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Matrix Computations of Corwin−Greenleaf Multiplicity Functions for Special Unitary Group ($G = SU(m, n)$)

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Abstract

In this paper, Corwin−Greenleaf multiplicity functions for special unitary group have been studied in the light of the Kirillov–Kostant Theory. This was pioneered by the famous mathematician L. Corwin and F.Greenleaf. The multiplicity function is defined as $n(\mathcal{O}_\pi^G, \mathcal{O}_v^H) = \#((\mathcal{O}_\pi^G \cap pr^{-1}(\mathcal{O}_v^H))/H)$. In the case where $G = SU(m, n)$, it has been shown that $n(\mathcal{O}_\pi^G, \mathcal{O}_v^H)$ is at most one.

Keywords: Hermitian symmetric space, Corwin−Greenleaf multiplicity function, orbit method, special unitary group, coadjoint orbit, symmetric pair.

I. Introduction

The orbit method pioneered by Kirillov and Kostant[4] seeks to understand irreducible unitary representation by analogy with "quantization" procedures in mechanics. Physically, the idea of quantization is to replace a classical mechanical model (a phase space modeled by a symplectic manifold M) with a quantum mechanical model (a state space modeled by a Hilbert space $\mathcal{H}$) of the same system. The natural quantum analogue of the action of a group G on M by symplectomorphisms is a unitary representation of G on $\mathcal{H}$.

For a Lie group G coadjoint orbits are symplectic manifolds, and the philosophy of the orbit method says that there should be a method of "quantization" to pass from coadjoint orbits for G to irreducible unitary representations of G. Kirillov proved that this works perfectly for nilpotent Lie groups. But many specialists have pointed out that the orbit method does not work very well for semisimple Lie groups[4,5,6]. However, we can still expect an intimate relation between the unitary dual of G and the set of (integral) coadjoint orbits even for a semi simple Lie group.

One of the fundamental problems in representation theory is to decompose a given representation into irreducible[5]. Branching laws are one of the most important cases. Here, by *branching laws* we mean the irreducible decomposition in terms of a direct integral of an irreducible unitary representation π of a group G when restricted to a subgroup H:

$$\pi|_H \simeq \int_{\hat{H}}^{\oplus} m_\pi(v) v d\mu(v).$$

Such a decomposition is unique, and the multiplicity $m_\pi: \hat{H} \to \mathbb{N} \cup \{\infty\}$ makes sense as a measurable function on the unitary dual $\hat{H}$. There are two basic questions on multiplicities:

Problem 1.1. (Problem 1.1, [9])

(a) For which (G, H, π), the restriction $\pi|_H$ is multiplicity-free?

(b) Relate quantum and classical pictures in the spirit of Kirillov−Kostant orbit method.

As for (a), T. Kobayashi[7] has established a unified theory on multiplicity-free theorem of branching laws for both finite and infinite dimensional representations in a broad setting. This theorem gives a uniform explanation for many known cases of multiplicity-free results and also presents many new cases of multiplicity-free branching laws.

As for (b), it is well-known that the orbit method works well for nilpotent Lie groups, but only partially for reductive groups[4].

II. Corwin−Greenleaf Multiplicity Function

For simply connected nilpotent Lie group G, building on the Kirillov isomorphism

$$\frac{\sqrt{-1}\mathfrak{g}^*}{G} \simeq \hat{G},$$

Corwin and Greenleaf introduced the function $n(\mathcal{O}^G, \mathcal{O}^H)$[1]. For coadjoint orbits $\mathcal{O}^G \subset \sqrt{-1}\mathfrak{g}^*$ and $\mathcal{O}^H \subset \sqrt{-1}\mathfrak{h}^*$, the Corwin−Greenleaf multiplicity function $n(\mathcal{O}^G, \mathcal{O}^H)$ is the number of H-orbits in the intersection $\mathcal{O}^G \cap pr^{-1}(\mathcal{O}^H)$. If π is attached to $\mathcal{O}^G$ and v is attached to $\mathcal{O}^H$, then one expects that $n(\mathcal{O}^G, \mathcal{O}^H)$ coincides with $m_\pi(v)$. Research in this direction has been done extensively for nilpotent Lie groups and for certain solvable groups by Kirillov, Corwin, Greenleaf, Lipsman, and Fujiware among others[1].

III. Statement of Results

The multiplicity free theorems of branching laws of unitary representation theory and other considerations motivated by the Corwin−Greenleaf multiplicity function lead Kobayashi to pose a general conjecture (below) for reductive symmetric spaces. Mainly, as a combination of questions (a) and (b) (Problem 1.1), Kobayashi posed the following conjecture on the base of Corwin−Greenleaf multiplicity function.

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Conjecture 3.1. (Conjecture 1.2, [10]) Let(G,H) be a reductive symmetric pair. Then under certain condition $n(\mathcal{O}_\pi^G, \mathcal{O}_v^H) = \#(Ad^*(H) - orbit\ in\ \mathcal{O}_\pi^G \cap pr^{-1}(\mathcal{O}_v^H)) \leq 1$.

A positive answer to the above conjecture has been given for an arbitrary Riemannian symmetric pair[8,9,10]. The following result has been proved therein.

Theorem 3.1. (Theorem 2.8 [8]) Conjecture 3.1 is true if $H = K$.

Example 3.1. (Example 2.7, [8]) Suppose $G = SL(2,\mathbb{R})$, $SU(2)$, respectively and $H = K = SO(2)$. Let X be a central element in $\mathfrak{k}$, the Lie algebra of K. Then in each case, for any $y \in pr(\mathcal{O}_X^H)$

$$\mathcal{O}_X^H \cap pr^{-1}(y) = circle = single\ SO(2) - orbit.$$

In[8], the proofs have been provided on the structure theory of Hermitian symmetric spaces and their root systems[2,3]. In this paper an alternative and independent proof has been given by elementary matrix computation. Thus, the main result (of this paper) can be stated briefly as follows:

Theorem 3.2. Conjecture 3.1 is true when $G = SU(m,n)$.

This section is for the purpose of (or deals with) obtaining a key result (Proposition 3.1 below). This is done by reducing the main theorem Theorem 3.2 to various Lie algebra formulations.

Suppose G is a real reductive group with Cartan involution θ, maximal compact subgroup $K := \{g \in G: \theta g = g\}$ and Lie algebra $\mathfrak{g}$. Then the Cartan decomposition corresponding to the Cartan involution θ, can be written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

Contrary to nilpotent Lie groups, there is no reasonable bijection between $\hat{G}$ and (a subset of) $\mathfrak{g}^*/G$. Therefore, it is not obvious if an analogous statement of Corwin−Greenleaf's theorem makes sense for a semi simple Lie group G. But, the orbit method still gives a good approximation of the unitary dual $\hat{G}$. For example, to an 'integral' elliptic co-adjoint orbit $\mathcal{O}_\lambda^G = Ad^*(G)\lambda \subset \mathfrak{g}^*$, one can associate a unitary representation, denoted by π_λ, of G[2]. Furthermore, π_λ is nonzero and irreducible for 'most'λ of both geometric and algebraic results in this direction[2]. Namely, to such a co-adjoint orbit$\mathcal{O}_\lambda^G$, one can naturally attach an irreducible unitary representation $\pi_\lambda \in \hat{G}$.

In particular, if G/K is Hermitian, associated to an (integral) co-ad-joint orbit that goes through $([\mathfrak{k},\mathfrak{k}] + \mathfrak{p})^\perp (\subset \mathfrak{g}^*)$, the corresponding unitary representation becomes a highest weight module of scalar type.

By the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, the co-ad-joint orbit

$$\mathcal{O}_\lambda^G = Ad^*(G)\lambda \subset \mathfrak{g}^*$$

Corresponds to the ad-joint orbit given by

$$\mathcal{O}_z^G := Ad(G).z \subset \mathfrak{g},$$

Where z is a non-zero central element in $\mathfrak{k}$. Also write $pr: \mathfrak{g} \to \mathfrak{h}$ for the projection instead of $pr: \mathfrak{g}^* \to \mathfrak{h}^*$.

Now, consider the projection $pr^\theta: \mathfrak{g} \to \mathfrak{k}$ or simply, denoted by $pr: \mathfrak{g} \to \mathfrak{k}$. Then the pullback $(pr^\theta)^{-1}(y)$ $(y \in pr^\theta(\mathcal{O}^K))$ is $Ad(K)-$stable. Then Theorem 3.2 can be rewritten precisely as follows:

Theorem 3.3. Let G/K be a Hermitian symmetric space of non-compact type, and z a central element in $\mathfrak{k}$. Then the intersection $\mathcal{O}_z^G \cap pr^{-1}(\mathcal{O}^K)$ is a single K-orbit for any ad-joint orbit $\mathcal{O}^K \subset \mathfrak{k}$, whenever it is non-empty.

In fact, the above theorem will be reduced in this section.

The global Cartan decomposition is

$$G = K\ exp\ \mathfrak{p}$$

Then, take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and a positive system $\sum_+(\mathfrak{g},\mathfrak{a})$ of the restricted root system$\sum(\mathfrak{g},\mathfrak{a})$.

Let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots. Then the corresponding Weyl chamber[2,3] is defined by

$$\mathfrak{a}_+ = \{H \in \mathfrak{a}: \alpha_1(H) \geq 0, \ldots, \alpha_r(H) \geq 0\},$$

and the generalized Cartan decomposition can be written as

$$G = KA_+K, \text{ where } A_+ = \exp(\mathfrak{a}_+).$$

A key lemma in proving the main theorem is:

Proposition 3.1. (Proposition 3.2., [8]) For any $a, a' \in A_+$, $pr^\theta((Ad(a)z)$ and $pr^\theta((Ad(a')z)$ are conjugate under $Ad(K)$ if and only if $a = a'$.

Proposition 3.2.

Proposition 3.1 implies Theorem 3.3.

Proof: It is obvious that for any $a \in A_+$, $pr^\theta((Ad(a)z) \in pr^\theta(\mathcal{O}_z^G)$ $(\because Ad(a)z \in \mathcal{O}_z^G)$. Then for any $y \in \mathcal{O}^K$ we have,

$$\mathcal{O}_z^G \cap (pr^\theta)^{-1}(y) = Ad(K)Ad(A_+)z \cap (pr^\theta)^{-1}(y)).$$

This implies that for any $x, x' \in \mathcal{O}_z^G \cap (pr^\theta)^{-1}(y)$, there exist $a, a' \in A_+$ and $k_1 \in K$ such that

$$x = Ad(k_1)Ad(a)z, \text{ and } x' = Ad(k_1)Ad(a')z.$$

Now, $x \sim x'$ under $Ad(K)$ implies that there exist some $k_2 \in K$ satisfying

$$Ad(k_2)x = x'.$$

Substituting the value of x and x' the claim follows.

IV. Matrix Computations of Corwin-Greenleaf Multiplicity Function.

In this section, Proposition 3.1 has been proved independently by elementary matrix computations.

The most interesting and complicated case where $G = SU(m,n)$ is considered. The group $U(m,n) \simeq \{g \in$ *GLm+n,ℂ:g∗Im,ng=Im,n* is the group of complex matrices of size $m+n$ preserving the Hermitian form on $\mathbb{C}^{m+n} \times \mathbb{C}^{m+n}$ given by

$$\langle \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \rangle = x_1\overline{y_1} + \cdots + x_m\overline{y_m} - x_{m+1}\overline{y_{m+1}} - \cdots - x_{m+n}\overline{y_{m+n}}$$

, and $SU(m,n)$ is the subgroup of members of $U(m,n)$ of determinant 1. Here, write $(\cdot)^*$ for conjugate transpose and put $I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$ (I_m, I_n denote the identity matrices of order m and n, respectively).

Therefore, $SU(m,n)$ can be realized as the special indefinite unitary group in $GL(m+n,\mathbb{C})$:

$$G = SU(m,n) \simeq \{g \in SL(m+n,\mathbb{C}) : g^* I_{m,n} g = I_{m,n}\}.$$

A maximal compact subgroup K is given by $K = S(U(m) \times U(n))$. Then the Lie algebra $\mathfrak{g}$ of G will be identified with

$$\mathfrak{g} \simeq \left\{ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \middle| A = -A^*, C = -C^*, \mathrm{Tr}(A) + \mathrm{Tr}(C) = 0 \right\}.$$

Corresponding to the Cartan involution θ (which is defined by $\theta(X) = I_{m,n} X I_{m,n}$), the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G is written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Here,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} : A \in u(m), B \in u(n) \right\},$$

$$\mathfrak{p} = \{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \text{ is a complex } m \times n \text{ matrix}\}.$$

The decomposition

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} A - \frac{1}{m}(TrA)I_m & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{m}(TrA)I_m & 0 \\ 0 & \frac{1}{n}(TrA)I_n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - \frac{1}{n}(TrA)I_n \end{pmatrix}$$

shows that $\mathfrak{k}$ is isomorphic to the sum $\mathfrak{su}(m) \oplus \mathfrak{c}(\mathfrak{k}) \oplus \mathfrak{su}(n)$, where $\mathfrak{c}(\mathfrak{k})$ is the center of $\mathfrak{k}$. Also $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{su}(m,n)$ and the corresponding symmetric space is $\frac{SU(m,n)}{S(U(m)\times U(n))}$.

Without loss of generality, from now one may assume that $m \geq n$. Consider the matrix unit $E_{i,j}$ to be 1 in the $(i,j)^{th}$ place and 0 elsewhere. Then, a maximal abelian subspace of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{\sum_{i=1}^{n} t_i(E_{i,m+i} + E_{m+i,i}) : t_1, \ldots t_n \in \mathbb{R}\}$$

Consequently, the rank is n. Suppose A denotes the connected subgroup of G having the Lie algebra $\mathfrak{a}$ and is defined by

$$A = \{\exp (\sum_{i=1}^{n} t_i(E_{i,m+i} + E_{m+i,i}) : t_1, \ldots t_n \in \mathbb{R}\}$$

$= \{\sum_{i=1}^{n} cosht_i(E_{i,i} + E_{m+i,m+i}) + \sum_{i=n+1}^{m} E_{i,i} +$ *i=1nsinhtiEi,m+i+Em+i,i:t1,…tn∈ℝ}.*

The space $SU(m,n)/S(U(m) \times U(n))$ is Hermitian symmetric means that the center $\mathfrak{c}(\mathfrak{k})$ is one dimensional. Let the center of $\mathfrak{k}$ be defined by

$$\mathfrak{c}(\mathfrak{k}) = \mathbb{R}z,$$

$$z = \sqrt{-1}(\sum_{i=1}^{m} nE_{i,j} - \sum_{i=1}^{n} mE_{i,i}) = \sqrt{-1}\begin{pmatrix} nI_m & 0 \\ 0 & -mI_n \end{pmatrix}$$

Remark 4.1. The normalization of the central element z is different from the previous sections.

Now let us take $z \in \mathfrak{c}(\mathfrak{k})$ and for $t_1, \ldots, t_n \in \mathbb{R}$, write $a \in A$ as

$a = \sum_{i=1}^{n} cosht_i(E_{i,i} + E_{m+i,m+i}) + \sum_{i=n+1}^{m} E_{i,i} +$ *i=1nsinhtiEi,m+i+Em+i,i:t1,…tn∈ℝ.*

Then, $Ad(a)z = \sqrt{-1}(\sum_{i=1}^{n}\Big((ncosh^2t_i + msinh^2t_i)E_{i,i} -$ *mcosh2ti+nsinh2tiEm+i,m+i+i=n+1mEi,i+i=1n(−m +ncoshtisinhtiEi,m+i+m+ncoshtisinhtiEm+i,i))*

Let us define the functions

$$f: \mathbb{R} \geq 0 \to \mathbb{R} \geq 0, g: \mathbb{R} \geq 0 \to \mathbb{R} \geq 0, and\ h: \mathbb{R} \geq 0 \to \mathbb{R} \geq 0$$

by

$$f(t) = ncosh^2t + msinh^2t,$$
$$g(t) = mcosh^2t + nsinh^2t,$$
$$h(t) = (m+n)coshtsinht,$$

Then, $\mathrm{Ad}(a)z$ can be written as

$$Ad(a)z = \sqrt{-1}(\sum_{i=1}^{n}\big(f(t_i)E_{i,i} - g(t_i)E_{m+i,m+i}\big) + \sum_{i=n+1}^{m} E_{i,i} + \sum_{i=1}^{n} h(t_i)(E_{m+i,i} - E_{i,m+i}))$$

The Cartan projection $pr^{\theta}: \mathfrak{g} \to \mathfrak{k}$ is given as follows:

$$\mathfrak{g} \to \mathfrak{k}, \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

Therefore,

$$pr^{\theta}\big((Ad(a)\big)z = \sqrt{-1}(\sum_{i=1}^{n}\big(f(t_i)E_{i,i} - g(t_i)E_{m+i,m+i}\big) + \sum_{i=n+1}^{m} E_{i,i}) \dots\dots\dots\dots (4.1)$$

Likewise, for $t_1', \dots, t_n' \in \mathbb{R}$, let us take $a' \in A$ such that

$pr^{\theta}(Ad(a')z) = \sqrt{-1}(\sum_{i=1}^{n}\big(f(t_i')E_{i,i} - g(t_i')E_{m+i,m+i}\big) +$ *i=n+1mEi,i)...........(4.2)*

Then defining a positive system $\sum_{+}(\mathfrak{g}, \mathfrak{a})$ such that the corresponding dominant Weyl chamber is given by

$$\mathfrak{a}_{+} = \{\sum_{i=1}^{n} t_i\big(E_{i,m+i} + E_{m+i,i}\big): t_1 \geq \dots \geq t_n \geq 0\}.$$

Put $A_{+} = \exp(\mathfrak{a}_{+})$. Now let us assume that $pr^{\theta}(Ad(a')z) \in Ad(K)pr^{\theta}(Ad(a)z)$ for $a, a' \in A_{+}$. Then the following result is obtained.

Lemma 4.1. $pr^{\theta}(Ad(a)z)$ and $pr^{\theta}(Ad(a')z)$ are conjugate under $K \simeq S(U(m) \times U(n))$ if and only if

$$\sum_{i=1}^{n} f(t_i)E_{i,i} + \sum_{i=n+1}^{m} E_{i,i} \sim \sum_{i=1}^{n} f(t_i')E_{i,i} + \sum_{i=n+1}^{m} E_{i,i}$$

under the permutation groupS_m , and $\sum_{i=1}^{n} g(t_i)E_{m+i,m+i} \sim$ *i=1ng(ti′)Em+i,m+i* under the permutation group *Sn.*

Proof:

The Cartan subgroup $\mathfrak{t}$ in $\mathfrak{g}$ can be taken as

$$\mathfrak{t} = \sqrt{-1}\left\{\sum_{i=1}^{m+n} \mathbb{R}E_{i,i}\}\right\}$$

Then in the computations (4.1), (4.2) we observe that

$$pr^{\theta}\big(Ad(a)\big)z, pr^{\theta}(Ad(a')z) \in \mathfrak{t}.$$

Whence, the lemma easily follows.

Now by the lemma 4.1, it follows that there exist some $\sigma \in S_n$ such that

$f(t_j) = f\big(t_{\sigma(i)}\big) = f(t_i'),$

$g(t_j) = g\big(t_{\sigma(i)}\big) = g(t_i')$ for $1 \leq i \leq j \leq n$.

Again it follows from

$t_1 \geq \dots \geq t_n \geq 0$, and $t_1' \geq \dots \geq t_i' \geq 0$ that

$f(t_1) \geq \dots \geq f(t_n) \geq 0$, and $f(t_1') \geq \dots \geq f(t_i') \geq 0,$

$g(t_1) \geq \dots \geq g(t_n) \geq 0$, and $g(t_1') \geq \dots \geq g(t_i') \geq 0.$

Hence in either case it can be concluded that $t_j = t_{\sigma(i)} = t_i'$ for any $1 \leq i \leq j \leq n$.

Whence it is easily proved that $a = a'$ for any $a, a' \in A_{+}$.

Therefore Proposition 3.1 in the case where $G = SU(m, n)$ has been proved.

V. Conclusion

Recently around the world many experiments and theoretical observations have been made to understand how orbit method, multiplicity function are applied to another fields of science. This paper has made and attempt to describe first: the essence of the special unitary Lie groups for non-experts and second: to attract the younger generation of mathematicians to some old and still unsolved problems in multiplicity function where we believe the orbit method could be helpful. This paper could be a guideline of studying of modern approach in this area of matrix computations of Corwin Greenleaf multiplicity function for special unitary group.

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