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Numerical Approximations of Blasius Boundary Layer Equation

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Abstract

The Blasius equation is a well-known third-order nonlinear ordinary differential equation, which arises in certain boundary layer problems in the fluid dynamics. In the paper the solutions of the Blasius equation $2f''' + ff'' = 0$, with boundary conditions $f(0) = f'(0) = 0, f'(\infty) = 1$ are investigated by three numerical methods. The observed phenomena in numerical solutions of previous published work are theoretically analyzed.

Keywords: Fluid dynamics; Blasius equation; Boundary layers; Flat plate; Numerical solution.

f dimensionless stream function
u velocity in the x-direction
v velocity in the y-direction
x coordinates measuring distance along the x-axis
y coordinates measuring distance normal to the x-axis
U free stream velocity

Greek Symbol

ν kinematic viscosity
ψ stream function
η similarity variable
δ boundary layer thickness

I. Introduction

Fluid dynamics is a science which deals with the behavior of fluids when subjected to a system of forces. The concept of boundary layer was introduced by a German scientist Ludwing Parndtl [1904], provides a link between the ideal fluid and the real fluid. The fluid layer which has its velocity affected by boundary shear is called the boundary layer. Boundary layer is the region in which viscous forces and inertial forces are of comparable magnitude. Outside the boundary layer, velocity gradient is small and so viscous forces can be ignored completely. In the study of Prandtl boundary layer problems relevant to the motion of an incompressible viscous fluid, solutions of self-similar form naturally give rise to such equations as the Blasius equation. It describes the steady two-dimensional boundary layer that forms on a semi-infinite flat plate which is held parallel to a constant unidirectional flow. This equation is one of the basic equations of fluid dynamic and has been the focus of many studies.

The Blasius differential equation arises in the theory of fluid boundary layer, and in general, must be solved numerically is shown later. Concerning the Blasius equation, many researchers have been attempted and much progress has been made to solve this equation. We have been inspired by some recent work. H. Blasius [1] was the first to show that this problem provided a special solution to the Prandtl boundary layer equations. In fact, the Blasius equation is a particular case of that of Falkner-Skan [2]

$$f''' + \frac{m+1}{2} ff'' + m(1 - f'^2) = 0 \tag{1}$$

describing the same phenomena, when the velocity at infinity has some dependence on m. Abdul-Majid Wazwaz [3] use the variational iteration method for a reliable treatment of two forms of the third order nonlinear equation. The study represents the series solution without restrictions on the nonlinearity behavior. Parlange et al.[4] construct a decomposition technique to obtain a solution as a converging infinite series. Also recent work of Abbasbandy [5] to come up with a modified decomposition technique to solve the Blasius equation.

Most of the authors solve Blasius equation for particular method or extended the respective method of the others. Parlange et al. [4] has solved this nonlinear equation analytically to obtain an integral condition by variational methods such that the physically relevant features can be calculated accurately even with an approx-imate solution. The principle of this method is that they write an integral condition directly, i.e. loosen the link with a variational principle. This method was also used with considerable success to extend the validity of prescribed diffusion approximation to the Smoluehowski equation [6]. In the parallel time Singh and Battacharya [7] Gyarmati's variational principle was applied to deduce the skin friction for the laminar boundary layer over a flat plate.

The aim of this paper is to apply the some numerical techniques to solve the blasius equation which describes the velocity profile in the boundary layer when fluid flows along a flat plate. This is a classical problem in viscous boundary layer theory and we shall apply the present methods to the same fundamental problem and be able to compare the accuracy of the other approaches.

II. Mathematical Formulation

Consider the flow of a viscous incompressible fluid past a semi-infinite flat plate which is placed in the direction of a uniform flow *U*. Let the leading edge of the flat plate be at the origin, *x* axis be the direction of the uniform stream and *y* axis be the normal to the plate. The velocity of the potential flow is constant and in this case the pressure gradient is zero. Hence the boundary layer equations for steady flow are

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \tag{2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

with boundary conditions

$$u=v=0 \; at \; y=0,\, x>0 \tag{4}$$

$$u=U \; as \; y\to\infty, \forall x$$

To get the equation in convenient form for integration, we now define the following one parameter group of transformation for the dependent and the independent variables:

$$\eta = y\sqrt{\frac{U}{\nu x}} \text{ and } \psi(x,y)=\sqrt{\nu x U}\, f(\eta) \tag{5}$$

In the above relations, η is the similarity variable and ψ is the stream function that is defined by

$$(u,v)=\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$$

which satisfy the relation (2) automatically.

Using the above transformation equations (3) reduce to the following form

$$2f'''+ff''=0 \tag{6}$$

The appropriate boundary conditions to be satisfied by (6) is

$$f(0)=f'(0)=0, f'(\infty)=1 \tag{7}$$

Here prime denotes the differentiation with respect to η. The physical model and the flow configuration are shown in the following figure:

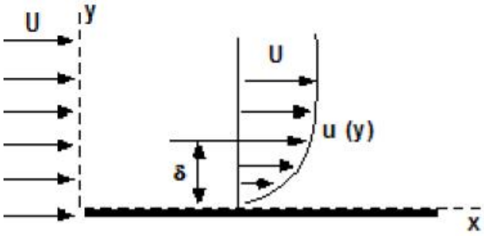

Fig. 1. Physical model and coordinate system

III. Solution Methodology

In the present analysis solution of the Blasius equation, (6), are obtained numerically using three numerical methods, namely, the Runge-Kutta method, Nonlinear Shoting method and Nachtsheim-Swigert method. A detail of these methods has been discussed below.

Runge-Kutta Method

Runge-Kutta methods have the high-order local truncation error of the Taylor methods while eliminating the need to compute and evaluate the derivatives of *f.* The most appropriate four-parameter form for approximating

$$T^{(3)}f(t,y)=f(t,y)+\frac{h}{2}f'(t,y)+\frac{h^2}{2}f''(t,y)$$

$$\alpha_1 f(t,y)+\alpha_2 f(t+\alpha_2, y+\delta_2 f(t,y)) \tag{8}$$

and even with this, there is insufficient flexibility to match the term

$$\frac{h^2}{6}\left[\frac{\partial f}{\partial y}(t,y)\right]^2 f(t,y)$$

Resulting from the expansion of $(O(h^2))f''(t,y)$. Consequently, the best that can be obtained from using (8) are methods with $O(h^2)$ local truncation error. The fact that (8) has four parameters, however, gives a flexibility in their choice, so a number of $O(h^2)$ methods can be derived. This method has local truncation error $O(h^4)$, provided the solution *y*(t) has five continuous derivatives which is essential to eliminate for successive nesting in the second variable of $f(t,\, y)$

INPUT

ENTER THE LOWER BOUNDARY = 0

ENTER THE UPPER BOUNDARY XLIM= 6

ENTER THE STEP SIZE H =0.02

ENTER THE ORDER OF THE DIFFEREANTIAL EQUATION =3

ENTER THE INITIAL VALUE OF Y<1> = 0.4696

ENTER THE INITIAL VALUE OF Y<2> = 0

ENTER THE INITIAL VALUE OF Y<3> = 0

And the output result are shown in the comparison table later.

Non-Linear Shooting Method

The Shooting technique for the nonlinear boundary-value problem is similar to the linear technique, except that the solution to a nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems. Instead, we approximate the solution to the boundary-value problem by using the solutions to a sequence of initial-value problems involving a parameter *t*. By analogy to the procedure of firing objects at a stationary target. we start with a parameter t_0 that determines the initial elevation at which the object is fired at a point and along the curve described by the solution to the initial-value problem. To use the secant method although many other methods are available to solve a algebraic problem, we need to choose initial approximations t_0 and t_1. Then generate the remaining terms of the sequence by

$$t_k = t_{k-1} - \frac{\left(y(b,t_{k-1}) - \beta\right)\left(t_{k-1} - t_{k-2}\right)}{y(b,t_{k-1}) - y(b,t_{k-2})}, k = 2,3,........$$

Nachtsheim-Swigert method

The transformed governing equation (6) is solved using Nachtsheim-Swigert [9] shooting iteration technique together with sixth order Runge-Kutta-Butcher initial value solver. Throughout the whole calculation we have fixed the value of step size $\Delta\eta = 0.001$ for sufficient accuracy of the solutions as the solutions are independent of step size. For a convergence of the solution, error 10^{-6} was considered in all cases.

The value of η_∞ has been obtained by $\eta_\infty = \eta_\infty + \Delta\eta$ to each iteration loop. The maximum value of η_∞ to each group of the prescribed parameters is determined, when the value of unknown boundary conditions at $\eta = 0$ not change to successful loop with error less than 10^{-6}.

IV. Result and Discussion

Here we have discussed three methods to solve Blasius boundary layer equation along a thin flat plate. These are Runge-Kutta method for solving boundary value problem, Non-linear Shooting method and Nachtsheim-Swigert method. We have presented the velocity profile for the steady state flow. The numerical values of the velocity profile, f', is obtained which is presented in tabular form in Table-1 against parameter η in the interval [0, 6]. In this table we have also compared the results with that obtained by different methods. The results are also shown graphically in figure-2. From figure it has been shown that velocity increases with the increasing value of η and near η=6, all the figures are coinciding. Comparing three methods, Nonlinear-Shooting method and Nachtsheim-Swigert method gives better accuracy. We have also compared the result with Parlange et. al [4] and the result gives an excellent agreement.

Published solution[4] indicate that the infinity boundary condition is easily met at η = 6. The analytical solution was obtained by Blasius by a series expansion at η = 0 and asymptotic expansion for large η. In our present investigation represent that at η = 6 infinite boundary conditions are satisfied because the velocity profiles are parallel from this point upto infinity.

Table. 1. Comparison of the numerical values found by different methods:

η	RUNGE-KUTTA METHOD	NONLINEAR-SHOOTING METHOD	PARLANGE et. al [4]	NACHTSHEIM-SWIGERT METHOD
0.00	0.00000	0.00000	0.00000	0.00000
0.20	0.09390	0.06645	0.06640	0.06648
0.60	0.28057	0.19906	0.19893	0.19912
0.80	0.37195	0.36488	0.264709	0.36494
1.00	0.46062	0.32994	0.32978	0.32999
1.40	0.62437	0.45654	0.45626	0.45663
1.80	0.76104	0.57511	0.574758	0.57517
2.00	0.81668	0.63014	0.62976	0.63021
2.40	0.90105	0.72940	0.72898	0.72951
2.80	0.95287	0.81196	0.81150	0.81199
3.00	0.96905	0.84650	0.84604	0.84660
3.40	0.98796	0.90222	0.90176	0.90231
3.80	0.99594	0.94157	0.94111	0.94164
4.00	0.99777	0.95585	0.95551	0.95597
4.40	0.99939	0.97631	0.97587	0.976642
4.80	0.99986	0.98822	0.98778	0.98840
5.00	0.99993	0.99190	0.99154	0.99193
5.40	0.99998	0.99658	0.99615	0.99661
5.80	0.99999	0.99879	0.99837	0.99880
6.00	0.99999	0.99939	0.99897	0.99939

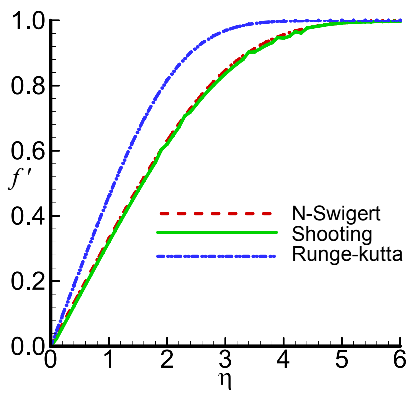

Fig. 2. Comparison of the velocity profile obtained in Runge-Kutta, Nonlinear Shooting and Nachtsheim-Swigert methods

V. Conclusion

The blasius equation is one of the basic equations of fluid dynamics and has been the focus of many studies. This nonlinear equation describes the velocity profile of the fluid in the boundary layer theory. The investigation carried out here is concerned with two dimensional laminar boundary layer flow of a viscous incompressible fluid past along a thin flat plate. Dimensional analysis and simple phenomenological models are also used. Using the appropriate transformation, the boundary layer momentum equation is reduced to local non-similarity equation, which is then integrated numerically, employing three numerical methods as mentioned before. Among these methods we observe that Nachtsheim-Swigert and Nonlinear shooting methods approximately close to each other but Runge-Kutta method gives nearly close to those methods. Velocity profile increases initially with the increase of the parameter η for all methods. When $\eta \to 6$ or more than this, all the methods gives the result approximately 0.99999.

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