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Pocklington Equation and Method of Moments with Non-Uniform Sampling

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Abstract

We present a non-equidistant sampling for the Method of Moments in the solution of Pocklington equation operator, to prove the reduction of segments junctions and conductor far end discontinuities. Comparison of equidistant and non-equidistant sampling is presented, obtaining E field over the surface of a λ/2 dipole.

Keywords: Method of Moments, Non-equidistant sampling, Pocklington Equation.

Introduction

For the solution of integral and differential equations using the Method of Moment (MM), the operator is divided in N segments, usually of the same size, each one besides the next, keeping the lineal independence of the resulting matrix equation; however, it is known that when the segments are of the same size sometimes the interpolation produces the Runge-Borel effect, incrementing the residual error in both the conductor surface and mainly in the open ends of the conductor, the problem is more acute when is applied to an integral and differential equation as the Pocklington equation, as is analyzed by literature [1-3]. In general, the error is not due to interpolation method itself but the way to divide the operator using segments of the same size. We propose the division of operator using non-equidistant segments, whose size and position is defined by the Legendre polynomials, the results show the reduction of interpolation error when is applied to the Pocklington equation solution, using the Method of Moments of a λ/2 dipole. For facility and as matter of comparison, in this paper is used piecewise sinusoidal, triangular and pulse as basis functions and pulse and Dirac's delta as weight functions, applying them for equidistant and non-equidistant segments, testing the electric field over the conductor surface, which theoretically should be zero, except in the source gap.

Pocklington Equation

In 1897 Pocklington deduced his equation for straight structures [4], and in 1965 Mei [5], used a heuristical procedure to define it for bent wires; for an arbitrary shaped wire as the one shown in Fig.1, it is possible to deduce the equation using a formal way, starting from Maxwell equations [6] getting:

$$E_S^I = -\frac{j}{\varpi\varepsilon}\int_{S'} I_S(s')\left[k^2 s \bullet s' + \frac{\partial^2}{\partial s \partial s'}\right]\frac{e^{-jk|r-r'|}}{4\pi|r-r'|}ds' \quad (1)$$

where E_S^I is the tangential incident electric field and *s'* & r' are the arc length and position vector over the wire surface, respectively. Considering the thin-wire approximation and skin effect, is possible to express the electric field as a linear integration over *s'*. The general Pocklington equation (1) can be used for any possible thin wire geometry. The wire's geometry is expressed by the dot product **s•s'**, where **s**(*s*) is the unit tangential vector for the wire's axis and **s'**(*s'*) the same for the parallel curve representing the current filament, shown by Fig.1:

$$s(s) = \frac{dx(s)}{ds}i + \frac{dy(s)}{ds}j + \frac{dz(s)}{ds}k \quad (2)$$

$$s'(s') = \frac{dx'(s')}{ds'}i + \frac{dy'(s')}{ds'}j + \frac{dz'(s')}{ds'}k$$

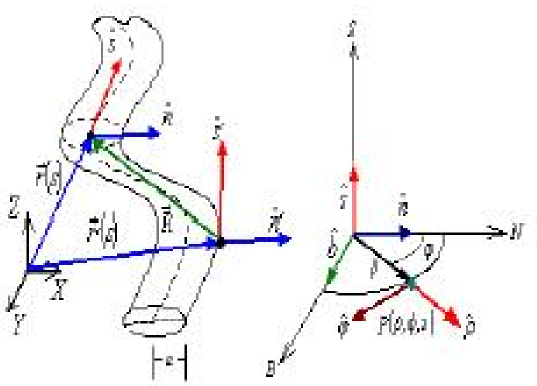

Fig. 1. Arbitrary shaped wire.

The geometry is also expressed by the difference between the vectors |r - r'| as:

$$R = |R| = |r - r'| = \sqrt{[x(s)-x'(s')]^2 + [y(s)-y'(s')]^2 + [z(s)-z'(s')]^2} \quad (3)$$

Defined all former equations, the work is reduced to find the vectors representing the parallel and axis curves for the considered wire and solve it by Method of Moments.

The MM Solution to Pocklington's Equation.

The objective of the MM applied to Pocklington's equation is to get the current distribution $I_s(s')$ in the wire, considering that unknown function is part of the integral operator. The MM formulation is used to get a numerical solution of (1) establishing that the unknown function must be expressed in terms of a linear combination of linearly independent functions $i_n(s')$ called basis functions:

$$I_s(s') = \sum_{n=1}^{N} c_n i_n(s') \quad (4)$$

c_n are the unknown coefficients to be determined and N the number of basis function. Substituting (4) into (1) results an equation with N unknowns:

$$E_S^I = -\frac{j}{\omega\varepsilon}\sum_{n=1}^{N} c_n \int_{s'} i_n(s')\left[k^2 s \bullet s' + \frac{\partial^2}{\partial s \partial s'}\right]\frac{e^{-jk|r-r'|}}{4\pi|r-r'|} ds' \quad (5)$$

for a consistent equation system, is necessary to find N linearly independent equations, obtained by taking the inner product of (5) with other set of N chosen linearly independent functions $w_m(s)$ named weighting function:

$$\left\langle w_m, E_s^I \right\rangle = -\frac{j}{\varpi\varepsilon}\sum_{n=1}^{N} c_n \left\langle W_m, \int_{s'} i_n(s')\left[k^2 s \bullet s' + \frac{\partial^2}{\partial s \partial s'}\right]\frac{e^{-jk|r-r'|}}{4\pi\,|\,r-r'\,|} ds' \right\rangle$$

$m=1,2,\ldots,N$ (6)

using inner product definition, (6) can be written as:

$$\int_S w_m E_S^I ds = -\frac{j}{\varpi\varepsilon}\sum_{n=1}^{N} c_n \int_S w_m \int_{S'} i_n(S')\left[k^2 s \bullet s' + \frac{\partial^2}{\partial s \partial s'}\right]\frac{e^{-jk|r-r'|}}{4\pi\,|\,r-r'\,|} ds' ds$$

$m=1,2,\ldots,N$ (7)

then the system in matricial form is:

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \quad (8)$$

$[Z_{mn}](c_n)=v_m)$

where the elements Zmn are obtained from:

$$Z_{mn} = -\frac{j}{\omega\varepsilon}\int_s w_m \int_{s'} i_n(s')\left[k^2 s \bullet s' + \frac{\partial^2}{\partial s \partial s'}\right]\frac{e^{-jk|r-r'|}}{4\pi|r-r'|} ds' ds \quad (9)$$

and the elements v_m are:

$$V_m = \int_S w_m E_S^I ds \quad (10)$$

The c_n are the system's unknowns. The matrices of (8) are known as impedance matrix $[Z_{mn}]$, current matrix (c_n), and voltage matrix (V_m). The solution for (8) is:

$$(c_n) = [Z_{mn}]^{-1}(v_m) \quad (11)$$

where the inverse matrix $[Z_{mn}]^{-1}$ is obtained by a numerical technique [6].

To solve (11) is necessary to define base and weight functions; as it is known both functions are selected arbitrarily. The most widely used subdomain functions have been a subject of research, some discussion of this may be found in [1,7], but the more often used are Dirac´s delta, pulse, triangle, piecewise sinusoidal and trigonometric functions. For facility and as matter of comparison, in this paper is used piecewise sinusoidal, triangular and pulse as basis functions and pulse and Dirac's delta as weight functions, applying them for equidistant and non-equidistant segments, testing the electric field over the conductor surface, which theoretically should be zero, except in the source gap.

Legendre Polynomials for non-equidistant Segments

To obtain equidistant segments implies only to divide by N times the length of the conductor, for the non-equidistant division we use the roots of the Legendre Polynomials to define position and length of segments.

The differential equation of Legendre is given by:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (12)$$

To solve (12) is used the classical method of series, getting two solutions, both linearly independent:

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x4 - \ldots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots \quad (13)$$

convergence of both series is in the range |x| <1. When n is an even positive number $y_1(x)$ is reduced to a polynomial of degree n ; if n is an odd positive number results the same for $y_2(x)$. The Legendre polynomials of degree n are:

$$P_n(x) = \sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} \quad (14)$$

where $M= n/2$ or $M= (n-1)/2$, any one of both to be an integer; the n real roots are in the convergence range of |x| < 1. The roots of equation (13) can be obtained by standard methods or using a computational algorithm as the one in [8].

Operator of equation (8) can be divided in N segments defined by the roots ξ_j of the Legendre polynomials P_n of (13), choosing:

$$s'_j = \frac{b+a}{2} + \frac{b-a}{2}\xi_j \qquad j = 1,2,\ldots,N \quad (15).$$

Numerical Results

To compare both techniques, we solve Pocklington equation for a λ/2 dipole, using pulse and Delta as weighting functions, comparing solution for pulse, triangular and piecewise sinusoidal as base functions, conductor is divided in N, equal equidistant and unequal non-equidistant segments for each one. The electric field over the conductor surface is obtained getting current from (11) and recalculating from (6). As is known the interface between two segments and open ends, are points of discontinuity, due interrupted integration, producing peak errors and reducing solution accuracy.

The testing object is a $\lambda/2$ dipole divided in 58 segments with 116 data points (end and middle points in each segment) with radius a=L/100 and a gap width of 0.05L, where L is the dipole length and a gap voltage of 1V. Figs. 2 and 3 show results for equidistant segments and Figs. 4 and 5 for non-equidistant segments, with delta and pulse weight functions, espectively, for the whole dipole:

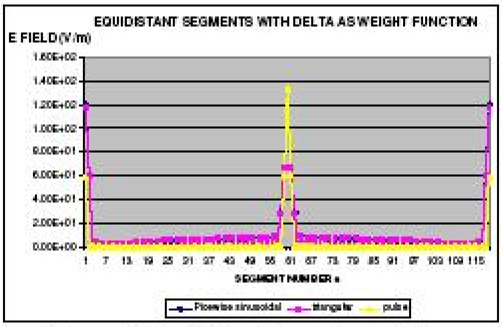

Figure 2. Equidistant Segments for Delta

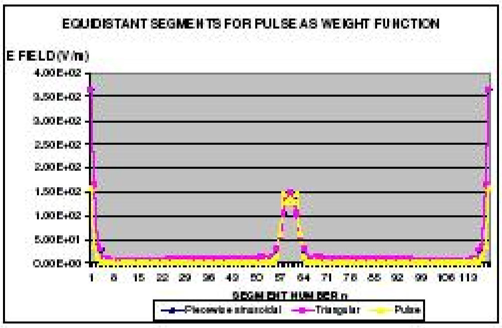

Figure 3. Equidistant Segments for pulse

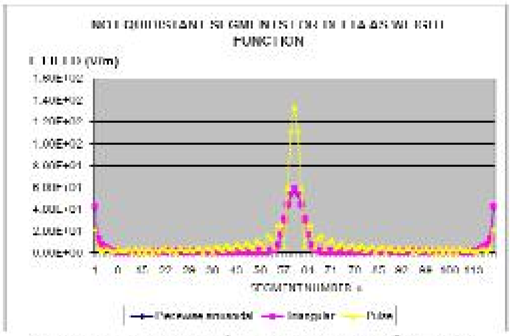

Figure 4. Non-equidistant Segments for Delta

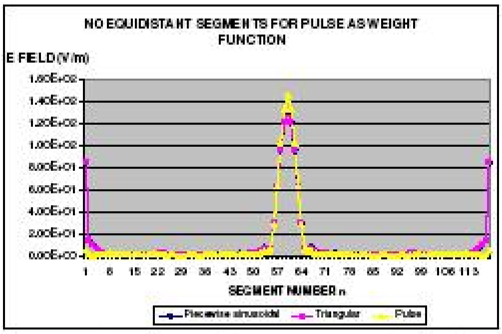

Figure 5. Non-equidistant Segments for Pulse

We note in all four figures, errors in the segment junctions (more clearly in delta functions), but mainly the great error at the end of structure, reduced notably with non-equidistant segmentation. For a more detailed view, we present graphics near the source gap and dipole end in following figures:

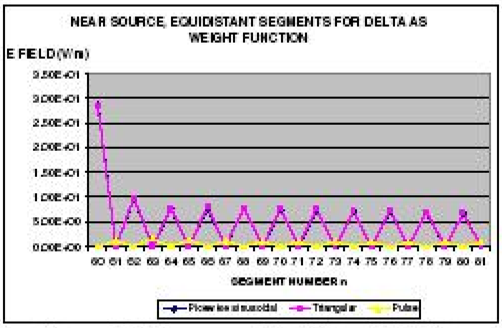

Figure 6. Near source Equidistant for Delta

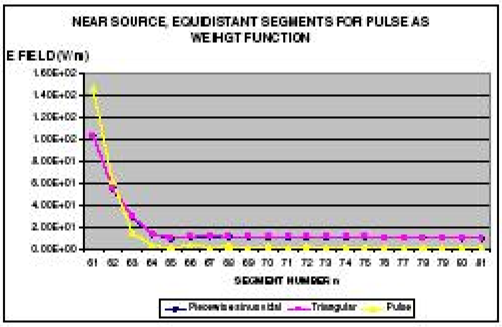

Figure 7. Near source equidistant for pulse

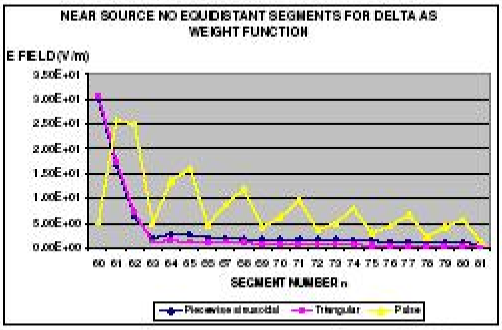

Figure 8. Near source non-equidistant for delta

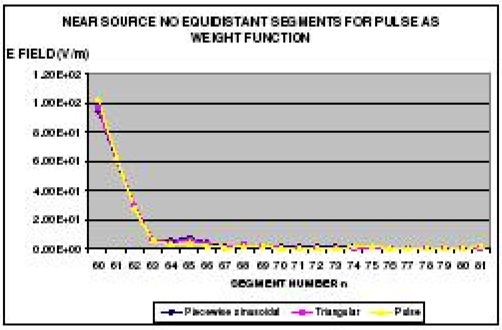

Figure 9. Near source non-equidistant for pulse

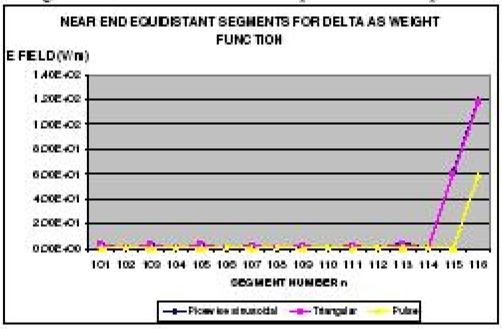

Figure 10. Near end equidistant for delta

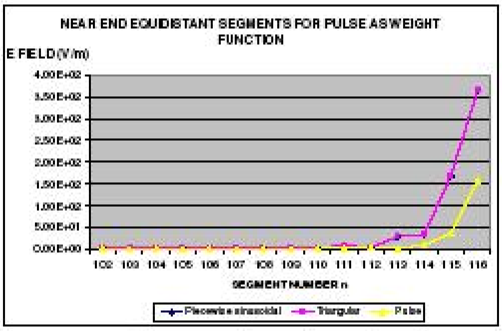

Figure 11. Near end equidistant for pulse

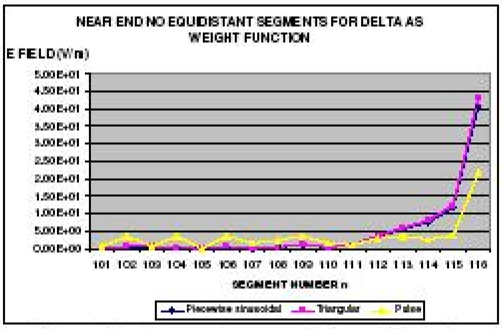

Figure 12. Near end non-equidistant for delta

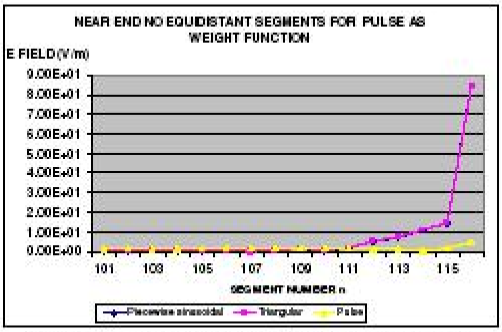

Figure 13. Near end non-equidistant for pulse

As we can see, best results are obtained with pulse weight function for both, equidistant and non-equidistant segments, although even better results are for the second one. As a matter of comparison the Figs. 14 and 15 show results, for an equidistant and non-equidistant, near source and end dipole, with pulse as weight function.

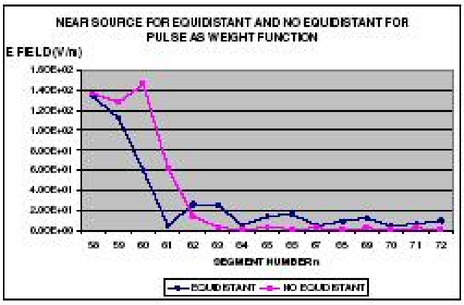

Figure 14. Near source equidistant and non- equidistant for pulse

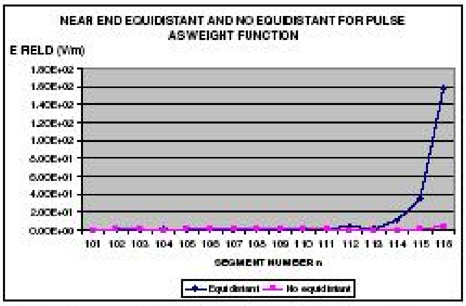

Figure 15. Near end equidistant and non-equidistant for pulse

Conclusion

Equidistant and non-equidistant division are presented in solution of Pocklington equation using the Method of Moments, comparing both with pulse and delta as weight functions; it is noticed the great reduction in discontinuities along the conductor, mainly in the far end, with a comparing reduction of almost 18 dB down for the best results for pulse as weight function, as shown by Fig. 15.

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1. Ch. M. Butler and D. R. Wilton, “Analysis of various Numerical Techniques Applied to Thin Wire Scatterers”, IEEE Trans. Antennas and Propagation **AP-23,** No.4 (1975) 534-540

2. T. K. Sarkar, A. Djodjevic and E. Arvas, “On the Choice of Expansion and Weighting Functions in the Numerical Solution of Operator Equations” IEEE Trans. Antennas and Propagation **AP-33**, No.9 (1985) 988-996

3. D. H. Werner, P.L. Werner and J. K. Breakall, “Some Computational Aspects of Pocklington Electric Field Integral Equation for Thin Wires”, IEEE Trans. on Antennas and Propagation **AP-42**, No. 4 (1985) 561-563

4. H. C. Pocklington, “Electrical Oscillations in Wires”, Cambridge Phil. Soc. Proc. **9** (1897) 324-332

5. K. K. Mei, “On the Integral Equation of Thin Wire Antennas”, IEEE Transactions on Antennas and Propagation **AP-13** (1965) 374-378

6. J. Sosa-Pedroza, J. López-Bonilla and V. Barrera-Figueroa, “La ecuación generalizada de Pocklington para antenas de alambre de forma arbitraria”, Revista Científica ESIME-IPN **9**, No.2 (2005) 83-86

7. J. Sosa-Pedroza, V. Barrera-Figueroa and J. López-Bonilla, “Base and Weight Functions Comparative Analysis for the Pocklington Equation Solution using Method of Moments”, Congress SoftCom 05, Split city, Croatia, September 2005.

8. V. Barrera-Figueroa, J. Sosa-Pedroza and J. López-Bonilla, “Multiple root finder algorithm for Legendre and Chebyshev polynomials via Newton`s method”, Annales Mathematicae et Informaticae, **33** (2006) 3-13.