# **Jacobi Fields on Normal Homogeneous Riemannian Manifolds**

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Received on 10. 07. 2008. Accepted for Publication on 03. 02. 2009

#### **Abstract**

We prove that a normal homogeneous space with the property that every Jacobi field along a geodesic vanishing at two points is the restriction of a Killing field along that geodesic is a globally symmetric space.

## **I. Introduction**

On a symmetric space, the Jacobi equation can be easily solved, and it follows from the work of Boot and Samelson [2] that a Jacobi field along a geodesic vanishing at two distinct points is the restriction of a Killing field along that geodesic. A Jacobi field which is the restriction of a Killing field along a geodesic is called *isotropic* .

In the case of a naturally reductive space, Chavel [4], [16] showed how to write the Jacobi equation in an efficient manner. By means of explicit calculations, he then proved that a simply connected normal homogeneous space of rank one with the property that any Jacobi field vanishing at two points is isotropic is homeomorphic to a rank one symmetric space [5].

Later, Ziller [20] studied the Jacobi equation on naturally reductive spaces, and obtained several interesting results about the structure of the space of Jacobi fields. This led him to suggest the following conjecture:

A naturally reductive homogeneous space with the property that every Jacobi field vanishing at two points is isotropic, is locally symmetric. In this paper, we prove that this conjecture is true in the case of a normal homogeneous space.

**Theorem 1.** *A normal homogeneous space with the property that every Jacobi field vanishing at two points is isotropic*  $J'(0) + A J(0)$ *is a globally symmetric space.*

As a preparation for the proof above theorem, the following theorem of [8]

**Theorem 2.** *An isotropic action of a compact Lie group on a normal homogeneous space is variationally complete if and only if it is hyperpolar.*

## **II. Preliminaries**

We first recall the notion of variationally completeness introduced by Bott and Samelson in [2]. Let *G* be a Lie group acting properly and isometrically on a complete Riemannian manifold  $X$ .<br>A geodesic  $\gamma$  in  $X$  is called G-*transversal* if it is

perpendicular to every G-orbit it meets. A Jacobi field along a geodesic is called G-*isotropic* if it is the restriction of a G killing field along that geodesic. A Jacobi field along a Gtransversal geodesic  $\gamma$  is called G-transversal if it is the variational vector field associated to a variation of  $\gamma$  through

G-transversal geodesics which are at time zero perpendicular to the same G-orbit. Notice that a G-isotropic Jacobi field along a G-transversal geodesic is a Gtransversal Jacobi field which is moreover tangent to the G orbit at every point. The action of G on *X* is called *variationally complete* if every G-transversal Jacobi field that is tangent to the G-orbits at two diffeent points is Gisotropic. It is proved in [2] is enough to impose this condition on G-transversal Jacobi fields that are tangent to the G-orbit at one point and vanish at another one.

Variational completeness can also be defined for proper Fredholm isometric actions of Hilbert Lie group on complete Riemannian Hilbert manifolds [15], [17]. It is not difficult to show that an orbit of a Fredholm isometric action has the property that the normal exponential map is a nonlinear Fredholm map of index zero. This has as pleasant consequence inter alia that monofocal and epifocal points along a normal geodesic must coincide and have finite multiplicities, no clustering of focal points along finite normal geodesic segments is permitted, and the set of focal points to the orbit is of first Baire category [14]. (a) *v* learn there are the interest and goods in the damage and the distance is a transversal gacobi field which is moreover tangent to the G-<br>orbit at every point. The action of G on X is called<br>antarianally complete if

It is known and easy to see that a G-transversal Jacobi field *J* along a G-transversal geodesic  $\gamma$  in a Riemannian manifold  $X$  ( of finite or infinite dimension ) has the property that  $J(0)$  is tangent to the orbit  $Gx$  and  $\gamma(0) = x$ ,  $\gamma'(0) = v$ , and  $A_v$  is the Weingarten operator of  $Gx$  with respect to  $v$ .

The next proposition 1 generalizes Lemmas 1 and 2 in [8].

**Proposition 1.** Let  $\hat{G}$  and  $G$  be two Hilbert Lie groups equipped with proper Fredholm isometric actions on the complete Riemannian Hilbert manifolds  $\hat{X}$  and  $X$ , and suppose that there is an equivariant Riemannian submersion  $\pi: \hat{X} \to X$  with respect to an epimorphism  $\rho: \hat{G} \to G$ difficult to show that an orbit of a Fredbolm isometric action<br>thas the property that the normal exponential map is a<br>nonlinear Fredholm map of index zero. This has as pleasant<br>nonlinear Fredholm map of index zero. This h metric action<br>al map is a<br>s as pleasant<br>ifocal points<br>have finite<br>along finite<br>e set of focal<br>1 Jacobi field<br>Riemannian<br>n ) has the<br>it  $Gx$  and<br> $x$ , where<br>rten operator<br>nd 2 in [8].<br>Lie groups<br>tions on the<br>md X, and<br>n sub such that  $\pi : \hat{X} \to X$  is a principal  $\hat{H}$ -bundle for a redholm map of index zero. This has as pleasant<br>redholm map of index zero. This has as pleasant<br>rmal geodesic must coincide and chircal points<br>s, no clustering of focal points along finite<br>lessic segments is permitted, an Hilbert Lie subgroup  $\hat{H}$  of  $\hat{G}$  . Then the action of  $\hat{G}$  on  $\hat{X}$  is *variationally complete* if and only the action of  $G$  on *X* is variationally complete.

**Proof.** We assume that the action of  $G$  on  $X$  is a way that the intersections between  $\Sigma$  and the orbits of  $G$ variationally complete and prove that so is the action of  $\hat{G}$  on  $\hat{X}$ . The other direction is easier.

Let  $\hat{\gamma}$  be a  $\hat{G}$ -transversal geodesic in  $\hat{X}$  defined on that a hyper [0,1] and let  $\hat{J}$  be a  $\hat{G}$ -transversal Jacobi field along  $\hat{\gamma}$  and it was proved in [8] the such that  $\hat{J}(0)$  is tangent to the  $\hat{G}$ -orbit through symmetric restriction along  $\hat{\gamma}$  of a  $\hat{G}$  -Killing field.

There is a variation  $\{\hat{\gamma}_\alpha\}$  of  $\hat{\gamma}$  through  $\hat{G}$ -transversal groups on Riemannian geodesic whose associated variational vector field is  $\hat{J}$ . Since the  $\hat{G}$ -orbits are the preimages under  $\pi$  of the  $\hat{G}$ - Proposition 1. If the a orbits, the normal spaces to the  $\hat{G}$ -orbits are horizontal hyperwith respect to  $\pi$ , so that each  $\hat{\gamma}_{\alpha}$  is a horizontal curve. It follows that  $\{\gamma_{\alpha}\}\$ , where  $\gamma_{\alpha} = \pi \hat{\gamma}_{\alpha}$ , defines a variation of **Proof.** Let  $\Sigma$  be a  $\gamma = \pi \hat{\gamma}$  through  $\hat{G}$ -transversal geodesics in *X*. If is not *X* is the respect to *n*, we re Moreover, the associated  $\hat{G}$ -transversal Jacobi field  $\frac{\pi}{\sigma}$ aristionally complete and prove that so is the action of  $\alpha$  *J* and orthogonal. An action is called  $\alpha$  *J* and  $\lambda$  *J* and  $\$ *J*(0) is tangent to the  $\hat{G}$ -orbit through  $x := \pi(\hat{x})$  and It is caser.<br>
induces metric is called hyperpolar. Conlon proved in [6]<br>
geodesic in  $\hat{X}$  defined on early and the a hyperpolar action of a computer lie group on a<br>
and twas proved in [8] that the concert in is a compu Let  $\hat{\gamma}$  be a  $\hat{G}$ -transversal geodesic in  $\hat{X}$  defined on that a hyperploat action of a completeness of the divergeness of the completeness of the action of  $\hat{J}(0) = \hat{x}$  and  $\hat{J}(1) = 0$ . We must show that  $\hat$ *G* on *X*, there is a *G* -Killing field  $\xi$  on *X* such that [0,1] and let  $\hat{J}$  be a  $\hat{G}$ -transversal Jacobi field along  $\hat{\gamma}$  and it was proved in [8] that the converse of Figure 1.1 (a) is tangent to the  $\hat{G}$ -orbit through symmetric space. Notice that neither do  $\hat{\gamma}($ mgent to the  $\hat{G}$ -orbit through in the case in which the Riemannian manifold is a compact<br>  $\hat{G}$ -orbit through  $\hat{G}$ -orbit through in [6] that  $\Sigma$  is closed not do we assume that it is<br>  $\therefore$  K is the conductive c there is a  $\hat{G}$ -Killing field  $\xi$  on X such that  $\rho_* \hat{\xi} = \xi$ , and  $\chi$  is an effective, transition since the  $\hat{H}$ -orbits coincide with the fibers of  $\pi$ , we may poin subtract an  $\hat{H}$ -Killing field from  $\hat{\xi}$  if necessary and algebra assume furthermore that  $\hat{\mathcal{E}} \cdot \hat{x} = \hat{J}(0)$ .<br>Ad(K) -invariant vector space direct sum g = f + p; this is There is a variation  $\{\hat{Y}_a\}$  of  $\hat{Y}$  through  $\hat{G}$ -transversal enobed effering from the contribution of thitler Lie<br>geodesic whose associated variational vector field is  $\hat{J}$ . **Proposition 2.** Let  $\hat{G}, G, \hat{X$ with respect to *π*, so that each  $\hat{y}_\alpha$  is a borizontal curve. It by<br>Perpolar).<br>
follows that  $\langle y_\alpha \rangle$ , where  $\gamma_\alpha = \pi \hat{y}$ , defines a wariston of **Proof**. *r* **F** *c F*  $\alpha$  $\pi$  *is*  $\pi$  *is a consider inversi* 

 $\tilde{J}(t) = \hat{\xi} \cdot \hat{\gamma}(t)$ . Then  $\tilde{J}$  is a  $\hat{G}$ -isotropic Jacobi field the pr along  $\hat{\gamma}$ . Since the Jacobi field  $\hat{J}$  and  $\tilde{J}$  are  $\hat{G}$ -trans-  $T_{x0}M$ , and the given versal, we have that  $\hat{J}'(0) + A_{\hat{v}} \hat{u}$  and  $\tilde{J}'(0) + A_{\hat{v}} \hat{u}$  are <sup>in</sup> normal to  $\hat{G}\hat{x}$ , where  $\hat{v} = \hat{v}'(0)$  and  $\hat{u} = \hat{J}(0) = \tilde{J}(0)$ . positive definite symmetric bilinear from on p. Now M is Since the normal spaces to the  $\hat{G}$ -orbits are horizontal, it follows that both  $\hat{J}'(0) + A_{\hat{v}}\hat{u}$  and  $\tilde{J}'(0) + A_{\hat{v}}\hat{u}$  are Hence  $\hat{J}'(0) = \tilde{J}'(0)$ , from where we conclude that *J*(1) = 0. By variational completeness of the action of **III. The proof of Theorem 2**<br> *G* on *X*, there is a *G*-Killing field  $\zeta$  on *X* such that **Let** *M* be a comeded Riemannia<br> *J*(*t*)=  $\xi^2 \cdot \gamma$ (*t*). Since *c*  $\hat{J} = \tilde{J}$ 

Hence completes the proof of the Proposition.

Next we need the concept of a hyperpolar action, [15], [12]. Let *G* be a lie group acting properly and isometrically on a complete Riemannian manifold *X* . A *section* is a connected, complete (necessarily totally geodesic ) submanifold  $\Sigma$  of  $X$  that meets all the orbits of  $G$  in such

**Example 1** *Md.* **Showkat Ali and Md. Mazharul A:<br>
<b>Proof.** We assume that the action of  $G$  on  $X$  is a way that the intersections between  $\Sigma$  and the orbits or variationally complete and prove that so is the action o are all orthogonal. An action admitting a section is called *polar*, and an action admitting a section that is flat in the induces metric is called hyperpolar. Conlon proved in [6] that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete, and it was proved in [8] that the converse of his result is true in the case in which the Riemannian manifold is a compact symmetric space. Notice that neither do we assume as Conlon in [6] that  $\Sigma$  is closed nor do we assume that it is properly embedded as is usually required in the recent literature on the subject. Polar and hyperpolar actions can also be defined for proper Fredholm actions of Hilbert Lie groups on Riemannian Hilbert manifolds [17].

- Proposition 1. If the action of  $\hat{G}$  on  $\hat{X}$  is polar (resp. **Proposition** 2. Let  $\hat{G}$ ,  $G$ ,  $\hat{X}$ ,  $X$ , and  $\pi$  be as in hyperpolar), then the action of  $G$  on  $X$  is polar (resp. hyperpolar).

**Proof.** Let  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\Sigma$ is horizontal with respect to  $\pi$ , we have that  $\pi|_{\Sigma}:\Sigma\to X$  is an isometric immersion. It is clear that  $\pi(\Sigma)$  is an immersed submanofold of X that meets all the *G* -orbits perpendicularly. Moreover,  $\pi(\Sigma)$  is flat if  $\Sigma$  is flat. Hence completes the proof.

## **III. The proof of Theorem 2**

 $\rho_* \hat{\xi} = \xi$ , and  $\hat{G}$  is an effective, transitive connected group of isometries are  $\hat{G}$  -trans-<br> $T_{x0}M$ , and the given Riemannian metric in M is the G that each  $\hat{\gamma}_u$  is a horizonal curve. It **byperpolar**).<br>
there  $\gamma_u = \pi \hat{\gamma}_u$ , defines a variation of **Proof**, let  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\Sigma$ <br>  $\hat{G}$ -transversal geodesics in  $X$ orizontal curve. It hyperpolar).<br> *ines* a variation of **Proof.** Let  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\Sigma$ <br>
desises in  $X$ . In bright with respect to  $\pi$ , we have that<br> **z**  $\pi |_{\Sigma} : \Sigma \rightarrow X$  is ciated  $\hat{G}$ -transversal Jacobi field  $\pi_{\Sigma} \Sigma > 7$ . an influence tunnitation. The such and  $\pi(\Sigma)$  is an influence tunnitation of  $\pi(\Sigma)$  is flat if  $\Sigma$  is the  $\hat{G}$ -orbit through  $x := \pi(\hat{X})$  and flat. Hence complet *vrsal* Jacobi field  $\pi | \Sigma^2 \rightarrow \pi \times \pi$  an immersed submanofold of  $X$  that meets all the<br>
This implies that  $G$ -orbits perpendicularly. Moreover,  $\pi(\Sigma)$  is flat if  $\Sigma$  is<br>  $h \times := \pi(\hat{x})$  and that. Hence completes the proo J along  $\gamma$  satisfies  $J(t) = \pi \sqrt{J(t)}$ . This implies that  $G = \text{Orbit}$  *A*  $\rightarrow$  *C*  $\rightarrow$  *C* as *A* horizontal completes the proof.<br>  $J(0)$  is tangent to the  $\vec{G}$ -orbit through  $x := \pi(\hat{x})$  and  $\vec{B}$  and  $\vec{B}$  and  $\vec{B$ Let *M* be a connected Riemannian homogeneous space. in the case in which the Riemannian manifold is a compact<br>
Symmetric space. Notice that neither dowe assume that it is<br>
Conlon in [6] that  $\Sigma$  is closed nor dowe assume that it is<br>
properly embedded as is usually require of *M* and *K* is the isotropy subgroup at a chosen base point  $x_0$ . Notice that *K* is compact. Let g and f be the Lie algebras of  $G$  and  $K$ , respectively. Then there is an **Proposition** 2. Let  $\hat{G}, G, \hat{X}, X$ , and  $\pi$  be as in<br>Proposition 1. If the action of  $\hat{G}$  on  $\hat{X}$  is polar (resp.<br>phyperpolar), then the action of  $G$  on  $\hat{X}$  is polar (resp.<br>phyperpolar).<br>**Proof.** Let  $\Sigma$  be called a reductive decomposition of *M* . The differential of **Proposition 1.** If the action of  $\hat{G}$  on  $\hat{X}$  is polar (resp.<br>hyperpolar), then the action of  $G$  on  $X$  is polar (resp.<br>hyperpolar).<br>**Proof.** Let  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\Sigma$ <br>i *T*(*Posistion* 1: The diaction of  $G$  on  $A$  is polar (resp.<br> *T*(*Positional*). (hen the action of  $G$  on  $X$  is polar (resp.<br> *T*(*T*) *T*) *T*(*T*) *Let*  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\S$ hyperpolar).<br> **Proof.** Let  $\Sigma$  be a section of the action of  $\hat{G}$  on  $\hat{X}$ . Since  $\Sigma$ <br>
is horizontal with respect to  $\pi$ , we have that<br>  $\pi|_{\Sigma}:\Sigma \rightarrow X$  is an isometric immersion. It is clear that<br>  $\pi(\Sigma)$  is an imm positive definite symmetric bilinear from on p. Now *M* is **Profit.** Let  $\Sigma$  oe a secono of nee action of  $\nabla$  on  $\Lambda$ . Since  $\Lambda$ <br>is horizontal with respect to  $\pi$ , we have that  $\pi|_{\Sigma}:\Sigma \rightarrow X$  is an isometric immersion. It is clear that  $\pi(\Sigma)$  is an immersed submanofold of invariant positive definite symmetric bilinear from on g such that f and p are orthogonal complements and whose restriction to p induces the given metric in *M* . *G*-orbits perpendicularly. Moreover,  $\pi(\Sigma)$  is flat if  $\Sigma$  is<br>flat. Hence completes the proof. <br>**III.** The proof of Theorem 2<br>Let *M* be a connected Riemannian homogeneous space.<br>Then *M* can be identified with a coset

*H* be a compact Lie group acting by isometries on a *M* . Without loss of generality, we may assume that  $G$  is a compact semisimple Lie group, and  $H$  is a closed subgroup of *G* . We will prove that the action of *H* on *M* is hyperpolar if and only if it is variationally complete.

A hyperpolar action is always variationally complete according to Colon's theorem. In order to prove the converse, we follow the same idea of the proof of the main

result of [8]. We first lift the action of *H* on *M* to a **Proof.** Suppose that *J* is a *K* -transversal Jacobi field variationally complete action of a path group on a Hilbert space. A Lemma in [8] asserts that this action is hyperpolar.

*M* to an action of a path group on a Hilbert space, [17],<br>
[18] Fel for details about this construction Recall that  $G$  is hypothesis. In case  $J(1) \neq 0$ , since  $J(1)$  is tangent to the [18], [8] for details about this construction. Recall that *G* is equipped with a bi-invariant Riemannian metric. Let Jacobi Fields On Normal Homogeneous Riemannian Manifolds<br>
result of [8]. We first lift the action of *H* on *M* to a<br> **Proof.** Suppose that *J* is a *K*-transvers<br>
variationally complete action of a path group on a Hilber Jacobi Fields On Normal Homogeneous Riemannian Manifolds<br>
result of [8]. We first lift the action of *II* on *M* to a **Proof.** Suppose that *J* is a *K*-transversal Jacobi field<br>
variationally complete action of a path gr Hilbert Lie group of absolutely continuous paths By hypothesis,  $\overline{J}$  is K-isotropic It follows that J is also  $u:[0,1] \to G$  whose derivative is square integrable. Then Jacobi Fields On Normal Homogeneous Riemannian Manifolds<br>
result of [8]. We first lift the action of *H* on *M* to a<br>
result of [8] asserts that this scient in [8] asserts on *S* are point  $x_0$ ,  $J(0) = 0$  and  $J(0) = 0$  a by affine isometries via  $g^*u = g u g^{-1}$ , As we said above. Theorem 2 now Jacobi Fields On Normal Homogeneous Riemannian Manifolds<br>
result of [8]. We first lift the action of *H* on *M* to a **Proof.** Suppose that *J* is a *K*-transversal Jacobi field<br>
variationally complete action of a path gro the parallel transport map defined by  $\varphi k(u) = g_u$  (1) K, where  $g_u \in p(G)$  is the unique **Excelure Fields On Normal Hormogeneous Riemannian Manifolds<br>
<b>Excelure of Fig. We first lift the action of H on M to a Proof.** Suppose that J is a *K*-transversal Jacobi field<br>
synchology and Hormogeneous of a particul solution of  $g_{u}^{-1}g_{u} = u$ ,  $g_{u}(0) = 1$ . Bacobi Fields On Normal Homogeneous Riemannian Manifolds<br>
y result of [8]. We first lift the action of *H* on *M* to a **Proof.** Suppose that *J* is a *K*-transversal Jacobi field<br>
symmetric lighting the close of and negat staton reads of consists in a monogeneous selenamian valuitions<br>
result of  $\{S\}$ . We first this decision is hyperpolar.<br>
variationally complete action of a path group on a Hilbert along a K-transversal geodesic  $\gamma$  suc variationally complete action of a poth group on a Hilbert linear points and  $K$ -transversal geodesic  $Y$  such that space. A Lemma in [8] spaces that this action is hyperpolar.<br>
So we begin by constructing a lift of the a base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangento the equal of the action of  $H$  on a critic of a path group on a Hilbert space, [17], by rorbith trough  $\gamma(1)$ . If  $J(1) = 0$ , then  $J$  is  $S$  is for details about this co **Example 1.** If  $X_0$ ,  $J(0) = 0$  and  $J(1)$  is tangen<br>
For the causar in the action of  $I$  on  $M$  is a River thirtong  $\gamma(1)$ . If  $J(1) = 0$ , then  $J$  is  $K$ <br>  $K = L^2$  ([0,1],  $L$ ] (18) consider that is construction. Recall

 $g(0) \in H$  and  $g_u(1)K$ . Then it is known that: the action of  $p(G, H \times K)$  on  $V_G$  is proper, Fredhlom and isometric; of K on  $T_{x0}M \cong p$  is polar with Abelian subalgebras as  $g(0) = 1$ . solution of  $g_a^{-1}g_a = u$ ,  $g_u(0) = 1$ .<br>
Let  $p(G, H \times K)$  denote the closed, finite codimensional remark that in [11] remarks the proof *F* on *K* is the proof *F* on *K* is the proof *H* on *K* and the proof *F* on *K* is the

Now we can apply Proposition 1 to the equivariant the action of  $p(G, H \times K)$  on  $V_G$  is variationally complete.  $V_G$  is hyperpolar, and Proposition 2 implies that the action  $\tau$ Theorem 2.

#### **IV. The proof of Theorem 1**

*G* is compact and semisimple as in the last section, and suppose that *M* satisfies the hypothesis of Theorem 1. We first prove a following Lemma 1 that states that the isotropy action of *K* on *M* is variationally complete. It then follows from Theorem 2 that this action is hyperpolar. Therefore we can rely on the results of [11] to deduce that *M* is a symmetric space.

**Lemma 1.** If every Jacobi field on *M* vanishing at two points is isotropic, then the isotropy action of *K* on *M* is variationally complete.

So we begin by constructing a lift of the action of *H* on<br>So we begin by constructing a lift of the action of *H* on<br>orbit through  $\gamma(1)$ . If  $J(1) = 0$ , then *J* is *K*-isotropic by **bus Riemannian Manifolds**<br> **Solution** 18 Becommental Manifolds<br> **Solution** 18 Becommental Proof. Suppose that *J* is a *K* -transversal Jacobi field<br> **Example 18 A -transversal geodesic** *y* such that  $\gamma(0)$  is the<br>
ins along a *K*-transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the **Base 10** *M* and *J* is a *K* -transversal Jacobi field along a *K* -transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the *K* - orbit through  $\gamma(1)$ . If  $J(1) = 0$ **Proof.** Suppose that  $J$  is a  $K$ -transversal Jacobi field<br>along a  $K$ -transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the<br>base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the  $K$ -<br>orbit through  $\gamma(1)$ . If  $J(1) = 0$ , K -orbit through  $\gamma(1)$ , there is a K -Killing field X on **Proof.** Suppose that *J* is a *K* -transversal Jacobi field<br>along a *K* -transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the<br>base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the *K* -<br>orbit through  $\gamma(1)$ . If  $J(1) =$ **Proof.** Suppose that *J* is a *K* -transversal Jacobi field<br>along a *K* -transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the<br>base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the *K* -<br>orbit through  $\gamma(1)$ . If  $J(1) =$ *K* -isotropic. **33**<br> **Proof.** Suppose that *J* is a *K*-transversal Jacobi field<br>
along a *K*-transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the<br>
sase point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is targent to the *K*-<br>
by typothesis. In case  $J(1$ 

so we regar to you within the matter in the section in the section of the state in the section of a path gas on the section in the section of the epimor of the epimor of the path at  $G$  is the control of equiperate to the 18], [8] for details show this construction. Recall that  $G$  is hypothesis. In case  $J(1) \neq 0$ , since  $J(0) = 0$ ,  $J(0) = 0$  and  $V_0 = L^2(0,1,1,0)$  is angle to the Hilbert space of  $L^2$ . *M* such that  $J(1) = X \cdot \gamma(1)$ . Now at this construction. Receal that  $\widetilde{G}$  is bypothesis. In case  $J(1) \neq 0$ , since  $J(1)$  is tangent to the<br>nivariant Nemannian metric. Let  $K$  crotic through  $y(1)$ , there is a  $K$ -Killing field  $X$  on<br>note the Hither equipped with a bi-invariant Reiemannian territ. I.e.  $K$ -orbit through  $\gamma(1)$ , there is a  $K$ -Killing field  $X$  on<br>thegrable paths as i. [0,1]  $\rightarrow$  g, and let  $p(G)$  denote the  $\gamma(1)$  and  $\lambda(1) = X$ ,  $\gamma(1)$ . Now  $\overline{J}($  $V_c = L^r([0,1],\Box)$  donote the Hilbert space of  $L^r$ . *M* such that  $J(1) = X$ .  $\gamma(1)$ . Now  $\overline{J}(t) = J(t) - \overline{J}$ , interfering the proposition of position of  $\gamma(t)$  is a notion of the proposition of the proposition of the pr p(G) acts on  $V_G$  by affine isometries via  $\frac{u}{g} + u = u \frac{1}{g} + \frac{1}{g}$ . As we said above, Theorem 2 now yields that the action of  $\frac{1}{g}$  of the particle of  $\frac{1}{g}$  of  $\frac{1}{g}$  of  $\frac{1}{g}$  of  $\frac{1}{g}$  of  $\frac{1}{g}$ **Example 12** and the equilibrium of the equivariant subsequent of the equivariant poperation of  $\mathbb{F}_G$ . Let  $\varphi k : V_G \to G / K$  be  $K$  on  $G / K$  is thyperpolar, in order to finish the proof, we equid the proof many defined b Extremely the action of  $\mu$  and  $\mu$  of  $\mu$  on  $\mu$  and  $\mu$  ( $\mu$  =  $\mu$  ( $G$ )  $H$ , where  $g \in \rho(G)$  and  $\mu \in V_G$ . The action of p  $\mu$  and  $\mu$  is Lemma 3 in the species of the properties that the equivariant species of (11). It implement to noise the parallel transport  $\phi K(t) = g$ , (1) *K*, where  $g_n \in p(G)$  is the unique upperpointly in [11] require section to be prop As we said above, Theorem 2 now yields that the action of need to invoke the results of [11]. It is important to notice that even through the definitions of polarity and hyperpolarity in [11] require sections to be properly embedded, this will not affect our argument. Indeed, we first remark that in [11] remains true even if the sections of the *K* -action on *M* are not properly embedded. This can be seen by noticing that the proof of Proposition in [1] does not use that sections are properly embedded. Now it follows as in Proposition in [11] that the linear isotropy representation along a K-transversal geodesic  $\gamma$  such that  $\gamma(0)$  is the<br>base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the K-<br>orbit through  $\gamma(1)$ . If  $J(1) = 0$ , then  $J$  is K-isotropic by<br>hypothesis. In case  $J(1) \neq 0$ , sin base point  $x_0$ ,  $J(0) = 0$  and  $J(1)$  is tangent to the  $K$ -<br>orbit through  $\gamma(1)$ . If  $J(1) = 0$ , then  $J$  is  $K$ -isotropic by<br>hypothesis. In case  $J(1) \neq 0$ , since  $J(1)$  is tangent to the<br> $K$ -orbit through  $\gamma(1)$ , th polar pair. Hence it follows from the classification of polar both anotal  $\mu(\gamma)$ . It  $v = v$ , then  $v = v$ , then  $v = v$  is  $A$ -solotope by<br>hypothesis. In case  $J(1) \neq 0$ , since  $J(1)$  is tangent to the<br>*K*-orbit through  $\gamma(1)$ , there is a *K*-Killing field *X* on<br>*M* such that  $J(1) = X$ space and this finishes the proof of Theorem 1.

#### **V. Other Remarks**

We would like to take this opportunity to make some remarks about variationally complete actions and the related class of taut submanifolds of complete Riemannian manifolds.

subgroup of P(U) consisting of those paths g such that the action are properly embedded. Not  $\left[\rho(G,H \times K) \text{ on } V_G$  is proper, Fredhlom and isometric; of K on  $T_{\gamma,0}M = p$  is polar with Abelian  $qk:V_G \rightarrow G/K$  is a Riemannian eq An embedded submanifold *M* of a complete Riemannian manifold  $N$  is called reflective if  $M$  is complete with respect to the induced metric and it is a connected component of the fixed point set of an involutive isometry  $\tau$  of N. By a well known result about fixed points of isometries, every reflective submanifold is automatically totally geodesic. Reflective submanifolds of simply connected symmetric spaces were completely classified by Leung [13]. It is not difficult to see that a reflective submanifold  $M$  of a compact symmetric space  $N$  is an orbit of a symmetric subgroup *H* of the connected component *G* of the isometry group of *N* . Namely, *H* can be taken to be the connected component of the centralizer of  $\tau$  in  $G$ . It follows from a theorem of Hermann [9] that the action of  $H$  on  $N$  is variationally complete [12]. It is easy to see that this implies that the conjugate locus of *M* in *N* is the union of the singular  $H$ -orbits in  $N$ , a result which was reproved by Burns in [3] by direct calculations.

> A properly embedded submanifold *M* of a complete Riemannian manifold *N* is called taut if, for some coefficient field, the energy functional

 $E_q$ :  $p(N, M \times q) \rightarrow R$  is a perfect Morse function for p ( $N, M \times q$ ) → R is a perfect Morse function for<br>  $q \in N$  that is not a focal point of  $M$ , where<br>  $M \times q$ ) denotes the space of  $H^1$ -paths 11. Heintze E, X. Lin, and C. Olmos, 2000, Isopara<br>  $M \times q$ ) denotes the space of Md. Showkat Ali and Md. Mazharul Anwar<br>  $E_q : p(N, M \times q) \rightarrow R$  is a perfect Morse function for<br>
every  $q \in N$  that is not a focal point of *M*, where<br>  $p(N, M \times q)$  denotes the space of *H*<sup>1</sup>-paths 11. Heintze E., R. Palais C.-L. Md. Showkat Ali and Md. Mazharul <br>  $E_q : p(N, M \times q) \rightarrow \mathbb{R}$  is a perfect Morse function for<br>  $\psi_q \in \mathbb{R}$  and C. Olmos, 2000, Isopara<br>  $p(N, M \times q)$  denotes the space of  $H^1$ -paths 11. Heintze E., X. Lin, and C. Olmos, 2000,  $\gamma$ :[0,1]  $\rightarrow$  *N* such that  $\gamma$ (0)  $\in$  *M* and  $\gamma$ (1) = *q* [7], [19]. Bott and Samelson [2] proved that the orbits of variationally  $\frac{1}{12}$ complete actions are taut submanifolds. Hence it follows from the remarks in the preceeding paragraph that reflective submanifolds of compact symmetric spaces are taut submanifolds. This partially answers a question raised by Terng and Thorbergsson in [19] that whether totally 13 geodesic submanifolds of compact symmetric spaces are always taut.

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