Dhaka Univ. J. Sci. 58(2): 161-164, 2010 (July)

Matrix Method of Designing Asymmetrical Factorial Experiment

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Received on 10. 03. 2009. Accepted for Publication on 28. 10. 2009.

Abstract

This article describes matrix method of designing asymmetrical factorial experiment with two or more factors. Using vector and matrix operations it has been constructed the layout with different levels of a $p \times q \times \cdots$ n - factors asymmetrical factorial experiments.

I. Introduction

In the construction of asymmetrical design, we use the combinatorial concepts to find all the level combinations in a manipulating manner[1, 2]. Dean and John (1975) described the construction of asymmetrical arrangements. The method appears to be well suited to the production of factorial designs on a computer. This procedure is incorporated in a program called DSIGN, written by Mrs. J. Tolmie and used both at Rothamsted and in Edinburgh. In this system, treatment factors are derived from levels of plot factors by a set of rules. Jalil (et al) developed a matrix method[3] of designing p^n - factorial (symmetrical) design. In this article, we developed matrix method to design asymmetrical factorial experiments with two or more factorial effects. Using the concept of matrix operations, we developed a general method of construction of $p \times q \times \cdots$ factorial experiment. The specific methods with examples for $p \times q$, $p \times q \times r$ and $p \times q \times r \times s$ asymmetrical factorial experiments have been described also.

II. Method

The proposed matrix method of designing a $p \times q \times \cdots$; n - factors asymmetrical design is given by the following matrix equation as shown below:

$$M_a = \left[V_1\{c_1\}\ \ V_2\{c_2\}\ \ \cdots\ \ V_n\{c_n\}\right]_{pq\cdots \times n}.$$ Where,

M_a: is the desired plan with the different level combinations;

$V_i\{c_i\} = c_i[\,0\,I_{k_i}\ \ 1I_{k_i}\ \ 2\,I_{k_i}\ \ \cdots(L-1)\,I_{k_i}\,]_{pq\cdots\times 1}$;

$i = 1, 2, \cdots, n$; each is a column vector of dimension $(p \times q \times \cdots)$. L takes the value equal to the number of levels of the $i-$th factor; for $i = 1$, the value of L is equal to p, the number of levels of first factor etc.

I_m : a sum vector of dimension m;

$\{c_i\} = c_i$ - times repetition of the elements of V_i in ascending ordered levels.

The constants c_i and k_i ($i = 1, 2, \cdots, n$) can be found with the formulas given below.

$$c_1 = \frac{pq\cdots}{pq\cdots} = 1; \qquad c_2 = \frac{pq\cdots}{qr\cdots} = p;$$

$$c_3 = \frac{pq\cdots}{rs\cdots} = pq\ \cdots;\ c_n = \frac{pq\cdots}{z} = pq\cdots$$ (up to $(n-1)$th factor).

$$k_1 = \frac{pq\cdots}{p} = qr\cdots;\ k_2 = \frac{pq\cdots}{pq} = rs\cdots;\ k_3 = \frac{pq\cdots}{pqr} = st\cdots;$$

$$k_n = \frac{pq\cdots}{pq\cdots} = 1\cdot$$

Thus, for a $p \times q \times \cdots$ n - factors asymmetrical design,

$$V_1\{c_1\} = V_1\{1\} = [\,0\,I_{qr\ldots}\ 1\,I_{qr\ldots}\ \cdots(p-1)\,I_{qr\ldots}\,]'_{pq\cdots\times 1};$$

$$V_2\{c_2\} = V_2\{p\} = p[\,0\,I_{rs\cdots}\ 1\,I_{rs}\ \cdots\cdots(q-1)\,I_{rs\cdots}\,]'_{pq\cdots\times 1};$$

$$V_3\{c_3\} = V_3\{pq\} = pq[\,0\,I_{st\cdots}\ 1\,I_{st\ldots}\ \cdots(r-1)\,I_{st\cdots}\,]'_{pq\cdots\times 1};$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$V_n\{c_n\} = V_n\{pq\cdots\} = pq\cdots[\,0\,I_1\ 1\,I_1\ \cdots(L-1)\,I_1\,]'_{pq\cdots\times 1}$$

[$pq\cdots$ means multiplication of the levels up to the $(n-1)$th factor].

The method is illustrated specifically with two, three and four factors asymmetrical factorial experiments in the following sections.

For a $p \times q$ factorial experiment the proposed matrix method to represent all the level combinations is given by,

$$M_a = \left[V_1\{c_1\}\ \ V_2\{c_2\}\right]_{pq\times 2}.$$ Where,

M_a: the matrix showing the desired plan with different level combinations;

$V_i\{c_i\} = c_i[\,0\,I_{k_i}\ \ 1I_{k_i}\ \ 2\,I_{k_i}\ \ \cdots(L-1)\,I_{k_i}\,]_{pq\times 1}$;

$i = 1, 2, \cdots, n$; each is a column vector of dimension pq.

$\{c_i\} = c_i$ - times repetition of the elements of V_i in ascending ordered levels.

The constants c_i and k_i ($i = 1, 2, \cdots, n$) can be found with the formulas given below.

$c_1 = \frac{pq}{pq} = 1$; $c_2 = \frac{pq}{q} = p$; and

$k_1 = \frac{pq}{p} = q$; $k_2 = \frac{pq}{pq} = 1$.

The method is illustrated with an example shown below.

For a 2×3 factorial experiment, the layout of the level combinations is shown below.

$$M_a = \left[V_1\{c_1\}\ \ V_2\{c_2\}\right]_{pq \times 2} = \left[V_1\{c_1\}\ \ V_2\{c_2\}\right]_{6 \times 2}$$

where,

$V_i\{c_i\} = c_i[0I_{k_i}\ 1I_{k_i}\ 2I_{k_i}\ \cdots(L-1)I_{k_i}]_{pq\times 1}$;

$i = 1, 2, \cdots, n$; each is a column vector of dimension $(p \times q)$.

$V_1\{c_1\} = c_1[0I_{k_1}\ 1I_{k_1}]_{6\times1} = [0I_3\ 1I_3]_{6\times1}$;

$c_1 = \frac{pq}{pq} = 1$, $k_1 = \frac{pq}{p} = q = 3$.

$V_2\{c_2\} = c_2[0I_{k_2}\ 1I_{k_2}]_{6\times1} = 2[0I_1\ 1I_1\ 2I_1]_{6\times1}$;

$c_2 = \frac{pq}{p} = p = 2$, $k_2 = \frac{pq}{pq} = 1$.

Therefore, $V_1\{c_1\} = c_1[0I_{k_1}\ 1I_{k_1}]_{6\times1} = [0I_3\ 1I_3]_{6\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

and

$$V_2\{c_2\} = c_2[0I_{k_2}\ 1I_{k_2}\ 2I_{k_2}]_{6\times1} = 2[0I_1\ 1I_1\ 2I_1]_{6\times1} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

.

The final plan of level combinations is given by, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ & \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Similarly for a 3×5 factorial experiment, the desired matrix is shown as:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 3 \\ 0 & 4 \\ & \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ & \\ 2 & 0 \\ 2 & 1 \\ 2 & 2 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$$

For a $p \times q \times r$ asymmetrical design the method to represent all the level combinations is given by,

$M_a = \left[V_1\{c_1\}\ \ V_2\{c_2\}\ \ V_3\{c_3\}\right]_{pqr \times 3}$. Where,

$V_i\{c_i\} = c_i[0I_{k_i}\ 1I_{k_i}\ 2I_{k_i}\ \cdots(L-1)I_{k_i}]_{pqr\times 1}$;

$i = 1, 2, \cdots, n$; each is a column vector of dimension $(p \times q \times r)$.

$\{c_i\} = c_i$ - times repetition of the elements of V_i in ascending ordered levels.

The constants c_i and k_i ($i = 1, 2, \cdots, n$) can be found with the formulas given below.

$c_1 = \frac{pqr}{pqr} = 1$; $c_2 = \frac{pqr}{qr} = p$ and $c_3 = \frac{pqr}{r} = pq$; and

$k_1 = \frac{pqr}{p} = qr$; $k_2 = \frac{pqr}{pq} = r$ and $k_3 = \frac{pqr}{pqr} = 1$.

Thus, for a $p \times q \times r$ factorial design,

$V_1\{c_1\} = V_1\{1\} = [0I_{qr}\ 1I_{qr}\ \cdots(p-1)I_{qr}]$;

$V_2\{c_2\} = V_2\{p\} = p[0I_r\ 1I_r\ \cdots(q-1)I_r]$ and

$V_3\{c_3\} = V_3\{c_3 = pq\} = pq[0I_1\ 1I_1\ \cdots(r-1)I_1]$.

The method is illustrated with an example described below.

For a $2 \times 3 \times 5$ design, we have the three column vectors as given by,

$V_1\{c_1\} = V_1\{1\} = [0I_{qr}\ 1I_{qr}]_{pqr\times1} = [0I_{15}\ 1I_{15}]_{30\times1}$;

$V_2\{c_2\} = V_2\{p\} = p[0I_r\ 1I_r\ 2I_r]_{pqr\times 1} = 2[0I_5\ 1I_5\ 2I_5]_{30\times 1}$ and

$V_3\{c_3\} = V_3\{pq\} = pq[0I_1\ 1I_1\ 2I_1\ 3I_1\ 4I_1]_{pqr\times 1} = 6[0I_1\ 1I_1\ 2I_1\ 3I_1\ 4I_1]_{30\times 1}$.

Thus, we have the plan by the matrix, $M_a = [V_1\{c_1\}\ V_2\{c_2\}\ V_3\{c_3\}]_{pqr\times 3}$, which is shown below.

$$M_a = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&2\\0&0&3\\0&0&4\\0&1&0\\0&1&1\\0&1&2\\0&1&3\\0&1&4\\0&2&0\\0&2&1\\0&2&2\\0&2&3\\0&2&4\\1&0&0\\1&0&1\\1&0&2\\1&0&3\\1&0&4\\1&1&0\\1&1&1\\1&1&2\\1&1&3\\1&1&4\\1&2&0\\1&2&1\\1&2&2\\1&2&3\\1&2&4 \end{bmatrix}_{30\times 3}$$

For a $p\times q\times r\times s$ asymmetrical design the method to represent all the level combinations is given by,

$M_a = [V_1\{c_1\}\ V_2\{c_2\}\ V_3\{c_3\}\ V_4\{c_4\}]_{pqrs\times 4}$; where,

$V_i\{c_i\} = c_i[0I_{k_i}\ 1I_{k_i}\ 2I_{k_i}\ \cdots (L-1)I_{k_i}]_{pqr\times 1}$; $i = 1,2,\cdots,n$; each is a column vector of dimension $(p\times q\times r)$.

$\{c_i\} = c_i$ - times repetition of the elements of V_i in ascending ordered levels.

The constants c_i and k_i ($i = 1,2,\cdots,n$) can be found with the formulas given below.

$c_1 = \frac{pqrs}{pqrs} = 1$; $c_2 = \frac{pqrs}{qrs} = p$; $c_3 = \frac{pqrs}{rs} = pq$; and $c_4 = \frac{pqrs}{s} = pqr$.

$k_1 = \frac{pqrs}{p} = qrs$; $k_2 = \frac{pqrs}{pq} = rs$; $k_3 = \frac{pqrs}{pqr} = s$; and $k_4 = \frac{pqrs}{pqrs} = 1$.

Thus, for a $p\times q\times r\times s$ factorial design,

$V_1\{c_1\} = V_1\{1\} = [0I_{qrs}\ 1I_{qrs}\ \cdots (p-1)I_{qrs}]'_{pqrs\times 1}$;

$V_2\{c_2\} = V_2\{p\} = p[0I_{rs}\ 1I_{rs}\ \cdots (q-1)I_{rs}]'_{pqrs\times 1}$;

$V_3\{c_3\} = V_3\{pq\} = pq[0I_s\ 1I_s\ \cdots (r-1)I_s]'_{pqrs\times 1}$ and

$V_4\{c_4\} = V_4\{pqr\} = pqr[0I_1\ 1I_1\ \cdots (s-1)I_1]'_{pqrs\times 1}$.

The method is illustrated with an example described below.

For a $2\times 3\times 4\times 5$ design, we have the four column vectors as given by,

$V_1\{c_1\} = V_1\{1\} = [0I_{qrs}\ 1I_{qrs}]'_{pqrs\times 1} = [0I_{60}\ 1I_{60}]'_{120\times 1}$;

$V_2\{c_2\} = V_2\{p\} = p[0I_{rs}\ 1I_{rs}\ 2I_{rs}]'_{pqrs\times 1}$
$= 2[0I_{20}\ 1I_{20}\ 2I_{20}]'_{120\times 1}$;

$V_3\{c_3\} = V_3\{pq\} = pq[0I_s\ 1I_s\ 2I_s\ 3I_s]'_{pqrs\times 1}$
$= 6[0I_5\ 1I_5\ 2I_5\ 3I_5]'_{120\times 1}$;

and

$V_4\{c_4\} = V_4\{pqr\} = pqr[0I_1\ 1I_1\ \cdots (s-1)I_1]'_{pqrs\times 1}$
$= 24[0I_1\ 1I_1\ 2I_1\ 3I_1\ 4I_1]'_{120\times 1}$.

Thus, we have the plan by the matrix, $M_a = [V_1\{c_1\}\ V_2\{c_2\}\ V_3\{c_3\}\ V_4\{c_4\}]_{pqrs\times 4}$.

III. Conclusion

In this article, we have introduced a method of designing asymmetrical factorial experiments using matrix algebra. Construction of asymmetrical experiments become easier than the methods based on trial and error approach or manipulating manner. The method can be applied for any number of factors with different levels.

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