sup>6</sup> and Al-Said *et al*<sup>7</sup> demonstrated second order convergent method based on cubic and quartic polynomial spline functions for the solution of fourth order obstacle problems. Usmani<sup>8</sup> established and discussed convergent second order and fourth order methods for this problem with the change in the boundary conditions for first order instead of second order derivatives using quintic and sextic polynomial spline functions, respectively. Also, Rashidinia and Golbabaee<sup>9</sup>, and Siddiqi and Akram<sup>10</sup> generated a deference scheme via quintic spline functions for this problem. Loghmani and Alavizadeh<sup>11</sup> converted this problem into an optimal control problem and then constructed the approximate solution as a combination of quartic B-splines.

Van Daele *et al*<sup>12</sup> introduced a new second order method for solving the BVPs involving first derivatives based on nonpolynomial spline function. Siraj *et al*<sup>13</sup> solved a system of third order boundary value problems using nonpolynomial spline functions. Ramadan *et al*<sup>14</sup> proposed a second order convergent method for the numerical solution of second order BVP. Later Ramadan *et al*<sup>15</sup> and Siraj-ul-Islam *et al*<sup>16</sup> have solved fourth order two-point boundary value problems using quintic nonpolynomial spline

functions. Smith *et al*<sup>17</sup> solved only linear while El-Gamel *et al*<sup>18</sup> solved the nonlinear BVPs by the technique of Sinc-Galerkin method. Very recently Kasi *et al*<sup>19</sup> used Quintic B-splines for solving fourth order BVP by the Galerkin method<sup>3</sup>. Thus from the numerical survey, for fourth order BVPs, we note that a large number of authors have solved fourth order BVPs using spline functions which can be exploited easily but the numerical results converge slowly. Besides spline functions, there is another type of piecewise continuous polynomials, namely Legendre polynomials, which is available in the book<sup>2</sup> may be used for numerical solutions of fourth order BVPs. Therefore, in this paper we try to present a simple Galerkin approach with Legendre polynomials as basis functions to solve the fourth order BVPs with *essential* boundary conditions.

However, in section II of this paper, a short description on Legendre polynomials is mentioned. In section III, the formulation of the Galerkin method with Legendre polynomials as basis functions are to be presented for solving linear fourth order BVP. Then we deduce the similar formulation for nonlinear problems in the next section with particular problems. The proposed formulation is verified on four linear and two nonlinear BVPs in section IV. Finally, in the last section, the conclusion of the paper is presented.

### II. Legendre Polynomials

The general form of the Legendre polynomials [2] of degree  $n$  is defined by

$$L_n(x) = \sum_{r=0}^N (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \quad (1)$$

where  $N = \frac{n}{2}$  for  $n$  even and  $N = \frac{n-1}{2}$  for  $n$  odd.

The first few Legendre polynomials are given below:

$$L_1(x) = x$$

$$L_2(x) = (3x^2 - 1)/2$$

$$L_3(x) = (5x^3 - 3x)/2$$

$$L_4(x) = (35x^4 - 30x^2 + 3)/8$$

$$L_5(x) = (63x^5 - 70x^3 + 15x)/8$$

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$$L_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16$$

$$L_7(x) = (429x^7 - 693x^5 + 315x^3 - 35x)/16$$

### Modified Legendre polynomials

Here the Legendre polynomials are chosen such that they bijectively maps the interval  $[0, 1]$  to the interval  $[-1, 1]$ , implying that the polynomials are orthogonal on  $[0, 1]$ .

The analogue of Rodrigues' formula for the Legendre polynomials is

$$p_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - x)^n. \quad (2)$$

To satisfy the condition  $p_n(0) = p_n(1) = 0$ ,  $n \geq 1$ , we modify the Legendre polynomials in Eq. (2) as

$$p_n(x) = \left[ \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - x)^n - (-1)^n \right] (x-1). \quad (3)$$

Now from Eq. (3) we can write first few modified Legendre polynomials over the interval  $[0, 1]$ :

$$p_1(x) = -2x + 2x^2$$

$$p_2(x) = 6x - 12x^2 + 6x^3$$

$$p_3(x) = -12x + 42x^2 - 50x^3 + 20x^4$$

$$p_4(x) = 20x - 110x^2 + 230x^3 - 210x^4 + 70x^5$$

$$p_5(x) = -30x + 240x^2 - 770x^3 + 1190x^4 - 882x^5 + 252x^6$$

$$p_6(x) = 42x - 462x^2 + 2100x^3 - 4830x^4 + 5922x^5 - 3696x^6 + 924x^7$$

$$p_7(x) = -56x + 812x^2 - 4956x^3 + 15750x^4 - 28182x^5 + 28644x^6 - 15444x^7 + 3432x^8.$$

Since modified Legendre polynomials have special properties at  $x=0$  and  $x=1$ :  $p_n(0)=0$  and  $p_n(1)=0$ ,  $n \geq 1$  respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the *Dirichlet* boundary conditions to derive the matrix formulation of fourth order BVP over the interval  $[0,1]$ .

### III. Matrix Formulation

In this section we first obtain the rigorous formulation for fourth order linear BVP and then we extend our idea for solving nonlinear BVP. For this, we consider a general fourth order linear differential equation given by

$$a_4 \frac{d^4 u}{dx^4} + a_3 \frac{d^3 u}{dx^3} + a_2 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = r, \quad a < x < b \quad (4)$$

with *Dirichlet* boundary conditions

$$u(a) = A_0, u(b) = B_0 \quad (5)$$

and the first derivative boundary conditions

$$u'(a) = A_1, u'(b) = B_1, \quad (6)$$

where  $A_0, A_1, B_0, B_1$  are finite real constants and  $a_4, a_3, a_2, a_1, a_0$  and  $r$  are all continuous functions of  $x$  defined on the interval  $[a, b]$ . Since our aim is to use the Legendre polynomials as trial functions which are derived over the interval  $[0, 1]$ , so the BVP, defined in Eqs. (4) – (6), is to be converted to an equivalent problem on  $[0, 1]$  by replacing  $x$  by  $(b-a)x+a$ , and thus we have

$$c_4 \frac{d^4 u}{dx^4} + c_3 \frac{d^3 u}{dx^3} + c_2 \frac{d^2 u}{dx^2} + c_1 \frac{du}{dx} + c_0 u = b, \quad 0 < x < 1, \quad (7)$$

subject to the boundary conditions:

$$u(0) = A_0, u(1) = B_0, \quad \frac{1}{b-a} u'(0) = A_1, \frac{1}{b-a} u'(1) = B_1, \quad (8)$$

where

$$c_4 = \frac{a_4}{(b-a)^4} ((b-a)x+a),$$

$$c_3 = \frac{a_3}{(b-a)^3} ((b-a)x+a),$$

$$c_2 = \frac{a_2}{(b-a)^2} ((b-a)x+a),$$

$$c_1 = \frac{a_1}{b-a} ((b-a)x+a),$$

$$c_0 = a_0 ((b-a)x+a), \quad b = r((b-a)x+a). \quad (9)$$

Using Legendre polynomials  $p_i$ , we assume an approximate solution in a form

$$\tilde{u} = \theta_0 + \sum_{i=1}^n a_i p_i. \quad (10)$$

Here  $\theta_0$  is specified by the essential boundary conditions and  $p_i(0) = p_i(1) = 0$  for each  $i = 1, 2, \dots, n$ .

Using Eq. (10) into Eq. (7), the Galerkin weighted residual equations are

$$\int_0^1 \left[ c_4 \frac{d^4 \tilde{u}}{dx^4} + c_3 \frac{d^3 \tilde{u}}{dx^3} + c_2 \frac{d^2 \tilde{u}}{dx^2} + c_1 \frac{d\tilde{u}}{dx} + c_0 \tilde{u} - b \right] \times p_j dx = 0, \quad j = 1, 2, \dots, n. \quad (11)$$

Integrating by parts the terms up to second derivative on the left hand side of Eq. (11), we can obtain each term after applying the conditions prescribed in Eq. (8) as

$$\int_0^1 c_4 \frac{d^4 \tilde{u}}{dx^4} p_j dx = \left[ c_4 p_j \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d}{dx} [c_4 p_j] \frac{d^3 \tilde{u}}{dx^3} dx$$

[since  $p_j(0) = p_j(1) = 0$ ]

$$\begin{aligned}
 &= -\left[\frac{d}{dx}[c_4 p_j] \frac{d^2 \tilde{u}}{dx^2}\right]_0^1 + \int_0^1 \frac{d^2}{dx^2} [c_4 p_j] \frac{d^2 \tilde{u}}{dx^2} dx - \left[\frac{d}{dx}[c_4 p_j] \frac{d^2 p_i}{dx^2}\right]_{x=1} \\
 &= -\left[\frac{d}{dx}[c_4 p_j] \frac{d^2 \tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2} [c_4 p_j] \frac{d\tilde{u}}{dx}\right]_0^1 + \left[\frac{d}{dx}[c_4 p_j] \frac{d^2 p_i}{dx^2}\right]_{x=0}, \quad (16)
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^1 \frac{d^3}{dx^3} [c_4 p_j] \frac{d\tilde{u}}{dx} dx \\
 &= -\left[\frac{d}{dx}[c_4 p_j] \frac{d^2 \tilde{u}}{dx^2}\right]_0^1 + \left[\frac{d^2}{dx^2} [c_4 p_j]\right]_{x=1} \times (b-a)B_1 \\
 & \quad - \left[\frac{d^2}{dx^2} [c_4 p_j]\right]_{x=0} \times (b-a)A_1 - \int_0^1 \frac{d^3}{dx^3} [c_4 p_j] \frac{d\tilde{u}}{dx} dx \quad (12)
 \end{aligned}$$

$$\begin{aligned}
 & \int_0^1 c_3 \frac{d^3 \tilde{u}}{dx^3} p_j dx \\
 &= \left[c_3 p_j \frac{d^2 \tilde{u}}{dx^2}\right]_0^1 - \int_0^1 \frac{d}{dx} [c_3 p_j] \frac{d^2 \tilde{u}}{dx^2} dx \\
 &= -\left[\frac{d}{dx}[c_3 p_j]\right]_{x=1} \times (b-a)B_1 + \left[\frac{d}{dx}[c_3 p_j]\right]_{x=0} \\
 & \quad \times (b-a)A_1 + \int_0^1 \frac{d^2}{dx^2} [c_3 p_j] \frac{d\tilde{u}}{dx} dx \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 & \int_0^1 c_2 \frac{d^2 \tilde{u}}{dx^2} p_j dx \\
 &= \left[c_2 p_j \frac{d\tilde{u}}{dx}\right]_0^1 - \int_0^1 \frac{d}{dx} [c_2 p_j] \frac{d\tilde{u}}{dx} dx \\
 &= -\int_0^1 \frac{d}{dx} [c_2 p_j] \frac{d\tilde{u}}{dx} dx \quad (14)
 \end{aligned}$$

Substituting Eqs. (12) – (14) into Eq. (11) and using approximation for  $\tilde{u}(x)$  given in Eq. (10), and after rearranging the terms for the resulting equations, we get a system of equations in the matrix form as

$$\sum_{i=1}^n K_{i,j} a_i = F_j, \quad j=1,2,\dots,n, \quad (15)$$

where

$$\begin{aligned}
 K_{i,j} = \int_0^1 \left\{ \left[ -\frac{d^3}{dx^3} [c_4 p_j] + \frac{d^2}{dx^2} [c_3 p_j] \right] \right. \\
 \left. - \frac{d}{dx} [c_2 p_j] + c_1 p_j \right\} \frac{dp_i}{dx} + c_0 p_i p_j dx
 \end{aligned}$$

$$\begin{aligned}
 F_j = \int_0^1 \left\{ s p_j + \left[ \frac{d^3}{dx^3} [c_4 p_j] - \frac{d^2}{dx^2} [c_3 p_j] \right] \right. \\
 \left. + \frac{d}{dx} [c_2 p_j] - c_1 p_j \right\} \frac{d\theta_0}{dx} + c_0 p_j \theta_0 dx \\
 + \left[ \frac{d}{dx} [c_3 p_j] - \frac{d^2}{dx^2} [c_4 p_j] \right]_{x=1} \times (b-a)B_1 \\
 - \left[ \frac{d}{dx} [c_3 p_j] - \frac{d^2}{dx^2} [c_4 p_j] \right]_{x=0} \times (b-a)A_1 \\
 + \left[ \frac{d}{dx} [c_4 p_j] \frac{d^2 \theta_0}{dx^2} \right]_{x=1} - \left[ \frac{d}{dx} [c_4 p_j] \frac{d^2 \theta_0}{dx^2} \right]_{x=0}. \quad (17)
 \end{aligned}$$

Solving the system of equations specified by Eq. (15), we find the values of the parameters  $a_i$ , and then substituting these parameters into Eq. (10), we get the approximate solution of the BVP defined in Eqs. (7) – (9). If we replace  $x$  by  $\frac{x-a}{b-a}$  in  $\tilde{u}$ , then we get the desired approximate solution of the BVP defined in Eqs. (4) – (6).

For nonlinear fourth order BVP, we first compute the initial values neglecting the nonlinear terms and use the system (15). Then using the iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

#### IV. Numerical Results

To test the applicability of the proposed method, we consider two linear and two nonlinear problems with boundary conditions of the type (5) and (6) because the exact solutions for these problems are available in the literature. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the computations are performed using *MATLAB*. The convergence of linear BVP is calculated by

$$E = |\tilde{u}_{n+1}(x) - \tilde{u}_n(x)| < \delta,$$

where  $\tilde{u}_n(x)$  denotes the approximate solution using  $n$  polynomials and  $\delta$  depends on the problem which varies from  $10^{-7}$  to  $10^{-10}$ . In addition, the convergence of nonlinear BVP is assumed when the absolute error of two consecutive iterations,  $\delta$  satisfies

$$|\tilde{u}_n^{N+1} - \tilde{u}_n^N| < \delta,$$

where  $N$  is the Newton's iteration number and  $\delta$  varies from  $10^{-11}$  to  $10^{-13}$ .

*Example 1.* Consider the linear BVP<sup>9, 16, 19</sup>

$$\frac{d^4 u}{dx^4} + 4u = 1, \quad -1 < x < 1 \tag{18}$$

subject to the boundary conditions

$$u(-1) = u(1) = 0, \quad u'(-1) = -u'(1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)}, \tag{19}$$

whose exact solution is,

$$u(x) = \frac{1}{4} - \frac{1}{2} (\sinh 1 \sin 1 \sinh x \sin x + \cosh 1 \cos 1 \cosh x \cos x) / (\cosh 2 + \cos 2).$$

The equivalent BVP over  $[0, 1]$  to the BVP defined in Eqs. (18) and (19) is,

$$\frac{1}{2^4} \frac{d^4 u}{dx^4} + 4u = 1, \quad 0 < x < 1 \tag{20}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \quad \frac{1}{2} u'(0) = -\frac{1}{2} u'(1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)}. \tag{21}$$

Using the method illustrated in section III,

$$\sum_{i=1}^n K_{i,j} a_i = F_j, \quad j=1,2,\dots,n \tag{22}$$

where

$$K_{i,j} = \int_0^1 \left\{ -\frac{d^3 p_j}{dx^3} \frac{dp_i}{dx} + 64 p_i p_j \right\} dx - \left[ \frac{dp_j}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=1} + \left[ \frac{dp_j}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=0} \tag{23}$$

$$F_j = \int_0^1 16 p_j dx + \left[ \frac{d^2 p_j}{dx^2} \right]_{x=1} \left\{ \frac{\sinh 2 - \sin 2}{2(\cosh 2 + \cos 2)} \right\} + \left[ \frac{d^2 p_j}{dx^2} \right]_{x=0} \left\{ \frac{\sinh 2 - \sin 2}{2(\cosh 2 + \cos 2)} \right\} \tag{24}$$

**Table 1. Numerical results of example 1.**

$x$	Exact Solutions	Proposed method using 11 polynomials	Absolute Error
-1.0	0.0000000000	0.0000000000	0.000000000000
-0.8	0.0397692607	0.0397692607	4.6837534E-015
-0.6	0.0749849828	0.0749849828	1.7749691E-014
-0.4	0.1023106409	0.1023106409	7.2997164E-015
-0.2	0.1195382314	0.1195382314	7.9422580E-014
0.0	0.1254157424	0.1254157424	1.3106183E-016
0.2	0.1195382314	0.1195382314	9.5423669E-014
0.4	0.1023106409	0.1023106409	1.5751289E-014
0.6	0.0749849828	0.0749849828	2.3370195E-014
0.8	0.0397692607	0.0397692607	3.1780134E-015
1.0	0.0000000000	0.0000000000	0.000000000000

Solving the system in Eq. (22), we obtain the values of the parameters and then substituting these parameters into the similar form of Eq. (10), we get the approximate solution to

the BVP, defined in Eqs. (20) and (21), for different values of  $n$ . If we replace  $x$  by  $\frac{x+1}{2}$  in  $\tilde{u}(x)$ , then we get the

desired approximate solution of the BVP defined in Eqs. (18) and (19). The numerical results for this problem are shown in Table 1. The maximum absolute error obtained by the proposed method is  $9.542 \times 10^{-14}$ . On the contrary, the errors have been obtained by Rashidinia and Golbabaee<sup>9</sup>, Siraj-ul-Islam *et al*<sup>16</sup> and Kasi *et al*<sup>19</sup> are  $6.22 \times 10^{-6}$ ,  $2.82 \times 10^{-12}$  and  $1.303 \times 10^{-6}$ , respectively.

*Example 2.* Consider the linear BVP<sup>5, 9, 19</sup>

$$\frac{d^4 u}{dx^4} + xu = -(8 + 7x + x^3)e^x, \quad 0 < x < 1 \tag{25}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \quad u'(0) = 1, \quad u'(1) = -e. \tag{26}$$

whose exact solution is  $u(x) = x(1-x)e^x$ .

Applying the proposed method, mentioned in section III, the numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is  $2.502 \times 10^{-13}$ . It is observed that using splines the maximum errors have been found by Al-Said *et al*<sup>5</sup>, Rashidinia and Golbabaee<sup>9</sup>, and Kasi *et al*<sup>19</sup> are upto  $2.36 \times 10^{-7}$ ,  $5.37 \times 10^{-6}$  and  $5.99 \times 10^{-6}$ , respectively.

**Table 2. Numerical results of example 2.**

$x$	Exact Solutions	Proposed method using 11 polynomials	Absolute Error
0.0	0.0000000000	0.0000000000	0.000000000000
0.1	0.0994653826	0.0994653826	4.6490589E-014
0.2	0.1954244413	0.1954244413	1.2767565E-013
0.3	0.2834703496	0.2834703496	1.7824631E-013
0.4	0.3580379274	0.3580379274	2.8310687E-015
0.5	0.4121803177	0.4121803177	2.5018876E-013
0.6	0.4373085121	0.4373085121	1.8657298E-013
0.7	0.4228880686	0.4228880686	8.5431662E-014
0.8	0.3560865486	0.3560865486	2.0078383E-013
0.9	0.2213642800	0.2213642800	1.4432899E-013
1.0	0.0000000000	0.0000000000	0.000000000000

*Example 3.* Consider the nonlinear BVP<sup>19</sup>

$$\frac{d^4 u}{dx^4} = \sin x + \sin^2 x - \left( \frac{d^2 u}{dx^2} \right)^2, \quad 0 < x < 1 \tag{27}$$

subject to boundary conditions:

$$u(0) = 0, u(1) = \sin 1, \quad u'(0) = 1, u'(1) = \cos 1. \tag{28}$$

The exact solution of this BVP is  $u(x) = \sin x$ .

Consider the approximate solution of  $u(x)$  as

$$\tilde{u} = \theta_0 + \sum_{i=1}^n a_i p_i, \quad n \geq 1. \tag{29}$$

Here  $\theta_0 = x \sin 1$  is specified by the essential boundary conditions in Eq. (28). Also  $p_i(0) = p_i(1) = 0$  for each  $i = 1, 2, \dots, n$ .

Using Eq. (29) into Eq. (27), the Galerkin weighted residual equation is

$$\int_0^1 \left[ \frac{d^4 \tilde{u}}{dx^4} + \left( \frac{d^2 \tilde{u}}{dx^2} \right)^2 - \sin x - \sin^2 x \right] p_k dx = 0 \quad (30)$$

Integrating first term of Eq. (30) by parts, we obtain

$$\begin{aligned} \int_0^1 \frac{d^4 \tilde{u}}{dx^4} p_k dx &= \left[ p_k \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{dp_k}{dx} \frac{d^3 \tilde{u}}{dx^3} dx \\ & \quad [\text{since } p_k(1) = p_k(0) = 0] \\ &= - \left[ \frac{dp_k}{dx} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^2 p_k}{dx^2} \frac{d^2 \tilde{u}}{dx^2} dx \\ &= - \left[ \frac{dp_k}{dx} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 + \left[ \frac{d^2 p_k}{dx^2} \frac{d\tilde{u}}{dx} \right]_0^1 - \int_0^1 \frac{d^3 p_k}{dx^3} \frac{d\tilde{u}}{dx} dx \\ &= - \left[ \frac{dp_k}{dx} \right]_{x=1} \left[ \theta_0^* + \sum_{i=1}^n a_i p_i^* \right]_{x=1} + \left[ \frac{dp_k}{dx} \right]_{x=0} \left[ \theta_0^* + \sum_{i=1}^n a_i p_i^* \right]_{x=0} \\ & \quad + \left[ \frac{d^2 p_k}{dx^2} \right]_{x=1} \times \cos 1 - \left[ \frac{d^2 p_k}{dx^2} \right]_{x=0} - \int_0^1 \frac{d^3 p_k}{dx^3} \left[ \theta_0^* + \sum_{i=0}^1 a_i p_i^* \right] dx \end{aligned} \quad (31)$$

Putting Eq. (31) into Eq. (30) we obtain the equivalent matrix form

$$(D + B)A = G, \quad (32)$$

where the elements of  $A$ ,  $B$ ,  $D$ ,  $G$  are  $a_i, b_{i,k}, d_{i,k}$  and  $g_k$  respectively, given by

$$\begin{aligned} d_{i,k} &= \int_0^1 \left[ - \frac{d^3 p_k}{dx^3} \frac{dp_i}{dx} + 2 \frac{d^2 \theta_0}{dx^2} \frac{d^2 p_i}{dx^2} p_k \right] dx \\ & \quad - \left[ \frac{dp_k}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=1} + \left[ \frac{dp_k}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=0} \end{aligned} \quad (33)$$

$$b_{i,k} = \sum_{j=1}^n a_j \left( \frac{d^2 p_i}{dx^2} \frac{d^2 p_j}{dx^2} p_k \right) dx \quad (34)$$

$$\begin{aligned} g_k &= \int_0^1 \left[ (\sin x + \sin^2 x) p_k + \frac{d^3 p_k}{dx^3} \frac{d\theta_0}{dx} - \left( \frac{d^2 \theta_0}{dx^2} \right)^2 p_k \right] dx \\ & \quad + \left[ \frac{dp_k}{dx} \frac{d^2 \theta_0}{dx^2} \right]_{x=1} - \left[ \frac{dp_k}{dx} \frac{d^2 \theta_0}{dx^2} \right]_{x=0} \end{aligned}$$

$$- \left[ \frac{d^2 p_k}{dx^2} \right]_{x=1} \times \cos 1 + \left[ \frac{d^2 p_k}{dx^2} \right]_{x=0} \quad (35)$$

The initial values of these coefficients  $a_i$  are obtained by applying Galerkin method to the BVP neglecting the nonlinear terms in Eq. (27). That is, to find initial coefficients we will solve the system

$$D A = G \quad (36)$$

whose matrices are constructed from

$$\begin{aligned} d_{i,k} &= \int_0^1 \left[ - \frac{d^3 p_k}{dx^3} \frac{dp_i}{dx} \right] dx - \left[ \frac{dp_k}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=1} \\ & \quad + \left[ \frac{dp_k}{dx} \frac{d^2 p_i}{dx^2} \right]_{x=0} \\ g_k &= \int_0^1 \left[ (\sin x + \sin^2 x) p_k + \frac{d^3 p_k}{dx^3} \frac{d\theta_0}{dx} \right] dx \\ & \quad + \left[ \frac{dp_k}{dx} \frac{d^2 \theta_0}{dx^2} \right]_{x=1} - \left[ \frac{dp_k}{dx} \frac{d^2 \theta_0}{dx^2} \right]_{x=0} \\ & \quad - \left[ \frac{d^2 p_k}{dx^2} \right]_{x=1} \times \cos 1 + \left[ \frac{d^2 p_k}{dx^2} \right]_{x=0} \end{aligned} \quad (37)$$

**Table 3. Numerical results of example 3.**

$x$	Exact Solutions	Proposed method using 8 polynomials	Absolute Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.0998334166	0.0998334165	1.573325E-010
0.2	0.1986693308	0.1986693307	1.140406E-010
0.3	0.2955202067	0.2955202065	1.492606E-010
0.4	0.3894183423	0.3894183423	9.988721E-011
0.5	0.4794255386	0.4794255385	6.413814E-011
0.6	0.5646424734	0.5646424733	7.907730E-011
0.7	0.6442176872	0.6442176872	6.589806E-011
0.8	0.7173560909	0.7173560909	2.166500E-011
0.9	0.7833269096	0.7833269096	1.818679E-011
1.0	0.8414709848	0.8414709848	0.0000000000

Once the initial values of the  $a_i$  are obtained from Eq. (36), they are substituted into Eq. (32) to obtain new estimates for the values of  $a_i$ . This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into Eq. (29), we obtain an approximate solution of the BVP defined in Eqs. (27) and (28). The numerical results for this problem are presented in Table 3. The maximum absolute error obtained by the proposed method is  $1.573 \times 10^{-10}$  using 5 iterations. On the other hand, the maximum absolute error has been obtained by Kasi *et al*<sup>19</sup> is  $1.359 \times 10^{-5}$ .

*Example 4.* Consider the nonlinear BVP<sup>18, 19</sup>

$$\frac{d^4 u}{dx^4} - 6e^{-4u} = -12(1+x)^{-4}, \quad 0 < x < 1 \quad (39)$$

subject to the boundary conditions:

$$u(0) = 0, u(1) = \ln 2, \quad u'(0) = 1, u'(1) = 0.5. \quad (40)$$

The exact solution is  $u(x) = \ln(1+x)$ .

Following the proposed method illustrated as in example 3 the numerical results for this problem are summarized in Table 4. The maximum absolute error obtained by the present method is  $4.075 \times 10^{-8}$  using 5 iterations. The maximum absolute error has been obtained so far by El-Gamel *et al*<sup>18</sup> and Kasi *et al*<sup>19</sup> are  $2.2 \times 10^{-8}$  and  $4.917 \times 10^{-6}$  respectively.

**Table 4. Numerical results of example 4.**

$x$	Exact Solutions	Proposed method using 9 polynomials	Absolute error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	0.0953101798	0.0953101410	3.883659E-008
0.2	0.1823215568	0.1823215160	4.074934E-008
0.3	0.2623642645	0.2623642240	4.050873E-008
0.4	0.3364722366	0.3364722108	2.582027E-008
0.5	0.4054651081	0.4054650836	2.446640E-008
0.6	0.4700036292	0.4700036072	2.206729E-008
0.7	0.5306282511	0.5306282371	1.392392E-008
0.8	0.5877866649	0.5877866546	1.032887E-008
0.9	0.6418538862	0.6418538815	4.673076E-009
1.0	0.6931471806	0.6931471806	0.0000000000

## V. Conclusion

We have presented a simple and accurate method to solve a fourth order BVPs. We have extended a Galerkin method with Legendre polynomials as basis functions for solving fourth order BVPs with essential boundary conditions. The concentration has given not only on the performance of the results but also on the formulation. The proposed method is applied to solve a several number of linear and nonlinear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions and also confirm with great accuracy than the results obtained by the previous methods so far. In addition, to the knowledge of the present authors', none has attempted to solve the fourth order BVP using Legendre polynomials. Further we wish to extend our idea for solving any higher even order BVPs by the proposed method.

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