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Numerical Solutions of Fredholm Integral Equations of Second Kind Using Piecewise Bernoulli Polynomials

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Abstract

The aim of this paper is to solve the Fredholm integral equations numerically using piecewise Bernoulli polynomials. The modified Bernoulli polynomials are derived explicitly over the unit interval. A matrix formulation for non-singular linear Fredholm integral equations is derived by the technique of Galerkin method. In this method, the Bernoulli polynomials are exploited as basis functions in the approximation. Numerical examples are considered to verify the accuracy of the proposed derivations.

Keywords: Fredholm integral equation, Galerkin method, Bernoulli polynomials, Numerical solutions.

I. Introduction

In the survey of solutions of integral equations, a large number of analytical but a few approximate methods are available for solving numerically various classes of integral equations [1, 2, 7, 8]. Since the piecewise polynomials are differentiable and integrable, the Bernstein polynomials [5 – 8] have been used for solving differential and integral equations numerically. Recently, integral equations have been solved by the well known variational iteration method [9]. In the literature [7], Mandal and Bhattacharya have attempted to solve integral equations numerically using Bernstein polynomials, but they obtained the results in terms of finite series solutions. In contrast to this, we solve the linear Fredholm integral equation by exploiting very well known Galerkin method [3], and Bernoulli polynomials [4] are used as trial functions. For this, we give a short introduction of Bernoulli polynomials first. Then we derive a matrix formulation by the technique of Galerkin method. To verify the accuracy of our formulation we consider four examples, in which we obtain exact solutions for three examples even using a few and lower order polynomials. The error estimation for the last example shows an excellent agreement of accuracy compared to exact solution, and verifies the features of convergence. All the computations are performed using *MATHEMATICA*.

II. Bernoulli Polynomials

The Bernoulli polynomials [4] upto degree n can be defined over the interval $[0,1]$ implicitly by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k \; x^{n-k}$$

where b_k are Bernoulli numbers given by

$$b_0 = 1 \quad \text{and} \quad b_k = -\int_0^1 B_k(x)dx \qquad k \ge 1 .$$

These Bernoulli polynomials may be defined explicitly as

$$B_0(x) = 1$$

$$B_m(x) = \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m - \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m \quad , m \ge 1 \qquad (1)$$

The first 11 Bernoulli polynomials $(n = 10)$ are given bellow for using in this paper:

$B_0(x) = 1$

$B_1(x) = x$

$B_2(x) = -x + x^2$

$B_3(x) = \frac{x}{2} - \frac{3x^2}{2} + x^3$

$B_4(x) = x^2 - 2x^3 + x^4$

$B_5(x) = -\frac{x}{6} + \frac{5x^3}{3} - \frac{5x^4}{2} + x^5$

$Br_6(x) = -\frac{x^2}{2} + \frac{5x^4}{2} - 3x^5 + x^6$

$Br_7(x) = \frac{x}{6} - \frac{7x^3}{6} + \frac{7x^5}{2} - \frac{7x^6}{2} + x^7$

$Br_8(x) = \frac{2x^2}{3} - \frac{7x^4}{3} + \frac{14x^6}{3} - 4x^7 + x^8$

$Br_9(x) = -\frac{3x}{10} + 2x^3 - \frac{21x^5}{5} + 6x^7 - \frac{9x^8}{2} + x^9$

$Br_{10}(x) = -\frac{3x^2}{2} + 5x^4 - 7x^6 + \frac{15x^8}{2} - 5x^9 + x^{10}$

Note that Bernoulli polynomials have a special property at $x = 0$ and $x = 1$, respectively,

$Br_n(0) = 0, \qquad n \ge 1 \qquad \text{and} \qquad B_n(1) = 0, \qquad n \ge 2.$

Now the first six polynomials over $[0,1]$ are shown in Fig. 1(a), and the remaining five polynomials are shown in Fig. 1(b).

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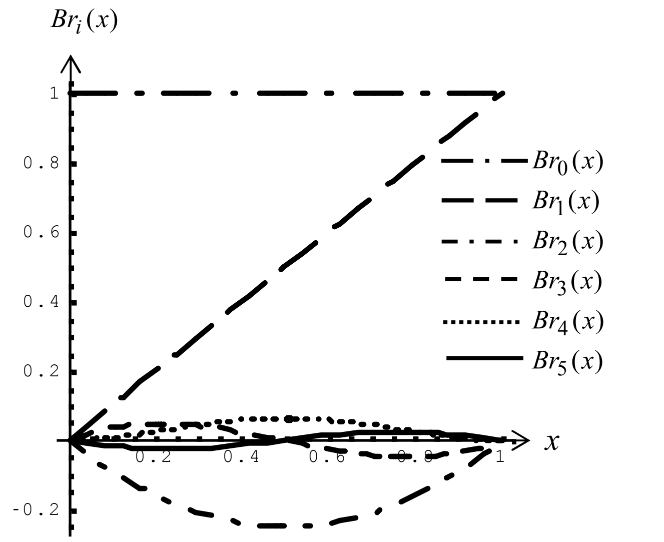

Fig. 1(*a*). Graphical representations of Bernoulli polynomials up to degree 5.

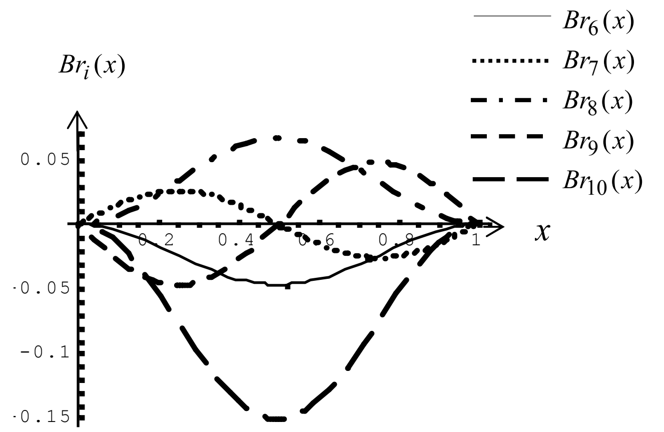

Fig. 1(*b*). Graphical representations of Bernoulli polynomials from degree 6 to degree 10.

III. Formulation of Integral Equation in Matrix Form

Consider a general linear Fredholm integral equation (FIE) of second kind [1, 2] is given by

$$a(x)\phi(x)+\lambda\int_a^b k(t,x)\phi(t)dt = f(x)\,,\ \ a\le x\le b \tag{2}$$

where $a(x)$ and $f(x)$ are given functions, $k(t,x)$ is the kernel, and $\phi(x)$ is the unknown function or exact solution of (2), which is to be determined.

Now we use the technique of Galerkin method [Lewis, 3] to find an approximate solution $\widetilde{\phi}(x)$ of (2). For this, we assume that

$$\widetilde{\phi}(x)=\sum_{i=0}^{n} a_i B_i(x) \tag{3}$$

where $B_i(x)$ are Bernoulli polynomials (basis) of degree i defined in eqn. (1), and a_i are unknown parameters, to be determined. Substituting (3) into (2), we obtain

$$a(x)\sum_{i=0}^{n} a_i B_i(x)+\lambda\int_a^b\left[k(t,x)\sum_{i=0}^{n} a_i B_i(t)\right]dt = f(x)$$

or,

$$\sum_{i=0}^{n} a_i\left[a(x)B_i(x)+\lambda\int_a^b k(t,x)B_i(t)dt\right]=f(x) \tag{4}$$

Then the Galerkin equations [Lewis, 3] are obtained by multiplying both sides of (4) by $B_j(x)$ and then integrating with respect to x from a to b, we have

$$\int_a^b\left[\sum_{i=0}^{n} a_i\left[a(x)B_i(x)+\lambda\int_a^b k(t,x)B_i(t)dt\right]B_j(x)dx\right.$$
$$=\int_a^b B_j(x)f(x)dx$$

or, equivalently

$$\sum_{i=0}^{n} a_i\left[\int_a^b\left[a(x)B_i(x)+\lambda\int_a^b k(t,x)B_i(t)dt\right]B_j(x)dx\right]$$
$$=\int_a^b B_j(x)f(x)dx,\quad j=0,1,\ldots,n$$

Since in each equation, there are three integrals. The inner integrand of the left side is a function of x and t, and is integrated with respect to t from a to b. As a result the outer integrand becomes a function of x only and integration with respect to x yields a constant. Thus for each j $(=0,1,\ldots,n)$ we have a linear equation with $n+1$ unknowns a_i $(i=0,1,\ldots,n)$. Finally (5a) represents the system of $n+1$ linear equations in $n+1$ unknowns, are given by

$$\sum_{i=0}^{n} a_i C_{i,j}=F_j,\quad j=0,1,2,\ldots,n, \tag{5a}$$

where

$$C_{i,j}=\int_a^b\left[a(x)B_i(x)+\lambda\int_a^b k(t,x)B_i(t)dt\right]B_j(x)\,dx$$
$$i,j=0,1,2,\ldots,n. \tag{5b}$$

$$F_j=\int_a^b B_i(x)f(x)dx,\qquad j=0,1,2,\ldots,n \tag{5c}$$

Now the unknown parameters a_i are determined by solving the system of equations (5), and substituting these values of parameters in (3), we get the approximate solution $\widetilde{\phi}(x)$ of the integral equation (2). The absolute relative error E for this formulation is defined by

$$E = \left| \frac{\varphi(x) - \widetilde{\varphi}(x)}{\varphi(x)} \right|.$$

IV. Numerical Examples

In this section, we explain three integral equations which are available in the existing literatures [1, 2, 7]. For each example we find the approximate solutions using Bernoulli polynomials.

Example 1: We consider the FIE of 2nd kind given by [7]

$$\phi(x) - \int_{-1}^{1} (xt + x^2 t^2)\phi(t)dt = 1, \qquad -1 \le x \le 1, \tag{6}$$

having the exact solution $\phi(x) = 1 + \frac{10}{9}x^2$.

Using the formulation described in the previous section, the equations (5) lead us, respectively,

$$C_{i,j} = \int_{-1}^{1} B_i(x)B_j(x)\ dx - \int_{-1}^{1}\left[\int_{-1}^{1}(xt + x^2t^2)B_i(t)dt\right]B_j(x)\ dx \qquad i,j = 0,1,2,\ldots,n \tag{7a}$$

$$F_j = \int_{-1}^{1} B_j(x)\ dx, \qquad j = 0,1,2,\ldots,n \tag{7b}$$

Solving the system (7) for $n = 3$, the values of the parameters are:

$$a_0 = 1,\ a_1 = \frac{10}{9},\ a_2 = \frac{10}{9},\ a_3 = 0$$

and the approximate solution is

$$\widetilde{\phi}(x) = 1 + \frac{10}{9}x^2$$

which is the exact solution.

Example 2: Consider a FIE of 2nd kind given by [7]

$$\phi(x) - \int_{-1}^{1}(x^4 - t^4)\phi(t)dt = x,\ -1 \le x \le 1 \tag{8}$$

having the exact solution $\phi(x) = x$

Proceeding as the example 1, the system of equations becomes as

$$\sum_{i=0}^{n} a_i \left[a(x)B_i(x) + \lambda \int_a^b k(t,x)B_i(t)dt \right] = f(x) \tag{9a}$$

where,

$$C_{i,j} = \int_{-1}^{1} B_i(x)B_j(x)dx - \int_{-1}^{1}\left[\int_{-1}^{1}(x^4 - t^4)B_i(t)dt\right]B_j(x)dx \qquad i,j = 0,1,2,\ldots,n, \tag{9b}$$

$$F_j = \int_{-1}^{1} xB_j(x)dx \qquad j = 0,1,2,\ldots,n, \tag{9c}$$

For $n = 3$, solving system (9), the values of the parameters (a_i) are:

$$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = 0,$$

and the approximate solution is $\widetilde{\phi}(x) = x$ which is the exact solution.

Example 3: Consider another FIE of 2nd kind given by [1, pp 213]

$$\phi(x) - \int_0^1 (tx^2 + xt^2)\phi(t)dt = x, \qquad 0 \le x \le 1 \tag{10}$$

having the exact solution $\phi(x) = \frac{180}{119}x + \frac{80}{119}x^2$

Proceeding as the previous examples, the equations (5b) and (5c) become as

$$C_{i,j} = \int_0^1 B_i(x)B_j(x)dx - \int_0^1\left[\int_0^1 (tx^2 + xt^2)B_i(t)dt\right]B_j(x)dx \qquad i,j = 0,1,2,\ldots,n, \tag{11a}$$

$$F_j = \int_0^1 xB_j(x)dx \qquad j = 0,1,2,\ldots,n, \tag{11b}$$

For $n = 3$, solving system (11), the values of the parameters (a_i) are:

$$a_0 = 0,\ a_1 = \frac{260}{119},\ a_2 = \frac{80}{119},\ a_3 = 0$$

and the approximate solution is

$$\widetilde{\phi}(x) = \frac{180}{119}x + \frac{80}{119}x^2$$

Again we have the exact solution

Example 4: Consider another FIE of 2nd kind given by [2, pp 124]

$$\phi(x) - \int_0^1 2e^x e^t \phi(t)dt = e^x,\ 0 \le x \le 1, \tag{12}$$

having the exact solution $\phi(x) = \frac{e^x}{2 - e^2}$.

Since the equations (5b) and (5c) are of the form

$$C_{i,j} = \int_0^1 B_{i,n}(x)B_{j,n}(x)dx - \int_0^1\left[\int_0^1 2e^x e^t B_{i,n}(t)dt\right]B_{j,n}(x)dx, \qquad i,j = 0,1,2,\ldots,n \tag{13a}$$

$$F_j = \int_0^1 e^x B_{j,n}(x)\ dx, \qquad j = 0,1,2,\ldots,n, \tag{13b}$$

Table. 1. Numerical solutions at various points and corresponding absolute errors of example 4.

x	Exact Solutions	Approximate Solutions	Error, E
		Polynomials used 4	
0.0	-0.1855612526	-0.1853868426	0.000940
0.1	-0.2050768999	-0.2051159200	0.000190
0.2	-0.2266450257	-0.2267185494	0.000324
0.3	-0.2504814912	-0.2505049431	0.000094
0.4	-0.2768248595	-0.2767853131	0.000143
0.5	-0.3059387842	-0.3058698717	0.000225
0.6	-0.3381146470	-0.3380688310	0.000136
0.7	-0.3736744748	-0.3736924032	0.000048
0.8	-0.4129741624	-0.4130508005	0.000186
0.9	-0.4564070342	-0.4564542350	0.000103
1.0	-0.5044077810	-0.5042129189	0.000386
x	Exact Solutions	Polynomials used 5	
0.0	-0.1855612526	-0.1855710208	5.264169×10^{-5}
0.1	-0.2050768999	-0.2050729963	1.903450×10^{-5}
0.2	-0.2266450257	-0.2266433924	7.206563×10^{-6}
0.3	-0.2504814912	-0.2504841199	1.049471×10^{-5}
0.4	-0.2768248595	-0.2768280330	1.146363×10^{-5}
0.5	-0.3059387842	-0.3059389289	4.732287×10^{-7}
0.6	-0.3381146470	-0.3381115484	9.164287×10^{-6}
0.7	-0.3736744748	-0.3736715751	7.760090×10^{-6}
0.8	-0.4129741624	-0.4129756359	3.568115×10^{-6}
0.9	-0.4564070342	-0.4564113012	9.349075×10^{-6}
1.0	-0.5044077810	-0.5043970842	2.120664×10^{-5}
x	Exact Solutions	Polynomials used 6	
0.0	-0.1855612526	-0.1855610006	2.405587×10^{-6}
0.1	-0.2050768999	-0.2050770088	8.649860×10^{-7}
0.2	-0.2266450257	-0.2266449063	3.273606×10^{-7}
0.3	-0.2504814912	-0.2504813833	5.420754×10^{-7}
0.4	-0.2768248595	-0.2768249425	1.162778×10^{-7}
0.5	-0.3059387842	-0.3059389458	4.731259×10^{-7}
0.6	-0.3381146470	-0.3381146594	1.259522×10^{-7}
0.7	-0.3736744748	-0.3736743009	3.620806×10^{-7}
0.8	-0.4129741624	-0.4129740843	2.074285×10^{-7}
0.9	-0.4564070342	-0.4564072670	4.075118×10^{-7}
1.0	-0.5044077810	-0.5044071950	9.560245×10^{-7}
x	Exact Solutions	Polynomials used 7	
0.0	-0.1855612526	-0.1855612694	9.049198×10^{-8}
0.1	-0.2050768999	-0.2050768958	1.990287×10^{-8}
0.2	-0.2266450257	-0.2266450312	2.425409×10^{-8}
0.3	-0.2504814912	-0.2504814909	9.600473×10^{-10}
0.4	-0.2768248595	-0.2768248544	1.846251×10^{-8}
0.5	-0.3059387842	-0.3059387842	7.752014×10^{-11}
0.6	-0.3381146470	-0.3381146522	1.554022×10^{-8}
0.7	-0.3736744748	-0.3736744750	5.635104×10^{-10}
0.8	-0.4129741624	-0.4129741564	1.441928×10^{-8}
0.9	-0.4564070342	-0.4564070387	9.997793×10^{-9}
1.0	-0.5044077810	-0.5044077618	3.797784×10^{-8}

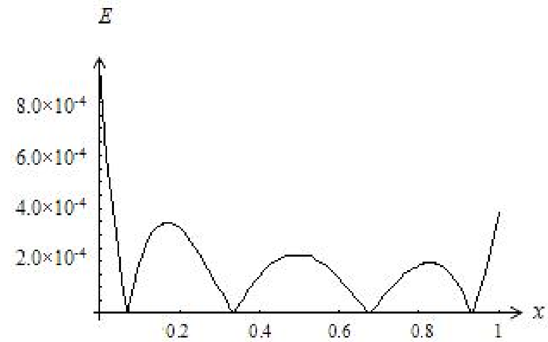

Fig.2a. Relative error E using 4 polynomials

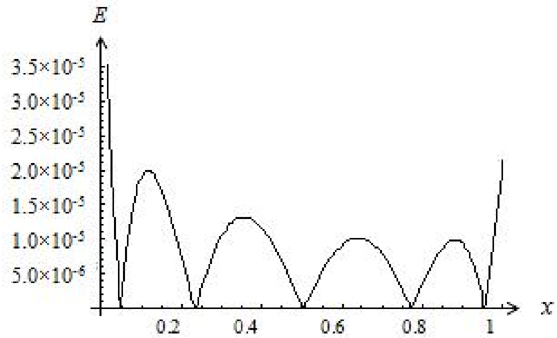

Fig.2b. Relative error E using 5 polynomials

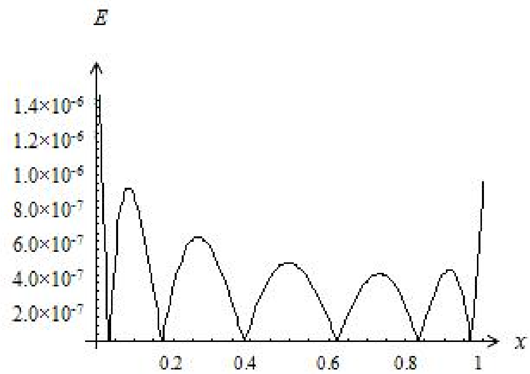

Fig.2c. Relative error E using 6 polynomials

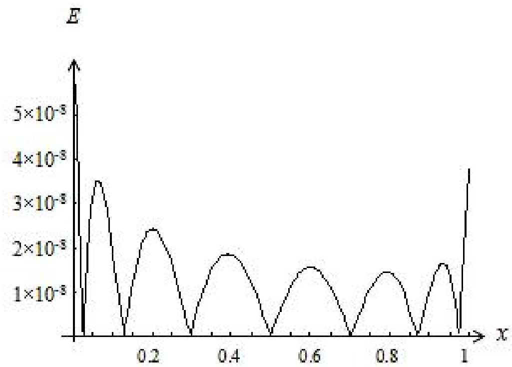

Fig.2d. Relative error E using 7 polynomials

Solving the system (5a) using equations (13), we have the approximate solutions for $n = 3, 4, 5$ and 6, respectively as:

$$\widetilde{\phi}(x) = -0.185387 - 0.188957x - 0.078167x^2 - 0.051702\,x^3$$

$$\widetilde{\phi}(x) = -0.185571 - 0.185273x - 0.0947437x^2 - 0.025916\,x^3 - 0.012893x^4$$

$$\widetilde{\phi}(x) = -0.185561 - 0.18558x - 0.0925986x^2 - 0.0316362x^3 - 0.00645779x^4 - 0.00257408x^5$$

and

$$\widetilde{\phi}(x) = -0.185561 - 0.18556x - 0.0925932x^2 - 0.0308581x^3 - 0.00791665x^4 - 0.0012903x^5 - 0.000427923x^6$$

Now the approximate solutions, exact solutions, and the absolute relative error *E*, between exact and approximate solutions, at various points of the domain using the Bernoulli polynomials 4 – 7 are displayed in Table 1.

Also plotting the absolute relative errors, shown in Fig.2, we see that the maximum errors are $10^{-4}, 10^{-5}$, 10^{-6} and 10^{-8} respectively, with 4, 5, 6 and 7 Bernoulli polynomials. These lead us that the convergence of the approximate solutions and the desired accuracy hinges on the size of the basis set chosen.

V. Conclusion

We have obtained the approximate solution of the unknown function of the Fredholm integral equations of second kind by the well known Galerkin method using Bernoulli polynomials as trial functions. The computed solutions are compared with the exact solutions, and we have found a good agreement with the exact solution. In this connection, we note that the numerical solutions are coincided with the exact solutions even a few of the polynomials have been used in the approximation. Thus the authors' concluding remark is that this method may be applied to solve other integral equations numerically to get the desired accuracy.

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1. Abdul J. Jerri, 1999 – *Introduction to Integral Equations with Applications*, John Wiley & Sons Inc.
2. Shanti Swarup, 2007 – *Integral Equations*, Krishna Prakashan Media (P) Ltd, Fifteenth Edition.
3. Lewis, P. E., J. P. Ward, 1991 – *The Finite Element Method*, Principles and Applications, Addison-Wesley.
4. Kendal E Atkinson,1989 – *An Introduction to Numerical Analysis*, John Wiley and Sons, 2nd edition, pp-284.
5. Reinkenhof, J., 1977 – Differentiation and integration using Bernstein's polynomials, *Int. J. Numer. Methods Engrg*, **11**, 1627 – 1630.
6. Kreyszig, E., 1979 – Bernstein polynomials and numerical integration, *Int. J. Numer. Methods Engrg*, **14**, 292 – 295.
7. Mandal, B. N., S. Bhattacharya, 2007 – Numerical solution of some classes of integral equations using Bernstein polynomials, *Applied Mathematics and Computation*, **190**, 1707 – 1716.
8. Bhatti, M.I., P. Bracken, 2007 – Solutions of differential equations in a Bernstein polynomial basis, *J. Computational and Applied Mathematics,* **205,** 272– 280
9. Wang, S.Q., J.H. He, 2007 – Variational method for solving integro-differential equations, *Physics Letters* A, **367**, 188 – 191.

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