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Two-dimensional Unsteady Boundary Layer Flow with Mixed Convection Along a Semi-Infinite Symmetric Wedge

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Abstract

The mixed convection of a two-dimensional, unsteady boundary-layer flow of a viscous incompressible fluid past a symmetric wedge has been studied. The resulting system of non-linear ordinary coupled differential equations has been solved analytically for small time and numerically by three individual competent methodologies; namely (a) Series solutions method for small time, (b) Asymptotic solutions for large time and (c) Implicit *finite difference method* (FDM) together with the Keller-box elimination scheme. Numerical results obtained for skin-friction coefficient, heat transfer rate, streamlines and isotherms with the effect of dissimilar leading parameters such as different time τ , the exponent *m*, mixed convection parameter λ and Prandtl number Pr.

Keywords: Mixed convection flow, Semi-infinite symmetric wedge and unsteady flow.

Nomenclature

f	Dimensionless stream function	U_∞	Potential flow velocity
g	Dimensionless temperature function	$\bar{u}_e(\bar{x})$	Velocity of the potential flow
F	Dimensionless stream function f expressed in terms of η	$\bar{u}, \bar{v}$	Velocity components along the $\bar{x}, \bar{y}$ axes, respectively
G	Dimensionless temperature function g expressed in terms of η	$\bar{x}, \bar{y}$	Cartesian coordinate measured along the surface of the wedge and normal to it respectively
C_f	Skin-friction coefficient		***Greek symbols***
C_p	Specific heat at constant pressure	α	Thermal diffusivity
k	Thermal conductivity of the fluid	β	Constant defining the included angle $\pi\beta$ of the wedge
l	Characteristic length	ν	kinematic viscosity
m	Exponent	μ	Dynamic viscosity
Gr_l	Grashof number	ρ	Fluid density
Pr	Prandtl number	λ	ratio between *Gr* and Re^2
Re_l	Reynolds number	η	Transformed spatial variable
Nu	Local Nusselt number	ψ	Stream function
θ	Temperature of the fluid	τ	Dimensionless time
T_w	Surface temperature of the wedge		***Subscripts & Superscripts***
T_r	Reference temperature	w	surface conditions
T_∞	Ambient temperature	∞	ambient temperature conditions
t	Time	$'$	differentiation with respect to η

I. Introduction

Convective heat transfer in unsteady flows are of fundamental importance over a stationary for the thermal design of various types of industrial equipments such as aircraft response to atmospheric gusts, in aerofoil lift hysteretic at the stall, in flutter phenomena involving wing, spin stabilized missiles, canisters of nuclear waste disposal, nuclear reactor cooling system and geothermal reservoirs, panel and staling flutter as well as in the prediction of flow over helicopter rotor blades and through turbo-machinery blade cascades. Environmental and industrial applications including ground water pollution are studied along the symmetric semi-infinite wedge. The steady Falkner-Skan is one of the recognized problems and has been considered by many investigators; for example, Leal [1], Schlichting et al. [2] and Gersten et al. [3]. In recent years, fluctuating conditions of motion and heating of bodies in fluids have been increasingly essential in certain applications within some engineering fields of aerodynamics and hydrodynamics. There is a large body of literature on unsteady, forced convection, boundary-layer flows past bodies of different geometries that give rise to the Falkner-Skan equations, mentioned in Telionis [4], Riley [5], and Ludlow et al. [6]. However, fewer studies have been concerned with the heat transfer aspects, see, Pop [7]. The problem reduces to an uncoupled, laminar boundary layer-flow and the fluid velocity field is unaffected by any temperature changes when the fluid is assumed to have constant properties. However, the problem becomes coupled when the thermo physical fluid properties depend on the temperature, so that the fluid velocity is also a function of time.

Studies of the skin-friction and heat transfer in two-dimensional laminar flow over wedge-shaped bodies can accurately be calculated by solving the boundary layer differential equations. The momentum boundary layer equation for Falkner-Skan flow from a wedge, with potential flow velocity $u_e(x) = x^m$, was first deduced by Falkner and Skan [8]. Afterward Hartree [9] investigated the similarity solutions of the flow in details. He represented the solutions in terms of velocity distribution for different values of pressure gradient parameter. For flow over an arbitrary body shape with known pressure or velocity distribution where there exists no similarity, the skin-friction and heat transfer are conventionally found by an approximate method, either the integral method or the equivalent wedge flow approximation. Both of these two methods yield sufficiently accurate results for most engineering applications. Smith [10] initiated the unsteady, forced convection, boundary-layer flow past a semi-infinite wedge and this problem was subsequently solved numerically by Nanbu [11] using the method proposed by Hall [12] and in recent times that was modified by Harris et al. [13]. This method solves the untransformed equations directly using an iterative procedure and by implicit finite difference method, which is well documented and widely used by Keller and Cebeci [14] for unsteady boundary-layers. Watkins [15] has also solved this problem numerically following a second order; he has also studied the unsteady heat transfer aspects of the semi-infinite wedge started impulsively from rest to include solutions of the energy equation. The present work is concerned with the problem of heat transfer for an impulsively started Falkner-Skan flow due to Harris et al. [13].The majority of relevant cases are considered related to the acute semi-infinite wedge problem.

The situation considered here is that of heat transfer in the unsteady, thermal boundary-layer associated with the mixed convection (momentum) boundary-layer flow resulting from a transient Falkner-Skan problem with exponent *m*. This situation has physical relevance when $0 \le m \le 1$ and, for such cases; the flow is that of an incompressible fluid past a sharp, semi-infinite wedge of included angle $2m\pi/(m+1)$. In this article, the concentration has been given to a study of the mixed convection flow of an unsteady, two-dimensional, viscous flow of an incompressible fluid past a symmetrical, sharp wedge with variable surface temperature, and a distributed heat source of the form $g\beta(\overline{T} - T_\infty)$. In addition, at $t = 0$, the thermal boundary-layer is produced by the simultaneous sudden imposition of a constant temperature $T_w(> T_\infty)$ over the surface. Using appropriate transformations, the governing equations are reduced to non-linear partial differential equations for both the steady mean flow and unsteady flow, solutions of which are obtained numerically employing the finite difference method together with Keller-box elimination scheme. The initial development of the thermal boundary-layer has satisfactorily been represented by analytical solution for small times. Physically, at this initial stage of the transient process, diffusion dominates convection, which is affected only weakly by the velocity components close to the surface. The asymptotic solution for large time approaches steady state and is given by the Falkner-Skan equation. The very detailed numerical solution is presented for the whole transient from the initial ($\tau \to 0$) unsteady to the final ($\tau \to \infty$) steady state by using a modification of the step-by-step method, in combination with a finite-difference method. Effects of pertinent parameters such as varying the pressure gradient *m*, the Prandtl number Pr and mixed convection parameter λ, on the shearing stress and the rate of heat transfer in terms of skin-friction and Nusselt number respectively are shown in figures (2) and (3). Finally the flow pattern in terms of streamlines and isotherms has also been shown with varying the time τ for the effects of fixed $m = 0.5$, $\Pr = 0.7$ and $\lambda = 0.5$ in figures (4) and (5).

II. Mathematical Formalism

We consider the unsteady two-dimensional laminar mixed convection flow of a viscous incompressible fluid past a symmetrical, sharp wedge with a distributed heat source of the form $g\beta(\overline{T} - T_\infty)$. The inviscid flow over the wedge develops instantaneously and its velocity is given by

$$\overline{u}_e(\overline{x}) = U_\infty(\overline{x}/L)^m \text{ , for } m \le 1 \tag{1}$$

where *L* is a characteristic length and *m* is pressure gradient related to the included angle πβ by *m* = *β*/(2-*β*). It is clear that for negative values of *m* the solution becomes singular at $\overline{x} = 0$, whilst for positive *m* the solution can be defined for all values of $\overline{x}$, and this leads to a general difference between the solutions for the case of $m < 0$ and $m \ge 0$. It is assumed that the variable surface temperature of the wedge is $T_w(> T_\infty)$, where the ambient temperature of the fluid is assumed to be T_∞ and θ being the temperature of the fluid in the boundary layer. The physical configurations for this flow are shown in **Fig. 1**.

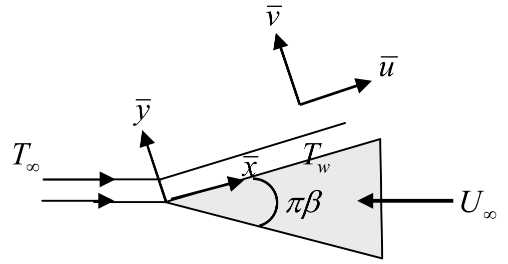


Fig. 1. Physical model and coordinate system

The flow is governed by the following boundary layer equations:

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0 \tag{2}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} =\bar{u}_e\frac{\partial \bar{u}_e}{\partial \bar{x}}+\nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+g\beta(\bar{T}-T_\infty) \tag{3}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}=\frac{k}{\rho C_p}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \tag{4}$$

where $(\bar{u},\bar{v})$ are the velocity components along the $(\bar{x},\bar{y})$ axes, ν is the kinematic viscosity, ρ is the density of the fluid, k is the thermal conductivity, C_p is the specific heat at constant pressure and μ is the constant viscosity of the fluid in the boundary layer region.

The boundary conditions are to be satisfied for the present problems are

$$\bar{u}=0,\ \bar{v}=0,\ \bar{T}=T_w(\bar{x})\quad at\quad \bar{y}=0 \tag{5a}$$

$$\bar{u}=\bar{u}_e(\bar{x}),\quad \bar{T}\to 0\quad as\quad \bar{y}\to\infty. \tag{5b}$$

We now introduce the following dimensionless dependent and independent variables according to,

$$x=\frac{\bar{x}}{L},\ y=\mathrm{Re}_L^{1/2}\frac{\bar{y}}{L},\ u=\frac{\bar{u}}{U_\infty},\quad v=\mathrm{Re}_L^{1/2}\frac{\bar{v}}{U_\infty},\ u_e=\frac{\bar{u}_e}{U_\infty},\quad t=\frac{U_\infty}{L}\bar{t},\ \theta=\frac{T_w-T_\infty}{T_r-T_\infty},\quad \mathrm{Re}_L=\frac{U_\infty L}{\nu}. \tag{6}$$

The velocity over the wedge is now given by

$$u_e(x)=x^m,\qquad for\quad m\le 1 \tag{7}$$

and, sufficiently far downstream from the apex, the governing equations (2)-(4) can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{8}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{\partial u_e}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\lambda\theta \tag{9}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial y^2} \tag{10}$$

In the above equations, λ is the mixed convection parameter and defined as the ratio of Gr and Re^2, where

Prandtl's number, $\mathrm{Pr}=\dfrac{\mu C_p}{k}$;

Grashof number, $\mathrm{Gr_L}=\dfrac{g\beta(T_r-T_\infty)L^3}{\nu^2}$; and Reynolds number, $\mathrm{Re_L}=\dfrac{U_\infty L}{\nu}$.

The boundary conditions then turn to

$$u=0,\ v=0,\ \theta=x^{2m-1}\quad at\quad y=0 \tag{11a}$$

$$u=u_e(x)=x^m,\ \theta\to 0\ as\ y\to\infty \tag{11b}$$

for $t\ge 0$, $m\le 1$.

The number of independent variables in the governing equations (8)-(10) can be reduced from three to two by introducing the dimensionless, reduced stream function $F(\eta,\tau)$ and temperature function $G(\eta,\tau)$ as

$$\psi=x^{\frac{m+1}{2}}(2\zeta)^{\frac{1}{2}}F(\eta,\zeta),\ \theta=x^{2m-1}G(\eta,\zeta)$$
$$\eta=x^{\frac{m-1}{2}}(2\zeta)^{-\frac{1}{2}}y,\ \zeta=1-e^{-\tau},\ \tau=x^{m-1}t \tag{12}$$

where η is a non-dimensional similarity variable and ψ is the stream function for unsteady flow, which is defined in the usual way, namely $u=\dfrac{\partial\psi}{\partial y}$ and $v=-\dfrac{\partial\psi}{\partial x}$. Thus the set of equations (9)-(10) are transformed to

$$\begin{aligned} &F'''+\eta(1-\zeta)F''+\big[(m+1)\zeta-(m-1)(1-\zeta)\ln|1-\zeta|\big]FF''+2m\zeta(1-F'^2)\\ &=2(1-m)\zeta(1-\zeta)\ln|1-\zeta|\left[F'\frac{\partial F'}{\partial\zeta}-F''\frac{\partial F}{\partial\zeta}\right]+2\zeta(1-\zeta)\frac{\partial F'}{\partial\zeta}-2\lambda\zeta G \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{1}{\mathrm{Pr}}G''+\eta(1-\zeta)G'+\big[(m+1)\zeta-(m-1)(1-\zeta)\ln|1-\zeta|\big]FG'-2(2m-1)\zeta GF'\\ &=2(1-m)\zeta(1-\zeta)\ln|1-\zeta|\left[F'\frac{\partial G}{\partial\zeta}-G'\frac{\partial F}{\partial\zeta}\right]+2\zeta(1-\zeta)\frac{\partial G}{\partial\zeta}. \end{aligned} \tag{14}$$

The appropriate boundary conditions to be satisfied by (13) and (14) are

$$F(0,\zeta)=F'(0,\zeta)=0,\quad G(0,\zeta)=1,\quad F'(\infty,\zeta)=1,\quad G(\infty,\zeta)=0. \tag{15}$$

Here prime denotes the differentiation of the functions with respect to η.

Now to find the numerical solutions we can get the easiest form by using the transformation $\zeta = 1 - e^{-\tau}$ from the equations (13) and (14)

$$\begin{aligned} &F''' + \eta e^{-\tau} F'' + \left[(m+1)(1-e^{-\tau}) \right. \\ &\left. + (m-1)\tau e^{-\tau}\right] FF'' + 2m(1-e^{-\tau})(1-F'^2) \\ &= 2(m-1)\tau(1-e^{-\tau})\left[F'\frac{\partial F'}{\partial \tau} - F''\frac{\partial F}{\partial \tau}\right] \\ &+ 2(1-e^{-\tau})\left[\frac{\partial F'}{\partial \tau} - \lambda G\right] \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{1}{\Pr} G'' + \eta e^{-\tau} G' + \left[(m+1)(1-e^{-\tau}) \right. \\ &\left. + (m-1)\tau e^{-\tau}\right] FG' - 2(2m-1)(1-e^{-\tau})GF' \\ &= 2(m-1)\tau(1-e^{-\tau})\left[F'\frac{\partial G}{\partial \tau} - G'\frac{\partial F}{\partial \tau}\right] \\ &+ 2(1-e^{-\tau})\frac{\partial G}{\partial \tau} \end{aligned} \tag{17}$$

The corresponding boundary conditions (15) becomes

$$\begin{aligned} &F(0,\tau) = F'(0,\tau) = 0, \quad G(0,\tau) = 1, \\ &F'(\infty,\tau) = 1, \quad G(\infty,\tau) = 0 \end{aligned} \tag{18}$$

The physical quantities that are important from application point of view are the surface shear stress and the surface rate of heat transfer, which can be measured in terms of the skin friction coefficient, $C_f = \tau_w(\bar{x}) / \rho(\bar{u}_e(\bar{x}))^2$ and the local Nusselt number, $N_u = q_w(\bar{x})\bar{x} / k_f(T_w - T_\infty)$, where, $\tau_w(\bar{x}) = \mu(\partial \bar{u} / \partial \bar{y})_{\bar{y}=0}$ is the skin friction along the surface, $q_w(\bar{x}) = -k_f(\partial \bar{T} / \partial \bar{y})_{\bar{y}=0}$ is the surface heat flux, and k_f is the thermal conductivity of the fluid. By introducing the dimensionless variables (6) and the transformation (12), the skin friction coefficient, $C_f \mathrm{Re}_x^{1/2}$ and the local Nusselt number, $Nu\,\mathrm{Re}_x^{-1/2}$ defined by

$$\begin{aligned} &C_f \mathrm{Re}_x^{1/2} = (2\zeta)^{-\frac{1}{2}} F''(0,\zeta) \text{ and} \\ &Nu\,\mathrm{Re}_x^{-1/2} = -(2\zeta)^{-\frac{1}{2}} G'(0,\zeta) \end{aligned} \tag{19-20}$$

Numerical values of $F''(0,\tau)$ and $G'(0,\tau)$ from the solutions of the governing equations (13)-(15) we obtain the values of the skin friction coefficient and the local Nusselt number from the relations (19-20) as mentioned above.

III. Analytical and Numerical Analysis

Now we employed three numerical methods; namely (i) 3-term Series solutions for small time, (ii) Asymptotic solutions for large time and (iii) Implicit *finite difference method* (FDM) together with the Keller-box scheme for all time, which are described below.

Analytical and Series solutions for small time τ<<1

At the present situation we must represent equations (16) and (17) in a form which is much more convenient for analysis at small times. The transforming equations become

$$\begin{aligned} &F''' + \eta F'' + 2m\tau FF'' + 2m\tau\left[1 - F'^2\right] \\ &= 2(m-1)\tau^2\left[F'\frac{\partial F'}{\partial \tau} - F''\frac{\partial F}{\partial \tau}\right] + 2\tau\left[\frac{\partial F'}{\partial \tau} - \lambda G\right] \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{1}{\Pr} G'' + \eta G' + 2m\tau FG' - 2(2m-1)\tau GF' \\ &= 2(m-1)\tau^2\left[F'\frac{\partial G}{\partial \tau} - G'\frac{\partial F}{\partial \tau}\right] + 2\tau\frac{\partial G}{\partial \tau} \end{aligned} \tag{22}$$

with boundary conditions

$$\begin{aligned} &F(0,\tau) = F'(0,\tau) = 0, \quad G(0,\tau) = 1, \\ &F'(\infty,\tau) = 1, \quad G(\infty,\tau) = 0 \end{aligned} \tag{23}$$

At small values of τ (<<1), it can easily verify that the solutions of equations (21) and (22) have the following forms:

$$F(\eta,\tau) = F_0(\eta) + \tau F_1(\eta) + \tau^2 F_2(\eta) + \cdots \tag{24}$$

$$G(\eta,\tau) = G_0(\eta) + \tau G_1(\eta) + \tau^2 G_2(\eta) + \cdots \tag{25}$$

Substituting the above two expressions into equations (21) and (22) and picking up the terms up to the order O(τ^2), we get the following three sets of equations with boundary conditions:

$$F_0''' + \eta F_0'' = 0 \tag{26}$$

$$\frac{1}{\Pr} G_0'' + \eta G_0' = 0 \tag{27}$$

$$\begin{aligned} &F_0(0) = F_0'(0) = 0, \quad G_0(0) = 1, \\ &F_0'(\infty) = 1, \quad G_0(\infty) = 0 \end{aligned} \tag{28}$$

$$F_1''' + \eta F_1'' - 2F_1' = -2m(1 - F_0'^2 + F_0 F_0'') - 2\lambda G_0 \tag{29}$$

$$\frac{1}{\Pr} G_1'' + \eta G_1' - 2G_1 = -2m F_0 G_0' + 2(2m-1)G_0 F_0' \tag{30}$$

$$\begin{aligned} &F_1(0) = F_1'(0) = 0, \quad G_1(0) = 0, \\ &F_1'(\infty) = 0, \quad G_1(\infty) = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} &F_2''' + \eta F_2'' - 4F_2' = 2\left[(1-2m)F_1 F_0'' \right. \\ &\left. + (3m-1)F_1' F_0' - m F_0 F_1'' - \lambda G_1\right] \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{1}{\Pr} G_2'' + \eta G_2' - 4G_2 = 2\left[(1-2m)F_1 G_0' \right. \\ &\left. + (3m-2)G_1 F_0' - m G_1' F_0 + (2m-1)G_0 F_1'\right] \end{aligned} \tag{33}$$

$$\begin{aligned} &F_2(0) = F_2'(0) = 0, \quad G_2(0) = 0, \\ &F_2'(\infty) = 0, \quad G_2(\infty) = 0 \end{aligned} \tag{34}$$

Solving the equations (26)-(31) analytically and the equations (32)-(34) numerically we get the skin friction coefficient and the local Nussult number as the form:

$$C_f \,\mathrm{Re}_x^{1/2} = (2\tau)^{-1/2}[F_0''(0) + \tau F_1''(0) + \tau^2 F_2''(0)] \quad (35)$$

$$Nu\mathrm{Re}_x^{-1/2} = (2\tau)^{-1/2}[G_0'(0) + \tau G_1'(0) + \tau^2 G_2'(0)] \quad (36)$$

After some simplification when η=0, the small time solutions for the skin friction coefficient $C_f \,\mathrm{Re}_x^{1/2}$ and the local Nusselt number $Nu\,\mathrm{Re}_x^{-1/2}$ can be written as follows:

$$C_f \,\mathrm{Re}_x^{1/2} = (2\tau)^{-1/2}[\sqrt{2/\pi} + \tau(1/\sqrt{2\pi})(3m - 4\lambda(1-\sqrt{pr})) + \tau^2 F_2''(0) + \cdots] \quad (37)$$

$$Nu\mathrm{Re}_x^{-1/2} = (2\tau)^{-1/2}[-\sqrt{2pr/\pi} - \tau(m/\sqrt{2\pi})(6\sqrt{pr} + pr) + \tau^2 G_2'(0) + \cdots] \quad (38)$$

Asymptotic solution for large time $\tau \to \infty$

Here the method of asymptotic expansion is used to solve the governing equations. When the transport of energy becomes steady as $\tau \to \infty$, then the governing equations (16)-(17) and respective boundary condition becomes,

$$F''' + \tau^{-2}\,\eta F'' + [m + 1 + (m-1)\tau^{-1}]FF'' + 2m\left[1 - F'^2\right] = 2(m-1)\tau\left[F'\frac{\partial F'}{\partial \tau} - F''\frac{\partial F}{\partial \tau}\right] + 2\left[\frac{\partial F'}{\partial \tau} - \lambda G\right] \quad (39)$$

$$\frac{1}{\mathrm{Pr}}G'' + \tau^{-2}\,\eta G' + [m+1+(m-1)\tau^{-1}]FG' - 2(2m-1)GF' = 2(m-1)\tau\left[F'\frac{\partial G}{\partial \tau} - G'\frac{\partial F}{\partial \tau}\right] + 2\frac{\partial G}{\partial \tau} \quad (40)$$

$$F(0,\tau) = F'(0,\tau) = 0, \quad G(0,\tau) = 1, \quad F'(\infty,\tau) = 1, \quad G(\infty,\tau) = 0 \quad (41)$$

For large time τ we can consider,

$$F(\eta,\tau) = F_0(\eta) + \tau^{-1}F_1(\eta) + \tau^{-2}F_2(\eta) + \cdots \quad (42)$$

$$G(\eta,\tau) = G_0(\eta) + \tau^{-1}G_1(\eta) + \tau^{-2}G_2(\eta) + \cdots \quad (43)$$

Substituting the above two expressions into equations (39)-(40) and picking up the terms up to the order O(τ^{-2}), we get three similar set of equations as described above in analytical and series solution with respective boundary conditions.

Numerically solving that equations gives the skin friction coefficient and the local Nusselt number as:

$$C_f \,\mathrm{Re}_x^{1/2} = (2)^{-1/2}[F_0''(0) + \tau^{-1} F_1''(0) + \tau^{-2} F_2''(0)] \quad (44)$$

$$Nu\mathrm{Re}_x^{-1/2} = (2)^{-1/2}[G_0'(0) + \tau^{-1} G_1'(0) + \tau^{-2} G_2'(0)] \quad (45)$$

Implicit finite difference method (FDM) for all time τ

We integrate the locally non-similar partial differential equations (16)-(17) by using an well established implicit finite difference method. To begin with, the partial differential equations (16)-(17) are first converted into a system of first order equations in y. Denoting the mesh points in the (x, y) plane by x_i and y_j, where i=1, 2, 3, …, M and j=1, 2, 3, …, N, central difference approximations are made such that the equations involving x explicitly are centered at ($x_{i-1/2}$, $y_{j-1/2}$) and the remainder at (x_i, $y_{j-1/2}$), where $y_{j-1/2}=(y_j+y_{j-1})/2$, etc. This results in a set of nonlinear difference equations for the unknowns at x_i in terms of their values at x_{i-1}. These equations are then linearised by the Newton's quasilinearization technique and are solved using a block-tridiagonal algorithm. Now to initiate the process at x=0 and use the Keller-box method to solve the governing ordinary differential equations. For any given value of x, the iterative procedure is stopped when the difference in computing the velocity and the temperature in the next iteration is less than 10^{-5}, i.e. when $\left|\delta f^{\,i}\right| \le 10^{-5}$, where the superscript denotes the number iterations. Throughout the computations a non-uniform grid has been used by setting $y_j = \sinh(j-1)/a$ with j=1, 2, …, 301 and a=125.

IV. Results and Discussion

In this article we present the results of calculations of the skin friction and rate of heat transfer in terms of Nusselt number for the problem of laminar mixed convection two-dimensional boundary-layer flow past a symmetric semi-infinite wedge with variable temperature. Implicit finite difference method being employed in finding the solutions of the equations (16)-(17) that govern the flow subject to the boundary conditions (18).

In absence of the effect of mixed convection parameter i.e when *λ=0,* these equations were solved by Harris et al. [13]. **Table 1** contains the present solutions for comparisons of the Skin-friction C_f and the Nusselt number Nu for small time that obtained by Harris et al. [13].

Table. 1. Numerical values of C_f and Nu for different values of small time τ for Pr = 1.0and m = 0.2

Harris et al. [13]			Present result		
τ	C_f	Nu	τ	C_f	Nu
0.01	5.65797	5.64360	0.01000	5.66886	5.64525
0.1	1.83491	1.78946	0.10017	1.83693	1.76140
0.2	1.33334	1.26902	0.20134	1.33189	1.22357
0.4	0.99341	0.90234	0.39996	0.99548	0.83937
1.0	0.72370	0.57925	0.99806	0.72665	0.46370

Table. 2. Numerical values of the skin-friction coefficient and the local Nusselt number obtain by different methods for different values of τ while Pr = 0.7, λ=0.25 and m = 0.0 (for flat plate), 0.5 (wedge angle 120^0).

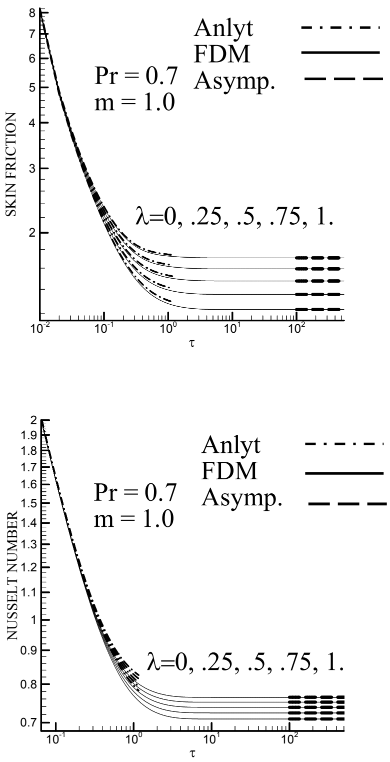
Pr = 0.7 λ=0.25	$C_f \mathrm{Re}_x^{1/2}$				$Nu \mathrm{Re}_x^{-1/2}$			
τ	m=0.0		m=0.5		m=0.0		m=0.5	
	FDM	AN & AS	FDM	AN & AS	FDM	AN & AS	FDM	AN & AS
0.10017	1.80291	1.78610an	1.95493	1.91241an	1.50910	1.49470an	1.52753	1.50784an
0.30452	1.00060	1.02438an	1.25047	1.24207an	0.83869	0.85725an	0.86929	0.88069an
0.52110	0.75900	0.78309an	1.07166	1.06433an	0.63750	0.65533an	0.67731	0.68673an
0.75858	0.63075	0.64903an	0.99176	0.98370an	0.53110	0.54314an	0.57953	0.58202an
1.02652	0.54809	0.55794an	0.94903	0.94113an	0.46277	0.46691an	0.51980	0.51343an
100.16	0.32187	0.33122as	0.89909	0.89948as	0.28303	0.29212as	0.41433	0.41696as
201.71	0.32570	0.33163as	0.89940	0.89959as	0.28668	0.29248as	0.41525	0.41717as
300.92	0.32731	0.33177as	0.89950	0.89964as	0.28820	0.29260as	0.41555	0.41724as
448.92	0.32855	0.33186as	0.89957	0.89966as	0.28938	0.29268as	0.41576	0.41728as
548.31	0.32906	0.33189as	0.89959	0.89967as	0.28985	0.29271as	0.41584	0.41730as

In Table 2 'AN' means analytical solution for small time and 'AS' means asymptotic solution for large time.

The numerical results of $C_f \mathrm{Re}_x^{1/2}$ and $Nu \mathrm{Re}_x^{-1/2}$, for Prandtl number Pr = 0.7, λ = 0.25 and for small and large time τ with different values of m, have been compared with the values of same physical quantities obtained by other methods are shown in the above **Table 2**.

The variations of skin-friction coefficient and local Nusselt number for different values of mixed convection parameter λ = 0.0, 0.25, 0.5, 0.75, 1.0 obtained by the aforementioned methods are drawn in **Fig. 2** for constant values of Pr = 0.7 (which represents air at 20^0C and exponent m = 1.0 (stagnation point). In **Fig. 3** skin-friction coefficient and local Nusselt number are plotted over a large Prandtl number Pr=50.0 with exponent m = 0.5 (wedge angle 120^0) and different mixed convection parameter as mentioned above. Considering all factors, we conclude from figures 2 to 3 that the agreement is as good as one may hope for the cases due to analytical solutions, the asymptotic solutions and the finite difference solutions.

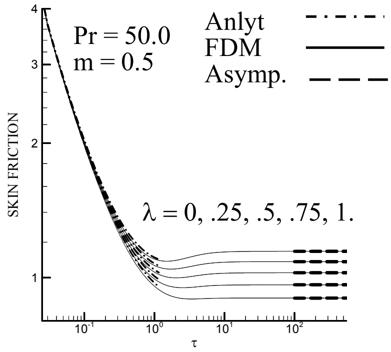
Effect of mixed convection parameter λ on Falkner-Skan boundary-layer flow along the wedge has been studied theoretically and the following conclusions may be drawn from the present investigation. The skin-friction coefficient $C_f \mathrm{Re}_x^{1/2}$ and local Nusselt number $Nu \mathrm{Re}_x^{-1/2}$ both decreases for small time increases and for the values of large time they become steady. It reveals from figures 2 and 3 that increases in the value of the mixed convection parameter leads to the decrease of the skin-friction coefficient and Nussult number.



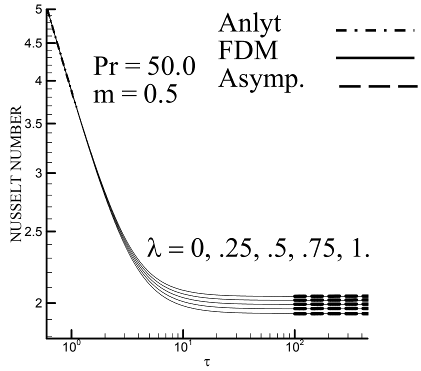
(a)

(b)

Fig. 2. (a) Skin friction coefficient and (b) local Nusselt number, for small and large time for λ = 0.0, 0.25, 0.5, 0.75, 1.0 at Pr = 0.7 and m = 1.0



(a)



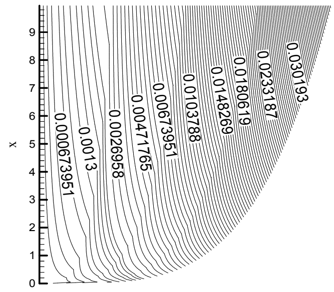
(b)

Fig. 3. (a) Skin friction coefficient and (b) local Nusselt number, for small and large time for λ = 0.0, 0.25, 0.5, 0.75, 1.0 at Pr = 50.0 and m = 0.5

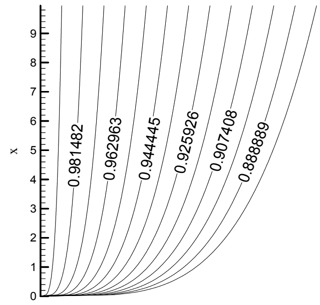
Now we see the effects of pertinent parameters controlling the present problem on the flow pattern and the temperature distribution in terms of the streamlines and isotherms. Following relations are considered to measure the values of the streamlines and isotherms at different times in x-y plane:

$$\psi = x^{\frac{m+1}{2}}(2\zeta)^{\frac{1}{2}}F(\eta,\zeta) \text{ and } \theta = x^{2m-1}G(\eta,\zeta)\text{, where } \eta = x^{\frac{m-1}{2}}(2\zeta)^{-\frac{1}{2}}y \tag{46}$$

We develop the figures for the fixed values of Pr = 0.7, m = 0.5 and λ = 0.5 and observed the effect of time τ. The physical implications of different profiles for streamlines and isotherms are clearly illustrated in **Fig. 4** and **Fig. 5** for the values of τ which are equal to 5 and 0.2 respectively. For τ = 0.2 the momentum boundary-layer becomes thinner and we have a weaker flow; on the other hand, for τ = 5 the momentum boundary-layer becomes higher and we have a stronger flow at the down stream region. For different τ isotherm shows significant changes. The fluid temperature is highest near the lower boundary and decreasing to the upper boundary and this is the region of lowest viscosity.

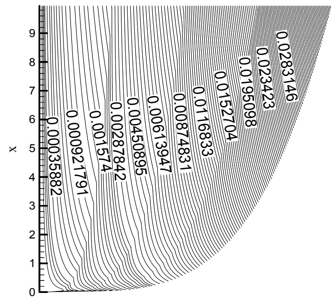


(a)

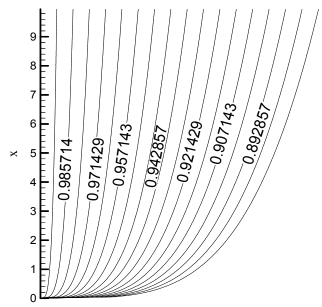


(b)

Fig. 4. (a) Streamlines and (b) Isotherms for τ =0.2 at $\text{Pr} = 0.7, m = 0.5, \lambda = 0.5$



(a)



(b)

Fig. 5. (a) Streamlines and (b) Isotherms for τ =5 at $\text{Pr} = 0.7, m = 0.5, \lambda = 0.5$

V. Conclusion

The analysis carried out here is concerned with two-dimensional mixed convection thermal boundary-layer flow which is produced by the sudden increase of the surface temperature as the motion is started. The initial development of these boundary-layers has satisfactorily been represented by an analytical solution for small time and the asymptotic solution for large time approaches steady state. From the present investigation, following conclusions may be drawn:

1. The values of skin-friction and Nusselt number positively decrease due to the increase of small time for different mixed convection parameter λ and Prandtl number Pr with exponent m. And when the time becomes large the values of skin-friction and Nusselt number become steady.
2. The streamlines and isotherm lines are densely situated at the down stream region. The momentum boundary-layer grows thinner to thicker when τ is increasing. The thermal boundary-layer is thicker near the down stream region.

The Falkner-skan one-parameter family of solutions of the boundary-layer equations has proved to be very useful in the interpretation of fluid flows at large Raynolds numbers. It is hoped that the present solution method can also be applied to other wedge and experimental data will be available in near future to verify the results of the present study.

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