Dhaka Univ. J. Sci. 59(2): 167-172, 2011 (July)

Comparison of Positive Solutions for Two Boundary Value Problems arising in the Boundary Layer Flow

Mohammad Riazuddin Molla, Md. Kamrujjaman and Mahmud-Ul-Hasan

Department of Mathematics, Dhaka University, Dhaka-1000, Bangladesh

Received on 27. 10. 2009. Accepted for Publication on 22.03.2011

Abstract

The existence and uniqueness of a positive solution of a singular nonlinear boundary value problem formulated from the Falkner-Skan boundary layer equation for $\alpha = 0$ and $\beta = -0.5$ are studied. To show the existence and uniqueness of a positive solution we use the constructive method such as the method of upper and lower solutions. We also compare between the positive upper and lower solutions obtained by Jun Yong Shin and present study. The positive solution obtained by Jun Yong Shin is less than or equal to the positive solution obtained from present study.

I. Introduction

The differential equation

$$f''' + \alpha ff'' + \beta(1 - f'^2) = 0 \tag{1.1}$$

with boundary conditions

$$\left.\begin{array}{l} f(\eta) = f'(\eta) = 0 \text{ at } \eta = 0 \\ f' \to 1 \text{ as } \eta \to \infty \end{array}\right\} \tag{1.2}$$

is known as Falkner-Skan boundary layer equation.

Shin [1] studied the differential equation (1.1) for $\alpha = 1$ and $\beta = 0.5$ with boundary conditions (1.2). In this article, we study the differential equation (1.1) for $\alpha = 0$ and $\beta = -0.5$ with boundary conditions (1.2).Let $y(= f''(\eta))$ be the dependent variable and $x = (f'(\eta))$ be the independent variable.

When $\alpha = 0$ and $\beta = -0.5$, then the case leads to a potential flows which are proportional to $\frac{1}{x}$ and it is applicable for all β. This corresponds to a case of a two-dimensional source or sink as potential flows is positive or negative. It may be represented as the flow in a divergent or convergent channel with flat walls.

For $\alpha = 0$ and $\beta = -0.5$, equation (1.1) reduces to

$$f''' - \frac{1}{2}(1 - f'^2) = 0, \tag{1.3}$$

with boundary conditions (1.2).

Now $y' = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\eta}}{\frac{dx}{d\eta}} = \dfrac{\frac{d}{d\eta}(f'')}{\frac{d}{d\eta}(f')} = \dfrac{f'''}{f''}$,

which implies

$$y'f'' = f'''. \tag{1.4}$$

Differentiating (1.4) with respect to η and simplifying we get

$$f^{iv} = y^2 y'' + y(y')^2.$$

As before differentiating (1.3) with respect to η we get

$$f^{iv} + f'f'' = 0.$$

This gives

$$y^2 y'' + y(y')^2 + xy = 0.$$

and finally we get

$$yy'' + (y')^2 + x = 0.$$

Here, we assume

$$f(\eta) = \eta - \ln(1+\eta) - \tfrac{3}{4}\eta^2 e^{-\eta}, \tag{1.5}$$

which satisfies the three boundary conditions of (1.2).

From (1.5), we get easily

$$f''(\eta) = \frac{1}{(\eta+1)^2} - \tfrac{3}{2}e^{-\eta} + 3\eta e^{-\eta} - \tfrac{3}{4}\eta^2 e^{-\eta},$$

from which we get for $\eta = 0$

$$f''(0) = -\tfrac{1}{2}.$$

Now

$$y(x) = f''(\eta) = \frac{d}{d\eta}[f'(\eta)],$$

from which

$$y(1) = [\frac{d}{d\eta}(f'(\eta))]_{x=1} = [\frac{d}{d\eta}(f'(\eta))]_{f'(\eta)=1} = 0 .$$

Again

$$y'(x) = \frac{dy}{dx} = \frac{f'''}{f''} = \frac{0.5(1 - f'^2)}{f''},$$

which implies

$$y'(0) = \left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{dy}{dx}\right|_{f'(\eta)=0} = \left.\frac{dy}{dx}\right|_{\eta=0} = \frac{0.5(1 - f'^2(0))}{f''(0)} = -1.$$

Thus by letting $y(= f''(\eta))$ and $x(= f'(\eta))$, equation (1.3) with boundary conditions (1.2) can be transformed into a second order singular nonlinear boundary value problem at $x = 1$:

$$\left.\begin{array}{l} yy'' + (y')^2 + x = 0,\ 0 < x < 1 \\ y'(0) = -1 \text{ and } y(1) = 0. \end{array}\right\} \quad (1.6)$$

For $\alpha = 1$ and $\beta = 0.5$, equation (1.1) with boundary conditions (1.2) takes the form

$$\left.\begin{array}{l} y^2 y'' - \frac{1}{2}(1 - x^2)y' = 0 \ ,\ 0 < x < 1 \\ y'(0) = -0.5 \quad \text{and } y(1) = 0 \end{array}\right\} \quad (1.7)$$

and its positive solution has been studied by Shin[1].

Shin [1] did not mention the value of $f''(0)$ when the boundary value problem (1.7) is formulated from (1.1) with boundary conditions (1.2) and did not state the details of this transformation in his article. Shin[1] transformation is possible if $f''(0) = 1$. In this article we deduced $f''(0) = -\frac{1}{2}$ for $\alpha = 0$ and $\beta = -0.5$.

The objectives of this article are as follows:
i) To establish the existence and uniqueness of a positive solution of (1.6) by using the constructive method such as the method of upper and lower solutions.
ii) To compare between the positive lower and upper solutions obtained by Shin [1] and the positive lower and upper solutions obtained from present study.
iii) To compare between the positive solution obtained by Shin [1] and the positive solution obtained from present study.

Definition 1.1: We call a function $\alpha_1 \in C^2[0,1]$ a positive upper solution of (1.6), if

$$\alpha_1 > 0 \quad \text{on } (0,1)$$
$$\alpha_1\alpha_1'' + (\alpha_1')^2 + x \le 0 \quad \text{on } (0,1)$$
$$\alpha_1'(0) \le -1 \text{ and } \alpha_1(1) \ge 0.$$

Definition 1.2: We call a function $\alpha_2 \in C^2[0,1]$ a positive Lower solution of (1.6), if

$$\alpha_2 > 0 \quad \text{on } (0,1)$$
$$\alpha_2\alpha_2'' + (\alpha_2')^2 + x \ge 0 \quad \text{on } (0,1)$$
$$\alpha_2'(0) \ge -1 \quad \text{and} \quad \alpha_2(1) \le 0.$$

Similar definitions hold for positive upper and lower solutions of a perturbation of (2.1) which will be given in the following section.

Definition 1.3: We call a function $y \in C[0,1] \cap C^2[0,1)$ a positive solution of (1.6) if

$$y > 0 \quad \text{on } (0,1)$$
$$yy'' + (y')^2 + x = 0 \quad \text{on } (0, 1)$$
$$y'(0) = -1 \quad \text{and } y(1) = 0.$$

II. Existence of a Unique Positive Solution

For each integer $p \ge 1$, we consider the nonlinear boundary value problem

$$\left.\begin{array}{l} yy'' + (y')^2 + x = 0, \quad 0 < x < 1 \\ y'(0) = -1 \text{ and } \quad y(1) = \dfrac{1}{p} \end{array}\right\} \quad (2.1)$$

which may be viewed as a perturbation of (1.6).

To prove the existence of positive solution of (1.6) we establish the existence of positive solution of (2.1).

Lemma 2.1: $y_{lp}(x) = (1 - x) + \frac{1}{p}$ is a positive lower solution of (2.1), for each $p \ge 1$.

Proof: It is clear that $y_{lp}(x) > 0$ on (0,1), $y_{lp}'(0) = -1$, which can be written as $y_{lp}'(0) = -1 \ge -1$, $y_{lp}(1) = \frac{1}{p}$, which can be written as $y_{lp}(1) = \frac{1}{p} \le \frac{1}{p}$ and

$y_{lp}y_{lp}'' + (y_{lp}')^2 + x = (1 + x) \ge 0$, for $0 < x < 1$ and $p \ge 1$.

Thus y_{lp} is a positive lower solution of (2.1).

Lemma 2.2: $y_{up}(x) = 2\ln(2 - x) + 4 + \frac{1}{p}$ is a positive upper solution of (2.1) for each $p \ge 1$.

Proof: It is clear that $y_{up}(x) > 0$ on (0,1), $y_{up}'(0) = -1$, which can be written as $y_{up}'(0) = -1 \le -1$, $y_{up}(1) = 4 + \frac{1}{p} \ge \frac{1}{p}$ and

$$y_{up}y_{up}'' + (y_{up}')^2 + x = \{2\ln(2 - x) + 4 + \tfrac{1}{p}\}$$
$$\{-\frac{2}{(2 - x)^2}\} + \frac{4}{(2 - x)^2} + x \le 0, \quad \text{for}$$

$0 < x < 1$ and $p \ge 1$.

Thus y_{up} is a positive upper solution of (2.1).

Hence we can formulate the following Lemma from an application of Schauder's Fixed Point Theorem [3].

Lemma 2.3: For any $p \geq 1$, there exists a positive solution $y_p \in C^2[0,1]$ of the problem (2.1) such that $y_{lp} \leq y_p \leq y_{up}$ on $0 \leq x \leq 1$, where y_{lp} and y_{up} are as given in Lemma 2.1 and Lemma 2.2 respectively.

Lemma 2.4: If y_p is a positive solution of (2.1), then $y'_p(x) < -1$ on (0, 1).

Proof: Since y_p is positive solution of (2.1), we get

$$y_p y''_p + (y'_p)^2 + x = 0.$$

This gives $y_p y''_p = -(y'_p)^2 - x$, which implies

$$y''_p < 0$$

Integrating 0 to x, we get

$$y'_p(x) - y'_p(0) < 0$$

$$\Rightarrow y'_p(x) < y'_p(0)$$

Since $y'_p(0) = -1$, it follows that $y'_p(x) < -1$ on (0, 1).

Lemma 2.5: If y_1 and y_2 are two positive solutions of (2.1), then $y_1 = y_2$.

Proof: Since y_1 and y_2 are two positive solutions of (2.1), we get

$y_1 y''_1 + (y'_1)^2 + x = 0$ and $y_2 y''_2 + (y'_2)^2 + x = 0.$

Therefore, we get

$$y''_1 = -\frac{(y'_1)^2}{y_1} - \frac{x}{y_1} \quad \text{and} \quad y''_2 = -\frac{(y'_2)^2}{y_2} - \frac{x}{y_2}.$$

Suppose that there exists an $\eta \in (0,1)$ such that $y_1(\eta) \neq y_2(\eta)$.

If $y_1(0) = y_2(0)$, then by the uniqueness theorem of the initial value problem, we get $y_1 = y_2$, which is a contradiction. Therefore, without loss of generality, we may assume that $y_1(0) < y_2(0)$.

Since $y_1(1) = y_2(1)$, by the continuity of $y_1 - y_2$, there exists a $\xi \in (0,1)$ such that $y_1(\xi) = y_2(\xi)$ and $y_1(x) < y_2(x)$ on $[0, \xi)$.

So the function $\varphi(x) = y_2(x) - y_1(x)$ has a maximum at $x = 0$ or at the interior of $[0, \xi]$. If the function $\varphi(x)$ takes its minimum at $x = 0$, then we obtain $\varphi''(0) > 0$.

Now,

$$\varphi''(0) = \lim_{x \to 0^+} \frac{\varphi'(x) - \varphi'(0)}{x} > 0.$$

It follows that $\varphi'(x) > \varphi'(0) = 0$ near $x = 0$ and so $\varphi(x) > \varphi(0)$ near $x = 0$. This is a contradiction.

Hence $\varphi(x)$ does not have a minimum at $x = 0$.

If the function $\varphi(x)$ takes its maximum at the interior of $[0, \xi]$, then there exists a $\xi_1 \in (0, \xi)$ such that $\varphi'(\xi_1) = 0$ and $\varphi''(\xi_1) < 0$.

But since $y_1(\xi_1)$, $y_2(\xi_1)$ are solutions of (2.1), $y_1(\xi_1) < y_2(\xi_1)$ and $y'_1(\xi_1) < -1$,

$$\varphi''(\xi_1) = y''_2(\xi_1) - y''_1(\xi_1)$$

$$= (y'_1)^2 \left(\frac{1}{y_1(\xi)} - \frac{1}{y_2(\xi)}\right) + \xi \left(\frac{1}{y_1(\xi)} - \frac{1}{y_2(\xi)}\right) > 0,$$

which again leads to a contradiction. Hence $\varphi(x)$ does not have a maximum at the interior of $[0, \xi]$.

This implies that $y_1 = y_2$.

Lemma 2.6: If y_p is a positive solution of (2.1), for each $p \geq 1$ then we obtain

$$y'_p(x) \geq -1 - \int_0^x \frac{s}{(1-s)} ds - \int_0^x \frac{(y'_p(s))^2}{(1-s)} ds \text{ on } (0,1).$$

Proof: Let y_{lp} be a positive lower solution of (2.1). Since $y_p \in C^2(0,1)$ is a positive solution of (2.1), we obtain $y_{lp} \leq y_p$ and $y_p y''_p + (y'_p)^2 + x = 0.$

Now,

$$0 = y''_p + \frac{(y'_p)^2}{y_p} + \frac{x}{y_p} \leq y''_p + \frac{(y'_p)^2 + x}{y_{lp}}$$

$$\text{or, } y''_p \geq -\frac{(y'_p)^2 + x}{y_{lp}},$$

which after integrating from 0 to x yields

$$y'_p(x) \geq y'_p(0) - \int_0^x \frac{(y'_p(s))^2 + s}{y_{lp}(s)} ds$$

which can be written as

$$y'_p(x) \geq -1 - \int_0^x \frac{s}{(1-s)} ds - \int_0^x \frac{(y'_p)^2}{(1-s)} ds \text{ on } (0, 1),$$

because $y_{lp}(s) = (1-s) + \frac{1}{p} \geq (1-s)$ on $[0,1)$ and $y'_p(0) = -1$.

This completes the proof.

Lemma 2.7: If $p_1 > p_2 \geq 1$ and y_{p_1}, y_{p_2} are positive solutions of (2.1), then we have $0 < y_{p_1}(x) < y_{p_2}(x)$ on (0, 1) and $y'_{p_1}(x) < y'_{p_2}(x) < 0$ on (0,1).

Proof: It is clear from the fact that y_{p_2} is a positive upper solution of (2.1) and so $y_{p_1}(x) \leq y_{p_2}(x)$ on (0,1].

If $y_{p_1}(0) = y_{p_2}(0)$, then by the uniqueness theorem of the initial value problem we obtain $y_{p_1}(x) \equiv y_{p_2}(x)$, which is a contradiction. So we may assume that $0 < y_{p_1}(0) < y_{p_2}(0)$. Then we have

$$y''_{p_2}(0) - y''_{p_1}(0) = (y'_{p_1}(0))^2 \left(\frac{1}{y_{p_1}(0)} - \frac{1}{y_{p_2}(0)}\right) > 0,$$

which implies that

$$y'_{p_2}(x) - y'_{p_1}(x) > y'_{p_2}(0) - y'_{p_1}(0) = -1 + 1 = 0,$$

$y_{p_2}(x) - y_{p_1}(x) > y_{p_2}(0) - y_{p_1}(0) > 0$, for x near to 0.

If there exists a $\xi \in (0,1]$ such that

$y'_{p_2}(\xi) - y'_{p_1}(\xi) = 0$ and $y'_{p_2}(x) - y'_{p_1}(x) > 0$, $0 < x < \xi$.

Then we obtain

$$y''_{p_2}(\xi) - y''_{p_1}(\xi) = \left(\frac{1}{y_{p_1}(\xi)} - \frac{1}{y_{p_2}(\xi)}\right)(\xi + (y'_{p_1}(\xi))^2) > 0,$$

because $y_{p_1}(\xi) < y_{p_2}(\xi)$.

Hence we have

$$y'_{p_2}(x) - y'_{p_1}(x) < y'_{p_2}(\xi) - y'_{p_1}(\xi) = 0,$$

$0 < x < \xi$,

which is a contradiction. Hence $y'_{p_1}(x) < y'_{p_2}(x) < 0$ on $(0,1)$.

Theorem 2.8: (Existence): If y_p is the positive solution of (2.1) for each $p = 1,2,3,4$............., then the sequence $\{y_p\}$ converges to a positive solution y of (1.6).

Proof: To prove this theorem, we prove the following steps:

Step 1. $y_p \to y$ as $p \to \infty$

Step 2. $y \in C[0,1] \cap C^2(0,1)$

Step 3. y is a positive solution of (1.6).

Our first step is to show that $y_p \to y$ as $p \to \infty$. From Lemma 2.3 and Lemma 2.7 we know that the sequence $\{y_p\}$ is monotone decreasing in p and bounded below by $(1-x)$. Therefore, $y_p \to y$ as $p \to \infty$ and $y(x) \geq (1-x)$ on [0, 1].

Also from Lemma 2.6 and Lemma 2.7 we know that the sequence $\{y'_p\}$ is monotone decreasing in p and bounded below by $-1 - \int_0^x \frac{s}{(1-s)} ds - \int_0^x \frac{(y'_p)^2}{(1-s)} ds$ on (0, 1).

Therefore, $y'_p \to y'$ as $p \to \infty$.

and

$$y'(x) \geq -1 - \int_0^x \frac{s}{(1-s)} ds - \int_0^x \frac{(y')^2}{(1-s)} ds \text{ on } (0,1).$$

Our second step is to show that $y \in C[0,1] \cap C^2(0,1)$. If we integrate $y''_p = -\frac{x}{y_p(x)} - \frac{(y'_p)^2}{y_p(x)}$ from 0 to x, then we have

$$y'_p(x) - y'_p(0) = -\int_0^x \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi$$

$$\Rightarrow y'_p(x) = -1 - \int_0^x \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi. \tag{2.2}$$

If we integrate both sides of (2.2) from 0 to x, then we obtain

$$y_p(x) - y_p(0) = -x - \int_0^x \int_0^s \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi ds.$$

Let

$$u(s) = \int_0^s \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi.$$

Then we have,

$$y_p(x) - y_p(0) = -x - \int_0^x u(s) ds$$

$$= -x - x\int_0^x \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi + \int_0^x s \frac{s + (y'_p(s))^2}{y_p(s)} ds.$$

Changing s to ξ ,we have

$$y_p(x) - y_p(0) = -x - x\int_0^x \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi + \int_0^x \xi \frac{\xi + (y'_p(\xi))^2}{y_p(\xi)} d\xi \tag{2.3}$$

.

If we let $p \to \infty$ on both sides of (2.2) and (2.3), then by Lebesgue's Dominated Convergence Theorem, we obtain

$$y'(x) = -1 - \int_0^x \frac{\xi + (y'(\xi))^2}{y(\xi)} d\xi$$

and

$$y(x) - y(0) = -x - x\int_0^x \frac{\xi + (y'(\xi))^2}{y(\xi)} d\xi + \int_0^x \xi \frac{\xi + (y'(\xi))^2}{y(\xi)} d\xi \quad , \qquad (2.4)$$

which implies that $y \in C^2(0,1)$. Since $y(x)$ converges to 0 as x approaches 1, y is continuous at $x = 1$ which implies $y \in C[0,1] \cap C^2(0,1)$. Finally, we shall show that y is a positive solution of (1.6).It is clear that $y'(0) = -1$ and $y(1) = 0$.

If we take second derivative on both sides of

$$y'(x) = -1 - \int_0^x \frac{\xi + (y'(\xi))^2}{y(\xi)} d\xi ,$$

then we obtain,

$y'' = -\dfrac{x + (y')^2}{y}$, which implies that y is a positive solution of (1.6).

Theorem 2.9: (Uniqueness) :Assume that y_1 and y_2 are positive solutions of (1.6). Then $y_1 = y_2$.

Proof: The proof of this theorem is similar to that of Lemma 2.5 .

III. Results and Discussion

Here we compare our positive upper and lower solutions for $\alpha = 0$ and $\beta = -0.5$ with the positive upper and lower solutions obtained by Shin [1] for $\alpha = 1$ and $\beta = 0.5$ and $p \geq 1$ in each case.In Table 1 and Table 2, results are given for $p = 1$, $p = 2$ and $p \to \infty$.

Table. 1.

x	$y_{lp} = (1-x) + \frac{1}{p}$ Positive lower solution [Present]			$y_{jlp} = \frac{1}{2}(1-x) + \frac{1}{p}$ Positive lower solution obtained by Shin[1]		
	p =1	p =2	$p \to \infty$	p =1	p =2	$p \to \infty$
0	2.00	0.50	1.00	1.50	1.00	0.50
0.4	1.60	0.62	0.60	1.30	0.80	0.30
0.8	1.20	0.58	0.20	1.10	0.60	0.10
1.0	1.00	0.50	0.00	1.00	0.50	0.00

Table. 2.

x	$y_{up} = 2\ln(2-x) + 4 + \frac{1}{p}$ Positive upper solution [Present]			$y_{jup} = 2\sqrt{1 - x + \frac{1}{p}}$ Positive upper solution obtained by Shin[1]		
	p =1	p =2	$p \to \infty$	p =1	p =2	$p \to \infty$
0	6.38629	4.84657	5.38629	2.82843	2.44949	2.00000
0.4	5.94001	4.73500	4.94001	2.52982	2.09672	1.54919
0.8	5.36464	4.59116	4.36464	2.19089	1.67332	0.89443
1.0	5.00000	4.50000	4.00000	2.00000	1.41421	0.00000

We form the following Table3 for y_{lp} and y_{up} as $p \to \infty$ and for exact unique positive solution of (1.6). The positive lower and upper solutions of (1.6) are $y_{lp} = (1-x)$ and $y_{up} = 2\ln(2-x) + 4$ respectively.

Table. 3.

x	$y_{lp}(x)$	$y_{up}(x)$	$y_p(x)$	x	$y'_p(x)$
0	1.00	5.38629	2.1547	0	-1.0000
0.4	0.60	4.94001	1.7022	0.4	-1.3128
0.8	0.20	4.36464	1.0122	0.8	-2.4449
1.0	0.00	4.00000	0.0000	0.9	-3.5453

We observe from Table 1 and Table 2 that $y_{jlp} \le y_{lp} \le y_{jup} \le y_{up}$, where y_{jlp} and y_{jup} are respectively the positive lower and upper solutions of (1.7) obtained by Shin [1] as $p \to \infty$. On the other hand y_{lp} and y_{up} are the positive lower and upper solutions of (1.6) obtained from present study as $p \to \infty$.

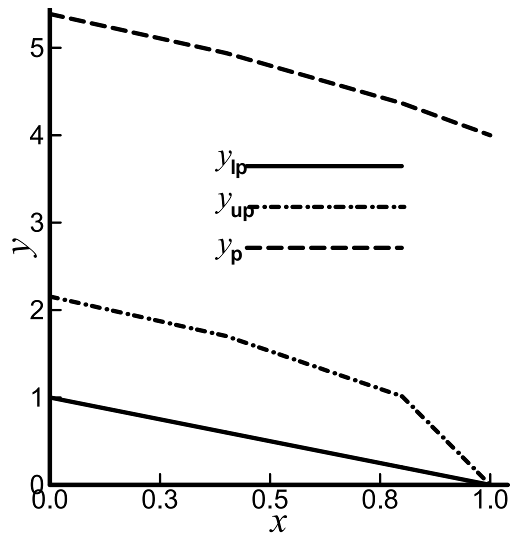

Fig.1. Different numerical solution such as (i) positive lower solution y_{lp} (ii) positive upper solution y_{up} and (iii) positive solution y_p.

Shin[1] did not find the numerical value of positive solution of (1.7),he only mentioned that there will exist a positive solution of (1.7) between y_{jlp} and y_{jup}.The main achievement of our present study is to find the numerical value of positive solution of (1.6).We established the relation between the positive solution obtained by Shin[1] and the positive solution obtained from present study that is also an achievement of us .In this article we also established the existence and uniqueness of a positive solution of (1.6) by using the method of upper and lower solutions and shown that the numerical value of the positive solution of (1.6) lies between y_{lp} and y_{up}.Equation (1.6) and (1.7) are different in form and physically. y_{jlp} and y_{jup} are not the positive upper solutions of (1.6) and y_{up} is not the positive upper solution of (1.7) as $p \to \infty$. Since the positive solution lies between positive lower and upper solutions so we can conclude that $y_{lp} \le y_{jlp} \le y_{jp} \le y_{jup} \le y_{up}$, where y_{jp} is the positive solution of (1.7) obtained by Shin [1] and y_p is the positive solution of (1.6) obtained from the present study as $p \to \infty$. Therefore it is clear from Table1,Table2 and Table3 that $y_{jp} \le y_p$, that is; the positive solution obtained by Shin is less than or equal to the positive solution obtained from present study. From the above Figure we see that the numerical value of the positive solution of (1.6) lies between y_{lp} and y_{up}.We think the results obtained in this article will be helpful to study the behavior of the boundary layer flow.

----

1. Jun Yong Shin, 1997.A Singular nonlinear differential equation arising in the Homann flow, J. Math. Anal. Appl. **212**, 443-451.
2. Shanti Swarup, 2000. Fluid Dynamics, Krishna Prakashan Media (P) Ltd. Merut, pp-630.
3. K. Schmidt, 1970. A nonlinear boundary value problem, J. Differential Equations **7**,527-537.