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Equivalence of Duals in Linear Fractional Programming

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Abstract

In this paper, different types of dual problems associated with a Linear Fractional Programming problem are discussed. A comparative study is made on different duals of Linear Fractional Program and is shown that some of the duals are equivalent to one another. Finally conclusions are drawn for different duals.

Keywords: Linear Fractional Programming, duality.

I. Introduction

Every mathematical programming problem has an associated dual problem. The relationship between these two problems is very useful when investigating properties of optimal solutions of both problems. For any LFP problem (primal problem), a dual problem can be constructed which is very closely connected with the original problem.

In this paper, different duals proposed by several authors are studied. Swarup[1], Bector[2,3], Chadha[4,5], Gol'estein[6,7], Sharma and Swarup[8], Seshan[9] and many other authors proposed different type of dual problems related to the primal LFP problem. A comparative study on duals proposed by several authors is made. It is shown that some of the duals are equivalent to one another. For, Gol'stein's[7] dual is equivalent to Seshan's[9] dual and also equivalent to Chadha's[5] dual. So Seshan's[9] dual and Chadha's[5] dual are equivalent. Also Chadha's dual and Bector's[2] linear dual are equivalent which implies the equivalence of Gol'stein's and Seshan's dual with that of Bector.

II. Dual of a Linear Fractional Program

Consider the following LFP problem (PP) as the primal problem:

(PP) : Maximize $Q(x)=\dfrac{C(x)}{D(x)}=\dfrac{c^t x+\alpha}{d^t x+\beta}$ (1)

subject to $Ax\leq b$ (2)

$x\geq 0$ (3)

where $d^t x+\beta>0$,

$\forall x=(x_1,x_2,...,x_n)^t\in S$, where $S=\{x: Ax\leq b, x\geq 0\}$ is the feasible set which has been assumed to be nonempty and bounded.

A is a m × n matrix,

$x,c,d\in\Re^n$

$b\in\Re^m$, $\alpha,\beta\in\Re$

c^t, d^t denotes transpose of vectors c and d respectively.

Bector's Dual

Bector[3] used Charnes and Cooper's[10] variable transformation technique and standard Lagrange function

$$L(x,y)=Q(x)+\sum_{i=1}^{m} y_i f_i(x)$$

to construct dual problem in fractional program in which

(a) The objective function, being the ratio of linear function to a strictly positive linear function, is a pseudo concave function.

(b) The constraint set is a closed convex polyhedral set in n-dimensional Euclidean space; such that all the duality theorem holds.

Bector[3] presented a dual problem in three equivalent forms. The major contribution of the work of Bector[3] is to introduce a linear dual program (LDP) using Charnes and Cooper's[10] transformation and then establish such relations among the dual problem (BDP), linear dual problem (LDP) and primal problem (PP). Solution of (LDP)gives the solution of (PP) and (BDP).

At first Bector[3] modified the primal problem (PP) as follows:

(MPP): Maximize $Q(x)=\dfrac{C(x)}{D(x)}=\dfrac{c^t x+\alpha}{d^t x+\beta}$ (4a)

Subject to $\dfrac{Ax-b}{d^t x+\beta}\leq 0$ (4b)

$x\geq 0$ (4c)

Where all symbols and conditions are same as in (PP) (1)-(3) and

$S_m=\{x:\dfrac{Ax-b}{d^t x+\beta}\leq 0, x\geq 0\}\subset S$, is the feasible set which has been assumed to be nonempty and bounded.

Bector proved the following result.

Theorem 1: If x^* maximizes Q over S in the primal problem (PP), then, x^* maximizes Q over S_m in the modified primal problem (MPP) also and vice versa.

Bector[3] applied **Kuhn – Tucker**[11] condition to the Lagrange function

$$\varphi(x,y)=Q(x)-\sum_{i=1}^{m} y_i\frac{Ax-b}{d^t x+\beta} \quad (5)$$

Bector's[3] dual problem (BDP) to the primal problem (PP) is

(BDP): Minimize $\varphi(x,y) = Q(x) - y^t \dfrac{Ax-b}{d^t x + \beta}$

$= \dfrac{c^t x + \alpha - y^t (Ax-b)}{d^t x + \beta}$

Subject to $\nabla_x \varphi(x,y) = 0$

$x \geq 0, \quad y \geq 0$

which is equivalent to

(BDP): Minimize $\varphi(x,y) = \dfrac{c^t x + \alpha - y^t (Ax-b)}{d^t x + \beta}$ (6a)

Subject to $\left.\begin{aligned} &\nabla_x Q(x) - \frac{y^t}{(d^t x+\beta)} \nabla_x (Ax-b) = 0 \\ &x \geq 0, \quad y \geq 0 \end{aligned}\right\}$ (6b)

where the feasible set

$S_{BDP} = \{(x,y) : \nabla_x Q(x) - \frac{y^t}{(d^t x+\beta)} \nabla_x (Ax-b) = 0, x \geq 0, y \geq 0\} \subset \Re^{n+m}$

Bector's Linear Dual Problem

Bector presented the linear dual problem to the primal problem (PP) by employing Charnes and Cooper's transformation $y = tx$ which is a homomorphism. According to Charnes and Cooper's linear transformation the primal problem (PP) is equivalent to the linear programming problem (ELP)

(ELP): Maximize $L(y,t) = c^t y + \alpha$ t (7a)

Subject to $\left.\begin{aligned} &Ay - bt \leq 0 \\ &d^t y + \beta\, t = 1 \\ &y \geq 0, t \geq 0 \end{aligned}\right\}$ (7b)

The dual problem to the above problem as in linear program is

(ELDP): Minimize $\phi = \lambda$ (8a)

Subject to $\left.\begin{aligned} &A^t \mu + d\lambda \geq c \\ &b^t \mu - \beta\lambda \leq - \alpha \\ &\mu \geq 0 \end{aligned}\right\}$ (8b)

Charnes and Cooper[10] also proved that, every (y,t) satisfying the constraints (7a) - (7b) have $t > 0$. Therefore by complementary slackness theorem of linear programming, at an optimal solution the inequality $b^t \mu - \beta\lambda \leq - \alpha$ must hold as an equation.

Hence the final form of the dual problem (8a)-(8b) becomes:

(ELDP): Minimize $\phi = \lambda$ (9a)

Subject to $\left.\begin{aligned} &A^t \mu + d\lambda \geq c \\ &b^t \mu - \beta\lambda = - \alpha \\ &\mu \geq 0 \end{aligned}\right\}$ (9b)

Bector's[3] linear dual problem (LDP) to the primal problem (PP) is

(LDP): Minimize $\psi(\mu) = \dfrac{\alpha + b^t \mu}{\beta}$ (10a)

Subject to $A^t \mu + d \dfrac{\alpha + b^t \mu}{\beta} \geq c$ (10b)

$\mu \geq 0$ (10c)

where the feasible set

$S_{LDP} = \{\mu : A^t \mu + d \dfrac{\alpha + b^t \mu}{\beta} \geq c, \mu \geq 0, \beta \neq 0\}$

Substituting $\lambda = \dfrac{\alpha + b^t \mu}{\beta}$ in LDP we have (LDP) is equivalent to (ELDP)

Since both the (LDP) and (ELP) are linear programming problems so a consequence of duality theory in LP implies that, if μ^* is an optimal solution to LDP then there exists (y^*, t^*) which is an optimal solution to ELP and according to Charnes and Cooper[10], $x^* = \dfrac{y^*}{t^*}$ is a solution to the primal problem (PP). Thus if μ^* is known then (y^*, t^*) and hence x^* can be obtained via simplex method. Bector[3] established the following result:

Theorem 2: For every $(x, y) \in S_{BDP}$, $\varphi(x,y) = \psi(\mu)$, and $\mu \in S_{LDP}$

Hence proved the weak, the direct and the converse duality theorems.

Chadha's Dual Problem

Chadha[5] presented the dual form to the primal problem (PP) which is a linear programming problem. Along with other duality theorems Chadha[5] proved complementary slackness theorem. Chadha has studied the duality for a restricted type of linear fractional functional programming problem under the assumption that

(a) The denominator of the objective function is strictly positive over the feasible region.

(b) $Ax \leq 0$ implies that $x = 0$.

And the feasible set is

$S_{CPP} = \{ x : Ax \leq b, x \geq 0 \}$

Chadha[5] proposed the following problem as the dual problem

(CDP): Minimize $g(y,z) = z$ (11a)

Subject to $A^t y + d\, z \geq c$ (11b)

$-b^t y + \beta\, z = \alpha$ (11c)

$y \geq 0$ (11d)

The feasible set to the dual problem is

$S_{CDP} = \{(y,z): A^t y + dz \geq c; -b^t y + \beta z = \alpha;\ y \geq 0\}$

Where $y \in \Re^m$ and $z \in \Re$.

Remark 1: Chadha[5] proved Weak Duality Theorem, Direct Duality Theorem, Complementary Slackness Theorem and Optimality Criteria Theorem. Chadha did not prove the converse duality theorem.

Gol'stein's Dual Problem

To define the dual to primal problem (PP) Gol'stein[7] introduced a special ratio type Lagrange function

$$L(x,y) = \frac{C(x) + \sum_{i=1}^{m} y_i f_i(x)}{D(x)}$$

Where $f_i(x) = b_i - \sum_{j=1}^{n} a_{ij} x_j$

And x and y are non-negetive.

A function $\psi(y)$ has been taken into consideration such that

$$\psi(y) = \max_{x \geq 0} L(x,y)$$

$$= \max_{x \geq 0} \frac{\alpha(y) + \sum_{j=1}^{n} c_j(y) x_j}{\sum_{j=1}^{n} d_j x_j + \beta} \tag{12}$$

where
$$\left.\begin{aligned} \alpha(y) &= \sum_{i=1}^{m} b_i y_i + \alpha \\ c_j(y) &= c_j - \sum_{i=1}^{m} a_{ij} y_i, \quad j = 1,2,\ldots n \end{aligned}\right\} \tag{13}$$

Equation (12) implies that for any fixed y, the function $\psi(y)$ is a linear fractional function that depends only on nonnegative variable $x_j,\ j = 1,2,\ldots n$.

Gol'stein[7] proposed the following dual problem (GDP) for the primal problem (PP) which in vector notation is

(GDP): Minimize $\psi(y) = y_0$ (14a)

Subject to $\beta y_0 - b^t y \geq \alpha$ (14b)

$dy_0 + A^t y \geq c$ (14c)

$y \geq 0$, y_0 is unrestricted in sign. (14d)

Remark 2: The dual problem (14a)- (14d) is a linear programming problem so its dual must be a linear programming problem. Which is

Maximize $L(u,t) = c^t u + \alpha\ t$ (15a)

Subject to
$$\left.\begin{aligned} Au - bt &\leq 0 \\ d^t u + \beta\ t &= 1 \\ u \geq 0, t &\geq 0 \end{aligned}\right\} \tag{15b}$$

It is clear that the dual form (15a) – (15b) of the dual problem (14a) – (14d) is nothing but the linear analogue (7a)-(7b) of the LFP problem (PP) obtained by Charnes and Cooper's transformation $u = xt$.

Remark 3: The dual problem (14a) – (14d) satisfies all the duality theorems.

Gol'stein[7] proved the following theorems

i) Weak duality theorem
ii) Direct duality theorem
iii) Converse duality theorem.
iv) Optimality criteria theorem

Gol'stein[7] stated the complementary slackness theorem without proof.

Dual Problem of Sharma and Swarup

In 1968 Swarup[1] worked on duality in linear fractional programming problem. In his work, the dual form of the primal problem (PP) contains nonlinear constraints and fractional objective function. Solution of such problem is more complicated. In 1972 Sharma and Swarup[8] together worked on this field and formulated the dual program for a linear fractional programming problem. The remarkable feature of this dual form is that the objective function is also a linear fractional function and the constraints are linear. So the dual program can be solved by any suitable existing technique. In their work they considered an LFP problem as a primal problem in which the objective function is in the form without the constant terms:

In vector notation which can be written as

(SPP): Maximize $S(x) = \dfrac{c^t x}{d^t x}$ (16a)

subject to $Ax \leq b$ (16b)

$x \geq 0$ (16c)

where A is a m × n matrix,

$x, c, d \in \Re^n, b \in \Re^m$, $\alpha, \beta \in \Re$ c^t, d^t denotes transpose of vectors c and d respectively.

Sharma and Swarup's[8] dual form to the primal problem is

(SDP): Minimize $g(u) = \dfrac{c^t u}{d^t u}$ (17a)

Subject to
$$\left.\begin{aligned} A^t v - cd^t u + dc^t u &\geq 0 \\ -b^t v &\geq 0 \\ u \geq 0, \quad v &\geq 0 \end{aligned}\right\} \tag{17b}$$

Where $u \in \Re^n$ and $v \in \Re^m$. $c^t u$ and $d^t u$ both do not vanish simultaneously.

Remark 4: Sharma and Swarup[8] proved Weak duality theorem, Direct duality theorem, Converse duality theorem and Optimality criteria theorem.

But they did not prove the complementary slackness theorem.

Remark 5: If (u^*, v^*) is an optimal solution to the dual problem (SDP) then

Minimum $g(u) = g(u^*) = \frac{c^t u^*}{d^t u^*}$

If $x = ku^*$, then

$$S(x) = \frac{c^t x}{d^t x} = \frac{c^t ku^*}{d^t ku^*} = \frac{c^t u^*}{d^t u^*}$$

By Optimality Criterion Theorem it implies that $x = ku^*$ is an optimal solution to the primal problem (SPP). Thus every scalar multiple of the optimal solution of (SDP) is an optimal solution of the (SPP) and vice versa.

Seshan's Dual Problem

In the dual proposed by Sharma and Swarup[8], constant terms do not appear in both the numerator and denominator of the objective function of the primal problem.

Seshan[9] extended their work to the general case where the constant term has been permitted to appear in both the numerator and denominator of the objective function of the primal and the constraints of the dual has been generalized.

Seshan[9] proposed the following dual form for the primal problem (PP):

(SeDP): Minimize $g(u,v) = \frac{c^t u + \alpha}{d^t u + \beta}$ (18a)

Subject to

$$\left.\begin{aligned} & cd^t u - dc^t u - A^t v \le \alpha d - \beta c \\ & \alpha d^t u - \beta c^t u + b^t v \le 0 \\ & u \ge 0, \quad v \ge 0, \\ & \text{Where } u \in \Re^n \text{ and } v \in \Re^m \end{aligned}\right\} \quad (18b)$$

Seshan proved that the dual of the dual (SeDP) (18a)-(18b) is not equivalent to (PP).

Remark 7: For a feasible solution x of the primal to be optimal the necessary and sufficient condition is that there exists a $v \ge 0$, $v \in \Re^m$, such that

$$cd^t x - dc^t x - A^t v \le \alpha d - \beta c \quad (19)$$

$$\alpha d^t x - \beta c^t x + b^t v \le 0 \quad (20)$$

Of the above two conditions, condition (19) is Kuhn-Tucker necessary optimality condition for (PP).

Remark 8: The problem (PP) is equivalent to the problem (SP)

(SP): Maximize $Q(x) = \frac{C(x)}{D(x)} = \frac{c^t x + \alpha x_{n+1}}{d^t x + \beta x_{n+1}}$

subject to

$$\left.\begin{aligned} Ax &\le b \\ x_{n+1} &= 1 \\ x \ge 0, \quad x_{n+1} &\ge 0 \end{aligned}\right\}$$

In this form the problem (PP) is in the same form as the LFP (16a)-(16c) considered by Sharma and Swarup[8]. The dual (SD) of (SP) as per definition of dual by Sharma and Swarup is as follows:

(SD): Minimize $g_1(u,v) = \frac{c^t u + \alpha u_{n+1}}{d^t u + \beta u_{n+1}}$

Subject to

$$\left.\begin{aligned} & cd^t u - dc^t u - A^t v \le (\alpha d - \beta c) u_{n+1} \\ & \alpha d^t u - \beta c^t u - v_{m+1} + v_{m+2} \le 0 \\ & b^t v + v_{m+1} - v_{m+2} \le 0 \\ & u \ge 0, \; v \ge 0, \; u_{n+1} \ge 0, \\ & v_{m+1} \ge 0, \quad v_{m+2} \ge 0 \end{aligned}\right\}$$

Seshan proved that the dual (SeDP) is not equivalent to the dual (SD). There is a one to many correspondence between solutions of (SeDP) and a subset of feasible solutions of (SD). Thus (SeDP) is much simpler dual of (PP) then (SD).

Remark 9 : Seshan[9] proved the duality theorems,

i) Weak duality theorem
ii) Direct duality theorem
iii) Converse duality theorem
iv) Optimality criteria theorem

And some other related theorems

Seshan[9] is silent about the complementary slackness theorem.

III. Equivalence of Dual Problems

In the previous section different type of duals proposed by several authors are discussed. In this section equivalence between different types of dual problems proposed by different authors are established.

Gol'stein's and Seshan's Dual Forms

Gol'stein's[7] dual (GDP) and Seshan's[9] dual (SeDP) are given by (14a)-(14d) and (18a)-(18b) respectively. Observing (12) and (18a), it is clear that $\psi(y)$ is equivalent to $g(u,v)$.

So if $\psi(y)$ is replaced by $g(u,v)$ that is if y_0 is replaced by $\frac{c^t u + \alpha}{d^t u + \beta}$, $u \ge 0$, in Gol'steins dual problem (14a)-(14d) then it becomes:

Minimize $g(u,v) = \frac{c^t u + \alpha}{d^t u + \beta}$ (21a)

(14b) gives $\beta y_0 - b^t y \ge \alpha$

$$\Rightarrow \beta \frac{c^t u + \alpha}{d^t u + \beta} - b^t y \ge \alpha$$

$$\Rightarrow \beta(c^t u + \alpha) - b^t y(d^t u + \beta) \ge \alpha(d^t u + \beta)$$

$$\Rightarrow \beta c^t u + \alpha\beta - b^t y(d^t u + \beta) - \alpha d^t u - \alpha\beta \ge 0$$

$$\Rightarrow \alpha d^t u - \beta c^t u + b^t y(d^t u + \beta) \le 0 \quad (21b)$$

And (14c) gives $dy_0 + A^t y \ge c$

$$\Rightarrow d\frac{c^t u + \alpha}{d^t u + \beta} + A^t y \ge c$$

$$=> d(c^t u+\alpha)+A^t y(d^t u+\beta)\geq c(d^t u+\beta)$$
$$=> dc^t u+d\alpha+A^t y(d^t u+\beta)-cd^t u-c\beta\geq 0$$
$$=> -dc^t u-d\alpha-A^t y(d^t u+\beta)+cd^t u+c\beta\leq 0$$
$$=> cd^t u-dc^t u-A^t y(d^t u+\beta)+\beta c-\alpha d\leq 0$$
$$=> cd^t u-dc^t u-A^t y(d^t u+\beta)\leq \alpha d-\beta c \quad (21c)$$

Since $y\geq 0$ and $(d^t u+\beta)>0$ so setting $y(d^t u+\beta)=v$, we get $v\geq 0$ and thus (21a)-(21c) gives the linear fractional programming problem

Minimize $g(u,v)=\dfrac{c^t u+\alpha}{d^t u+\beta}$

Subject to

$$\left.\begin{array}{l}\alpha d^t u-\beta c^t u+b^t v\leq 0\\ cd^t u-dc^t u-A^t v\leq \alpha d-\beta c\\ u\geq 0 \text{ and } v\geq 0\\ u\in\Re^n \text{ and } v\in\Re^m\end{array}\right\}\quad (22)$$

which is nothing but the dual form (SeDP) (18a)-(18b) of Seshan .

Bector's and Chadha's Dual Forms

Bector's[3] linear dual problem (LDP) and Chadha's[5] Dual form (CDP) to the primal problem (PP) is given by (10a)-(10c) and (11a)-(11d) respectively.

Setting $z=\dfrac{\alpha+b^t\mu}{\beta}$, in Chadha's dual form (CDP) we get

(11a) =>Minimize $\dfrac{\alpha+b^t\mu}{\beta}$ (23a)

(11b) => $A^t y+d\dfrac{\alpha+b^t\mu}{\beta}\geq c$ (23b)

(11c) => $-b^t y+\beta\dfrac{\alpha+b^t\mu}{\beta}=\alpha$

$=>-b^t y+\alpha+b^t\mu=\alpha$

$=>y=\mu$ (23c)

Thus (23a)-(23c) implies

Minimize $\psi(\mu)=\dfrac{\alpha+b^t\mu}{\beta}$

Subject to $A^t\mu+d\dfrac{\alpha+b^t\mu}{\beta}\geq c$

$\mu\geq 0$

This is nothing but Bector's Linear Dual form (LDP).

Relating theorem 1: The vector y^* is an optimal solution of

(CDP): $\underset{y}{Min}\{z: A^t y+dz\geq c,-b^t y+\beta z=\alpha\ , y\geq 0\}$ and

(LDP):

$$\underset{y}{Min}\ \{\frac{\alpha+b^t y}{\beta}: A^t y+d\,\frac{\alpha+b^t y}{\beta}\geq c, y\geq 0\}$$

Gol'stein's and Chadha's Dual Forms

Chadha's[5] Dual form is given by (11a)-(11d) And Gol'stein's[7] dual form to the primal problem (PP) is given by (14a)-(14d). As in **Remark 2,** the dual form of Gol'stein's dual (GDP) is

Maximize $L(u,t)=c^t u+\alpha\ t$

$$\left.\begin{array}{l}\text{Subject to } Au-bt\leq 0\\ d^t u+\beta\ t=1\\ u\geq 0, t\geq 0\end{array}\right\}$$

which is nothing but the linear analogue (7a)-(7b) of the LFP problem (PP) obtained by Charnes and Cooper's[10] transformation $u=xt$.

Now Charnes and Cooper proved that every

$(u,t)\in\{(u,t): Au+bt\leq 0, d^t u+\beta\ \ t=1,$

$u\geq 0, t\geq 0\}$ has t >0.

Therefore for the complementary slackness condition of linear programming problem to hold at an optimal solution the inequality $\beta y_0-b^t y\geq\alpha$ must hold as an equation only. Hence the Dual form of Gol'stein[7] (GDP) takes the form

(GDP): Minimize $\psi(y)=y_0$

Subject to $\beta y_0-b^t y=\alpha$

$dy_0+A^t y\geq c$

$y\geq 0$, y_0 is unrestricted in sign.

Substituting $y_0=z$ in (GDP) we have Chadha's[5] dual form (CDP) (11a)-(11d) to primal problem (PP).

Remark10: Since Gol'stein[7] dual is equivalent to Seshan's[9] dual and also equivalent to Chadha's[5] dual, so it follows immediately that Seshan's dual and Chadha's dual are equivalent. Also Chadha's dual and Bector's[2] linear dual are equivalent which indicates the equivalence of Gol'stein's and Seshan's dual with that of Bector.

IV. Conclusion

The dual of a linear fractional programming problem can be obtained by any of the method so far discussed. Several authors proposed different types of dual problems to the primal LFP problem. Some of the duals are equivalent to one another. In 1968 Swarup[1] first constructed a dual in which the constraints are nonlinear. In1972 Swarup and Sharma[8] proposed a dual which has a special feature that both the problem (Primal and dual) are linear fractional. But they considered a primal problem in which constant term does not appear in both the numerator and denominator of the objective function. In 1980 Seshan[9] extended their work to the general case where constant term has permitted to appear in both the numerator and denominator of the objective function and the constraints of the dual are also generalized. In 1971 Chadha[4] , Golestein[7], in 1968 Bector[3] proposed different duals.

Most of the authors proved all the duality theorems. Some of them did not prove Complementary Slackness Theorem. Gol'stein[7] stated this theorem without poof. Also Sharma and Swarup[8] , and Seshan[9] are silent about the Complementary Slackness Theorem. Chadha[5] did not prove the Converse Duality theorem.

Gol'stein used a ratio type Lagrange function to construct the dual problem. Both Gol'stein and Chadha proposed dual problems which are linear programming problem. Bector constructed a linear dual problem which is used to solve both the primal problem and the Bector's dual problem.

Most of the dual problems proposed by different authors are linear programming problem. Only Bector, Swarup and Sharma's and therefore of Seshan's dual problem is linear fractional problem.

In case of linear programming problem the dual of a dual problem is the primal problem. But in case linear fractional programming problem this property doesn't hold.

The dual of Gol'stein[7] is the linear analogue of primal problem obtained by Charnes and Cooper's[10] variable transformation technique. The dual of Seshan's[9] dual is a linear fractional programming problem which is not equivalent to the primal problem. But in Seshan's dual form less number of artificial variables is used to solve the LFP problem. So the problem is easy to solve and takes least iteration. Chadha's[5] dual, Gol'stein's[7] dual and Bector's[3] linear dual problems being linear programs are easy to solve and holds all the duality theorems.

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