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Jacobi Fields on Normal Homogeneous Riemannian Manifolds

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Abstract

We prove that a normal homogeneous space with the property that every Jacobi field along a geodesic vanishing at two points is the restriction of a Killing field along that geodesic is a globally symmetric space.

I. Introduction

On a symmetric space, the Jacobi equation can be easily solved, and it follows from the work of Boot and Samelson [2] that a Jacobi field along a geodesic vanishing at two distinct points is the restriction of a Killing field along that geodesic. A Jacobi field which is the restriction of a Killing field along a geodesic is called *isotropic*.

In the case of a naturally reductive space, Chavel [4], [16] showed how to write the Jacobi equation in an efficient manner. By means of explicit calculations, he then proved that a simply connected normal homogeneous space of rank one with the property that any Jacobi field vanishing at two points is isotropic is homeomorphic to a rank one symmetric space [5].

Later, Ziller [20] studied the Jacobi equation on naturally reductive spaces, and obtained several interesting results about the structure of the space of Jacobi fields. This led him to suggest the following conjecture:

A naturally reductive homogeneous space with the property that every Jacobi field vanishing at two points is isotropic, is locally symmetric. In this paper, we prove that this conjecture is true in the case of a normal homogeneous space.

Theorem 1. *A normal homogeneous space with the property that every Jacobi field vanishing at two points is isotropic is a globally symmetric space.*

As a preparation for the proof above theorem, the following theorem of [8]

Theorem 2. *An isotropic action of a compact Lie group on a normal homogeneous space is variationally complete if and only if it is hyperpolar.*

II. Preliminaries

We first recall the notion of variationally completeness introduced by Bott and Samelson in [2]. Let G be a Lie group acting properly and isometrically on a complete Riemannian manifold X.

A geodesic γ in X is called G-*transversal* if it is perpendicular to every G-orbit it meets. A Jacobi field along a geodesic is called G-*isotropic* if it is the restriction of a G-killing field along that geodesic. A Jacobi field along a G-transversal geodesic γ is called G-transversal if it is the variational vector field associated to a variation of γ through G-transversal geodesics which are at time zero perpendicular to the same G-orbit. Notice that a G-isotropic Jacobi field along a G-transversal geodesic is a G-transversal Jacobi field which is moreover tangent to the G-orbit at every point. The action of G on X is called *variationally complete* if every G-transversal Jacobi field that is tangent to the G-orbits at two diffeent points is G-isotropic. It is proved in [2] is enough to impose this condition on G-transversal Jacobi fields that are tangent to the G-orbit at one point and vanish at another one.

Variational completeness can also be defined for proper Fredholm isometric actions of Hilbert Lie group on complete Riemannian Hilbert manifolds [15], [17]. It is not difficult to show that an orbit of a Fredholm isometric action has the property that the normal exponential map is a nonlinear Fredholm map of index zero. This has as pleasant consequence inter alia that monofocal and epifocal points along a normal geodesic must coincide and have finite multiplicities, no clustering of focal points along finite normal geodesic segments is permitted, and the set of focal points to the orbit is of first Baire category [14].

It is known and easy to see that a G-transversal Jacobi field J along a G-transversal geodesic γ in a Riemannian manifold X (of finite or infinite dimension) has the property that $J(0)$ is tangent to the orbit Gx and $J'(0)+A_vJ(0)$ is normal to Gx, where $\gamma(0)=x$, $\gamma'(0)=v$, and A_v is the Weingarten operator of Gx with respect to v.

The next proposition 1 generalizes Lemmas 1 and 2 in [8].

Proposition 1. Let $\hat{G}$ and G be two Hilbert Lie groups equipped with proper Fredholm isometric actions on the complete Riemannian Hilbert manifolds $\hat{X}$ and X, and suppose that there is an equivariant Riemannian submersion $\pi:\hat{X}\to X$ with respect to an epimorphism $\rho:\hat{G}\to G$ such that $\pi:\hat{X}\to X$ is a principal $\hat{H}$-bundle for a Hilbert Lie subgroup $\hat{H}$ of $\hat{G}$. Then the action of $\hat{G}$ on $\hat{X}$ is *variationally complete* if and only the action of G on X is variationally complete.

Proof. We assume that the action of G on X is variationally complete and prove that so is the action of $\hat{G}$ on $\hat{X}$. The other direction is easier.

Let $\hat{\gamma}$ be a $\hat{G}$-transversal geodesic in $\hat{X}$ defined on $[0,1]$ and let $\hat{J}$ be a $\hat{G}$-transversal Jacobi field along $\hat{\gamma}$ such that $\hat{J}(0)$ is tangent to the $\hat{G}$-orbit through $\hat{\gamma}(0)=\hat{x}$ and $\hat{J}(1)=0$. We must show that $\hat{J}$ is the restriction along $\hat{\gamma}$ of a $\hat{G}$-Killing field.

There is a variation $\{\hat{\gamma}_\alpha\}$ of $\hat{\gamma}$ through $\hat{G}$-transversal geodesic whose associated variational vector field is $\hat{J}$. Since the $\hat{G}$-orbits are the preimages under π of the $\hat{G}$-orbits, the normal spaces to the $\hat{G}$-orbits are horizontal with respect to π, so that each $\hat{\gamma}_\alpha$ is a horizontal curve. It follows that $\{\gamma_\alpha\}$, where $\gamma_\alpha = \pi\hat{\gamma}_\alpha$, defines a variation of $\gamma = \pi\hat{\gamma}$ through $\hat{G}$-transversal geodesics in X. Moreover, the associated $\hat{G}$-transversal Jacobi field J along γ satisfies $J(t)=\pi_* \hat{J}(t)$. This implies that $J(0)$ is tangent to the $\hat{G}$-orbit through $x := \pi(\hat{x})$ and $J(1)=0$. By variational completeness of the action of G on X, there is a G-Killing field ξ on X such that $J(t)=\xi\cdot\gamma(t)$. Since $\rho:\hat{G}\to G$ is an epimorphism, there is a $\hat{G}$-Killing field ξ on X such that $\rho_*\hat{\xi}=\xi$, and since the $\hat{H}$-orbits coincide with the fibers of π, we may subtract an $\hat{H}$-Killing field from $\hat{\xi}$ if necessary and assume furthermore that $\hat{\xi}\cdot\hat{x}=\hat{J}(0)$.

Let $\tilde{J}(t)=\hat{\xi}\cdot\hat{\gamma}(t)$. Then $\tilde{J}$ is a $\hat{G}$-isotropic Jacobi field along $\hat{\gamma}$. Since the Jacobi field $\hat{J}$ and $\tilde{J}$ are $\hat{G}$-transversal, we have that $\hat{J}'(0)+A_{\hat{v}}\hat{u}$ and $\tilde{J}'(0)+A_{\hat{v}}\hat{u}$ are normal to $\hat{G}\hat{x}$, where $\hat{v}=\hat{\gamma}'(0)$ and $\hat{u}=\hat{J}(0)=\tilde{J}(0)$. Since the normal spaces to the $\hat{G}$-orbits are horizontal, it follows that both $\hat{J}'(0)+A_{\hat{v}}\hat{u}$ and $\tilde{J}'(0)+A_{\hat{v}}\hat{u}$ are horizontal lifts of $J'(0)+A_v J(0)$, where $v=\gamma'(0)$. Hence $\hat{J}'(0)=\tilde{J}'(0)$, from where we conclude that $\hat{J}=\tilde{J}$.

Hence completes the proof of the Proposition. □

Next we need the concept of a hyperpolar action, [15], [12]. Let G be a lie group acting properly and isometrically on a complete Riemannian manifold X. A *section* is a connected, complete (necessarily totally geodesic) submanifold Σ of X that meets all the orbits of G in such a way that the intersections between Σ and the orbits of G are all orthogonal. An action admitting a section is called *polar*, and an action admitting a section that is flat in the induces metric is called hyperpolar. Conlon proved in [6] that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete, and it was proved in [8] that the converse of his result is true in the case in which the Riemannian manifold is a compact symmetric space. Notice that neither do we assume as Conlon in [6] that Σ is closed nor do we assume that it is properly embedded as is usually required in the recent literature on the subject. Polar and hyperpolar actions can also be defined for proper Fredholm actions of Hilbert Lie groups on Riemannian Hilbert manifolds [17].

Proposition 2. Let $\hat{G}, G, \hat{X}, X$, and π be as in Proposition 1. If the action of $\hat{G}$ on $\hat{X}$ is polar (resp. hyperpolar), then the action of G on X is polar (resp. hyperpolar).

Proof. Let Σ be a section of the action of $\hat{G}$ on $\hat{X}$. Since Σ is horizontal with respect to π, we have that $\pi|_\Sigma : \Sigma \to X$ is an isometric immersion. It is clear that $\pi(\Sigma)$ is an immersed submanofold of X that meets all the G-orbits perpendicularly. Moreover, $\pi(\Sigma)$ is flat if Σ is flat. Hence completes the proof. □

III. The proof of Theorem 2

Let M be a connected Riemannian homogeneous space. Then M can be identified with a coset space G/K, where G is an effective, transitive connected group of isometries of M and K is the isotropy subgroup at a chosen base point x_0. Notice that K is compact. Let g and f be the Lie algebras of G and K, respectively. Then there is an $\mathrm{Ad}(K)$-invariant vector space direct sum g = f + p ; this is called a reductive decomposition of M. The differential of the projection $G \to G/K$ at the identity indentifies p and $T_{x0}M$, and the given Riemannian metric in M is the G-invariant metric induced from an $\mathrm{Ad}(K)$-invariant positive definite symmetric bilinear from on p. Now M is called normal homogeneous if there exists an $\mathrm{Ad}(G)$-invariant positive definite symmetric bilinear from on g such that f and p are orthogonal complements and whose restriction to p induces the given metric in M.

Let $M=G/K$ be a normal homogeneous space, and let H be a compact Lie group acting by isometries on a M. Without loss of generality, we may assume that G is a compact semisimple Lie group, and H is a closed subgroup of G. We will prove that the action of H on M is hyperpolar if and only if it is variationally complete.

A hyperpolar action is always variationally complete according to Colon's theorem. In order to prove the converse, we follow the same idea of the proof of the main result of [8]. We first lift the action of H on M to a variationally complete action of a path group on a Hilbert space. A Lemma in [8] asserts that this action is hyperpolar.

So we begin by constructing a lift of the action of H on M to an action of a path group on a Hilbert space, [17], [18], [8] for details about this construction. Recall that G is equipped with a bi-invariant Riemannian metric. Let $V_G = L^2([0,1], \square)$ donote the Hilbert space of L^2-integrable paths $u:[0,1] \to \mathrm{g}$, and let $\mathrm{p}(G)$ denote the Hilbert Lie group of absolutely continuous paths $\mathrm{u}:[0,1] \to G$ whose derivative is square integrable. Then $\mathrm{p}(G)$ acts on V_G by affine isometries via $\mathrm{g}*u = \mathrm{g}\ u\,\mathrm{g}^{-1}$, where $\mathrm{g} \in \mathrm{p}(G)$ and $u \in V_G$. Let $\varphi k : V_G \to G/K$ be the parallel transport map defined by $\varphi k(u) = \mathrm{g}_u\ (1)K$,where $\mathrm{g}_u \in\ \mathrm{p}(G)$ is the unique solution of $\mathrm{g}_u\ {}^{-1}\mathrm{g}_u = u$, $\mathrm{g}_u(0) = 1$.

Let $\mathrm{p}(G, H \times K)$ denote the closed, finite codimensional subgroup of $\mathrm{p}(G)$ consisting of those paths g such that $\mathrm{g}(0) \in H$ and $\mathrm{g}_u\ (1)K$. Then it is known that: the action of $\mathrm{p}(G, H \times K)$ on V_G is proper, Fredhlom and isometric; $\varphi k : V_G \to G/K$ is a Riemannian equivariant submersion with respect to the epimorphism $\mathrm{g} \in \mathrm{p}\ (G,H{\times}K) \mapsto \mathrm{g}(0) \in H$; and $\varphi k : V_G \to G/K$ is a principle $\mathrm{p}(G, 1\times K)$-bundle, where $\mathrm{p}(G, 1\times K)$ denotes the subgroup of $\mathrm{p}(G, H \times K)$ consisting of paths g such that $\mathrm{g}(0) = 1$.

Now we can apply Proposition 1 to the equivariant Riemannian submersion $\varphi k : V_G \to G/K$ to deduce that the action of $\mathrm{p}(G, H \times K)$ on V_G is variationally complete. Lemma 3 in [2] implies that the action of $\mathrm{p}(G, H \times K)$ on V_G is hyperpolar, and Proposition 2 implies that the action of H on G/K is hyperpolar. This finishes the proof of Theorem 2. □

IV. The proof of Theorem 1

Let $M = G/K$ be a normal homogeneous spaces where G is compact and semisimple as in the last section, and suppose that M satisfies the hypothesis of Theorem 1. We first prove a following Lemma 1 that states that the isotropy action of K on M is variationally complete. It then follows from Theorem 2 that this action is hyperpolar. Therefore we can rely on the results of [11] to deduce that M is a symmetric space.

Lemma 1. If every Jacobi field on M vanishing at two points is isotropic, then the isotropy action of K on M is variationally complete.

Proof. Suppose that J is a K-transversal Jacobi field along a K-transversal geodesic γ such that $\gamma(0)$ is the base point x_0, $J(0) = 0$ and $J(1)$ is tangent to the K-orbit through $\gamma(1)$. If $J(1) = 0$, then J is K-isotropic by hypothesis. In case $J(1) \neq 0$, since $J(1)$ is tangent to the K-orbit through $\gamma(1)$, there is a K-Killing field X on M such that $J(1) = X\,.\ \gamma(1)$. Now $\bar{J}(t) = J(t) - X\,.\ \gamma(t)$ is a Jacobi field along γ vanishing at $t = 0$ and $t = 1$. By hypothesis, $\bar{J}$ is K-isotropic. It follows that J is also K-isotropic. □

As we said above, Theorem 2 now yields that the action of K on G/K is hyperpolar. In order to finish the proof, we need to invoke the results of [11]. It is important to notice that even through the definitions of polarity and hyperpolarity in [11] require sections to be properly embedded, this will not affect our argument. Indeed, we first remark that in [11] remains true even if the sections of the K-action on M are not properly embedded. This can be seen by noticing that the proof of Proposition in [1] does not use that sections are properly embedded. Now it follows as in Proposition in [11] that the linear isotropy representation of K on $T_{x0}M \cong \mathrm{p}$ is polar with Abelian subalgebras as sections, or, in the language of that paper, that (G, K) is a polar pair. Hence it follows from the classification of polar pairs in that same paper that G/K is a globally symmetric space and this finishes the proof of Theorem 1. □

V. Other Remarks

We would like to take this opportunity to make some remarks about variationally complete actions and the related class of taut submanifolds of complete Riemannian manifolds.

An embedded submanifold M of a complete Riemannian manifold N is called reflective if M is complete with respect to the induced metric and it is a connected component of the fixed point set of an involutive isometry τ of N. By a well known result about fixed points of isometries, every reflective submanifold is automatically totally geodesic. Reflective submanifolds of simply connected symmetric spaces were completely classified by Leung [13]. It is not difficult to see that a reflective submanifold M of a compact symmetric space N is an orbit of a symmetric subgroup H of the connected component G of the isometry group of N. Namely, H can be taken to be the connected component of the centralizer of τ in G. It follows from a theorem of Hermann [9] that the action of H on N is variationally complete [12]. It is easy to see that this implies that the conjugate locus of M in N is the union of the singular H-orbits in N, a result which was reproved by Burns in [3] by direct calculations.

A properly embedded submanifold M of a complete Riemannian manifold N is called taut if, for some coefficient field, the energy functional

$E_q : \mathrm{p}(N, M \times q) \to \mathrm{R}$ is a perfect Morse function for every $q \in N$ that is not a focal point of M, where $\mathrm{p}(N, M \times q)$ denotes the space of H^1-paths $\gamma : [0,1] \to N$ such that $\gamma(0) \in M$ and $\gamma(1) = q$ [7], [19]. Bott and Samelson [2] proved that the orbits of variationally complete actions are taut submanifolds. Hence it follows from the remarks in the preceeding paragraph that reflective submanifolds of compact symmetric spaces are taut submanifolds. This partially answers a question raised by Terng and Thorbergsson in [19] that whether totally geodesic submanifolds of compact symmetric spaces are always taut.

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