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Associated polynomials of Chebyshev

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We introduce a new class of polynomials T_n^m (x) which permit to generate the Chebyshev polynomials T_n *(x)* and U_n *(x)*.
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The first-kind Chebyshev polynomials are given by [1,2]:

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}, \frac{1-x}{2}\right), \quad |x| \le 1 \tag{1}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function [3], such that:

$$\left(1-x^2\right)\frac{d^2Tn}{dx^2} - x\frac{dTn}{dx} + n^2Tn = 0\,, \tag{2}$$

with the important property

$$Tn(\cos\theta) = \cos(n\,\theta), \tag{3}$$

Similarly, for the second-kind Chebyshev polynomials we have that [2]

$$U_n(x) = (n+1)\ {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right), \tag{4.a}$$

$$\left(1-x^2\right)\frac{d^2U_N}{dx^2} - 3x\frac{dU_N}{dx} + N(N+2)U_N = 0, \tag{4.b}$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{4.c}$$

It is natural to search a generalization of (1) and (4.a), in fact, in this work we introduce the Associated Polynomials of Chebyshev, m=0,1,…,n:

$$T_n^m(x) = (-1)^m\binom{2n-m}{m}\,{}_2F_1\left(-m, 2n-m; n-m+\frac{1}{2}; \frac{1-x}{2}\right), \tag{5}$$

which are solutions of

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - (2n-2m+1)x\frac{dy}{dx} + m(2n-m)y = 0, \tag{6}$$

From (1), (4.a) and (5) we obtain the relations

$$T_n^0(x) = 1, T_n(x) = (-1)^n T_n^n(x), Un(x) = (-1)n\frac{2}{2+n}T_{n+1}^n(x), \tag{7}$$

and from (5) for m=n & m=N, n=N+1 we deduce (2) and (4.b), respectively. An open problem is to find the corresponding extension of (3) and (4.c), that is, to investigate if there is a closed expression for $T_n^m(\cos\theta)$.

To elucidate the meaning of (5), we remember that (1) can be generated as the determinant of Chebyshev matrices $T_{\sim n}$ (x) [2]

$$T_2(x) = Det\,\underset{\sim}{T_2}(x) = Det\begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix} = 2x^2 - 1,$$

$$T_3(x) = Det\,\underset{\sim}{T_3}(x) = Det\begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix} = 4x^3 - 3x, \tag{8}$$

$$T_4(x) = Det\,\underset{\sim}{T_4}(x) = Det\begin{pmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{pmatrix} = 8x^4 - 8x^2 + 1, \ldots$$

and we can obtain the characteristic equation [4,5] of $T_{\sim n}$

n **Characteristic Equation**

1: $\lambda - T_1 = 0$

2: $\lambda^2 - 3x\lambda + T_2 = 0$

3: $\lambda^3 - 5x\lambda^2 + (8x^2 - 2)\lambda - T_3 = 0$

4: $\lambda^4 - 7x\lambda^3 + (18x^2 - 3)\lambda^2 - (20x^3 - 10x)\lambda + T_4 = 0$

5: $\lambda^5 - 9x\lambda^4 + (32x^2 - 4)\lambda^3 - (56x^3 - 21x)\lambda^2 + (48x^4 - 36x^2 + 3)\lambda - T_5 = 0$

6: $\lambda^6 - 11x\lambda^5 + (50x^2 - 5)\lambda^4 - (120x^3 - 36x)\lambda^3 + (160x^4 - 96x^2 + 6)\lambda^2 -$
$- (112x^5 - 112x^3 + 21x)\lambda + T_6 = 0$, (9)

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or in compact form

$$\sum_{m=0}^{n} T_n^m \lambda^{n-m} = 0, \tag{10}$$

that is, the T_n^m *(x)* given by (5) are the polynomial coefficients in the characteristic equation of $T_{\sim n}$ (x).

Matlab program give us all roots x_j of each T_n^m in (9), thus we can see that they are real with $|x_j| < 1$, which also happens with the roots of (1) and (4.a). In another work we will study properties as recurrence, orthogonality and Rodrigues formula for the associated polynomials of Chebyshev.

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