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The Riemannian Metric on Tangent Bundle

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Abstract

Let *M* be an *n*-dimensional Riemannian manifold and *TM* its tangent bundle. The conformal and fiber preserving vector fields on *TM* have well-known physical interpretations and have been studied by physicists and geometricians. In this paper we define a Riemannian metric $\tilde{g}$ on *TM*, which is in some senses more general than other metrics previously defined on *TM* and also study the conformal vector fields on $(TM, \tilde{g})$ that every complete conformal vector filed on TM is homothetic and moreover every horizontal or vertical conformal vector filed on TM is a killing vector.

1. Introduction

Let *M* be an *n*-dimensional differential manifold with a Riemannian metric *g* and ϕ be a transformation on M. Then ϕ is called a *conformal* transformation if it preserves the angles. Let *V* be a vector field on *M* and $\{\varphi_t\}$ be the local one-parameter group of local transformations on *M* generated by *V*. Then *V* is called an *infinitesimal conformal transformation* on *M* if each ϕ_t is a local conformal transformation of *M*. It is well known that *V* is an infinitesimal conformal transformation or *conformal vector field* on *M* if and only if there is a scalar function ρ on M such that $\pounds_V g = 2\rho g$ where $\pounds_V$ denotes Lie derivation [2] with respect to the vector field *V*. *V* is called *homothetic* if *ρ* is constant and is called an *isometry* or *Killing vector field* when *ρ* vanishes.

Let *TM* be the tangent bundle over *M*, and Φ be a transformation on *TM*. Then Φ is called a *fiber preserving* transformation if it preserves the fibers. Fiber preserving transformations have well known applications in Physics [1]. Let *X* be a vector field on *TM* and $\{\Phi_t\}$ the local one parameter group of local transformation on *TM* generated by *X*. Then *X* is called an *infinitesimal fiber preserving transformation* or *fiber preserving vector field* on *TM* if each Φ_t is a local fiber preserving transformation of *TM*.

Let $\tilde{g}$ be a Riemannian metric on *TM*. The conformal vector field *X* on TM is said to be *essential* if the scalar function Ω on *TM* in $\pounds_X \tilde{g} = 2\Omega\tilde{g}$ depends only on (y^h) (with respect to the induced coordinates (x^i, y^i) on *TM*), and is said to be *inessential* if Ω depends only on (x^h). In other words, Ω is a function on M .

II. Preliminaries

Let (M, g) be a real n-dimensional Riemannian manifold and (U, x) a local chart on *M*, where the induced coordinates of the point $p \in U$ are denoted by its image on $\mathbf{R}^n$, $x(p)$ or briefly (x^i). Using the induced coordinates (x^i) on M , we have the local field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ on T_pM . Let ∇ be a Riemannian connection [2] on M with coefficients Γ^k_{ij}, where the indices $a, b, c, h, i, j, k, m, \cdots$ run over the range $1, 2, \cdots, n$. The Riemannian curvature tensor is defined by

$$K(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \forall X, Y, Z \in X(M).$$

Locally we have

$$K^m_{ijk} = \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ia} \Gamma^a_{jk} - \Gamma^m_{ja} \Gamma^a_{ik},$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $K(\partial_i, \partial_j, \partial_k) = K^m_{ijk} \partial_m$.

III. The Metric $\tilde{g}$ On Tangent Bundle

Let (M, g) be a Riemannian manifold. The Riemannian metric g has components g_{ij}, which are functions of variables x^i on *M*, and by means of the dual basis it is well known that;

$g_1 := g_{ij} dx^i dx^j$, $g_2 := 2 g_{ij} dx^i \delta y^j$ and $g_3 := g_{ij} \delta y^i \delta y^j$ are all bilinear differential forms defined globally on TM . The tensor field [3]:

$$\tilde{g} = a g_1 + b g_2 + c g_3,$$

on TM where a, b and c are certain positive real numbers, has components

$$\begin{pmatrix} a g_{ij} & b g_{ij} \\ b g_{ij} & c g_{ij} \end{pmatrix},$$

with respect to the dual basis of the adapted frame of TM . From linear algebra we have $\det \tilde{g} = (ac - b^2)^n \det g^2$. Therefore $\tilde{g}$ is nonsingular if $ac - b^2 \neq 0$ and positive definite if $ac - b^2 > 0$.

IV. Lie Derivative

Let M be an n-dimensional Riemannian manifold, V a vector field on M , and $\{\phi_t\}$ any local group of local transformations of M generated by V. Take any tensor field S on M, and denote by $\phi_t^*(S)$ the pull-back of S by ϕ_t [5]. Then Lie derivation of S with respect to V is a tensor field $£_v S$ on M defined by

$$£_v S = \frac{\partial}{\partial t}\phi_t^*(S)_{t=0} = \lim_{t \to 0} \frac{\phi_t^*(S)-(S)}{t},$$

on the domain of ϕ_t. The mapping $£_v$ which maps S to $£_v(S)$ is called the Lie derivative with respect to V.

Suppose that S is a tensor field of type (n,m). Then the components $(£_V S)^{j_1 \ldots j_n}_{i_1 \ldots i_m}$ of $£_v S$ may be expressed as [6]

$$(£_V S)^{j_1,\ldots j_n}_{i_1,\ldots i_m} = V^a \partial_a S^{j_1,\ldots j_n}_{i_1,\ldots i_m} + \sum_{k=1}^{m} \partial_{i_k} V^a S^{j_1,\ldots j_n}_{i_1,\ldots a,\ldots i_m} - \sum_{k=1}^{m} \partial_a V^{j_k} S^{j_1,\ldots a,\ldots j_n}_{i_1,\ldots a,\ldots i_m},$$

where, $S^{j_1,\ldots j_n}_{i_1,\ldots,i_m}$ and V^a denote the components of S and V.

The local expression of the Lie derivative $£_v(S)$ in terms of covariant derivatives on a Riemannian manifold for a tensor field of type (1, 2) is given by

$$£_V S^h_{ji} = v^a \nabla_a S^h_{ji} - S^a_{ji} \nabla_a v^h + S^h_{ai} \nabla_j v^a + S^h_{ja} \nabla_i v^a, \tag{1}$$

where, S^h_{ji} and v^h are components of S and V, and $\nabla_a S^h_{ji}$, $\nabla_a v^h$ are components of covariant derivatives of S and V, respectively.

Lemma 1. [1] The Lie bracket of adapted frame of TM satisfies the following relations

$$[X_i, X_j] = y^r K^m_{jir} X_{\bar{m}},$$

$$[X_i, X_{\bar{j}}] = \Gamma^m_{ji} X_{\bar{m}},$$

$$[X_{\bar{i}}, X_{\tilde{j}}] = 0,$$

where K^m_{jir} denotes the components of a Riemannian curvature tensor of M.

Lemma 2. [1] Let X be a vector field on TM with components $(X^h, X^{\bar{h}})$ with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then X is fiber-preserving vector field on TM if and only if X^h are functions on M.

Therefore, every fiber-preserving vector field X on TM induces a vector field $V = X^h \frac{\partial}{\partial x_h}$ on M.

Definition 1. Let V be a vector field on M with components V^h. We have the following vector fields on TM which are called respectively, complete, horizontal and vertical of V:

$$X^C := V^h X_h + y^m (\Gamma^h_{ma} V^a + \partial_m V^h) X_{\bar{h}},$$

$$X^H := V^h X_h,$$

$$X^V := V^h X_{\bar{h}}.$$

From Lemma 2 we know that X^C, X^H and X^V are fiber-preserving vector fields on TM.

Lemma 3. Let X be a fiber-preserving vector field on TM. Then the Lie derivative of the adapted frame and its dual basis are given by:

I)

$$£_X X_h = (-\partial_h X^a) X_a + \{y^b X^c K^a_{hcb} - X^{\bar{b}} \Gamma^a_{bh} - X_h(X^{\bar{a}})\} X_{\bar{a}},$$

II) $£_X X_{\bar{h}} = \{X^b \Gamma^a_{bh} - X_{\bar{h}}(X^{\bar{a}})\} X_a$,

III) $£_X dx^h = (\partial_m X^h) dx^m$,

IV) $£_X \delta y^h = -\{y^b X^c K^h_{mcb} - X^{\bar{b}} \Gamma^h_{bm} - X_m(X^{\bar{h}})\} dx^m - \{X^b \Gamma^h_{bm} - X_{\bar{m}}(X^{\bar{h}})\} \delta y^m$.

Lemma 4. Let X be a fiber-preserving vector field on TM, which induces a vector field V on M. Then Lie derivatives $£_X g_1$, $£_X g_2$ and $£_X g_3$ are given by:

I) $£_X g_1 = (£_V g_{ij}) dx^i dx^j$,

II) $£_X g_2 = 2[-g_{jm}\{y^b X^c K^m_{icb} - X^{\bar{b}} \Gamma^m_{bi} - X_i(X^{\bar{m}})\} dx^i dx^j +$
$\{£_V g_{ij} - g_{jm} \nabla_i X^m + g_{jm} X_{\bar{i}}(X^{\bar{m}})\} dx^j \delta y^i]$,

III) $£_X g_3 = -2g_{mi}\{y^b X^c K^m_{jcb} - X^{\bar{b}} \Gamma^m_{bj} - X_j(X^{\bar{m}})\} dx^j \delta y^i +$
$£_X g_3 = -2g_{mi}\{y^b X^c K^m_{jcb} - X^{\bar{b}} \Gamma^m_{bj} - X_j(X^{\bar{m}})\} dx^j \delta y^i +$
$\{£_V g_{ij} - 2g_{mj} \nabla_i X^m + 2g_{mj} X_{\bar{i}}(X^{\bar{m}})\} \delta y^i \delta y^j$,

where $£_V g_{ij}$ and $\nabla_i X^m$ denote the components of $£_V g$ and the covariant derivative of V respectively.

V. Main Results

Proposition 1. Let X be a complete (respectively horizontal or vertical) conformal field on TM. Then the scalar function $\Omega(x,y)$ in $£_x \tilde{g} = 2\Omega \tilde{g}$ is a function of position alone (respectively $\Omega = 0$).

Proof. Let TM be the tangent bundle over M with Riemannian metric $\tilde{g}$ and X be a complete (respectively horizontal or vertical) lift conformal vector field on TM. By definition, there is a scalar function Ω on TM such that

$$£_x \tilde{g} = 2\Omega \tilde{g}$$

Since the complete horizontal and vertical lift vector fields are fiber preserving, by applying $\pounds_x$ to the definition of $\tilde{g}$, using lemma 4 and the fact that $dx^i dx^j, dx^i \delta y^j$ and $\delta y^i \delta y^j$ are linearly independent, we have following three relations

$$a(\pounds_{\rm v} g_{ij} - 2\Omega g_{ij}) = b g_{im}(y^b X^c K^m_{jcb} - X^{\tilde{b}} \Gamma^m_{bj} - X_j(X^{\bar{m}})) + g_{jm}(y^b X^c K^m_{icb} - X^{\tilde{b}} \Gamma^m_{bi} - X_i(X^{\bar{m}})) \quad (2)$$

$$b(\pounds_{\rm v} g_{ij} - 2\Omega g_{ij}) = b g_{im}(\nabla_j X^m - X_{\bar{j}}(X^{\bar{m}})) + c g_{jm}(y^b X^c K^m_{icb} - X^{\tilde{b}} \Gamma^m_{bi} - X_i(X^{\bar{m}})) \quad (3)$$

Using relation 1, we have $\pounds_v g_{ij} = \nabla_i V_j + \nabla_j V_i$, from which we obtain

$$2\Omega g_{ij} = g_{mj} X_{\bar{i}}(X^{\bar{m}}) + g_{mi} X_{\bar{j}}(X^{\bar{m}}). \quad (4)$$

Applying $X_{\bar{k}}$ to the relation 4 and using the fact that g_{ij} is a function of position alone, we have

$$2 g_{ij} X_{\bar{k}}(\Omega) = g_{mj} X_{\bar{k}} X_{\bar{i}}(X^{\bar{m}}) + g_{mi} X_{\bar{k}} X_{\bar{j}}(X^{\bar{m}}) \quad (5)$$

By means of definition 1 for complete vector fields, and by replacing the value of $X^{\bar{m}}$ in relation 5, we have

$$2 g_{ij} X_{\bar{k}}(\Omega) = g_{mj} X_{\bar{k}} X_{\bar{i}}(y^l(\Gamma^m_{la} V^a + \partial_l V^m)) + g_{mi} X_{\bar{k}} X_{\bar{j}}(y^l(\Gamma^m_{la} V^a + \partial_l V^m)).$$

Since the coefficients of the Riemannian connection on *M*, and components of vector field V are functions of position alone, the right hand side of the above relation becomes zero, from which we have $X_{\bar{k}}(\Omega) = 0$. This means that the scalar function $\Omega(x, y)$ on *TM* depends only on the variables (x^h).

Similarly, for vertical vector fields, by using the fact that the components of V are functions of position alone and from relation 4, we have $\Omega = 0$. Finally, for horizontal lift vector field by means of relation 4, we have $\Omega = 0$. Hence completes the proof.

Proposition 2. Let *M* be a connected manifold and *X* be a complete lift conformal vector field on *TM*. Then the scalar function $\Omega(x, y)$ in $\pounds_x \tilde{g} = 2\Omega \tilde{g}$ is constant.

Proof. Let *X* be a complete conformal vector field on *TM* with components $(X^h, X^{\bar{h}})$, with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$.

Let us put

$$A^m_a = \Gamma^m_{ah} X^h + \partial_a X^m.$$

The coordinate transformation rule implies that A^m_a are the components of (1, 1) tensor field *A*.

Then its covariant derivative is

$$\nabla_i A^m_a = \partial_i A^m_a + \Gamma^m_{ik} A^k_a - \Gamma^k_{ia} A^m_k,$$

where $\nabla_i A^m_a$ is the component of the covariant derivative of tensor field *A*.

From definition 1, $X^{\bar{m}} = A^m_a y^a$. By means of relation 3, we have

$$b[\pounds_{\rm v} g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j X^m - A^m_j)] = c g_{jm}[y^a X^c K_{ica}{}^m - \Gamma^m_{ki} A^k_a y^a - X_i(A^m_h y^h)]$$

Note that the components of *A* are functions of position alone, from which the right hand side of this relation becomes

$$c g_{im}[y^a X^c K^m_{ica} - \Gamma^m_{ki} A^k_a y^a - (\frac{\partial}{\partial x^i} - y^a \Gamma^k_{ai} \frac{\partial}{\partial y^k})(A^m_h y^h)]$$
$$= c g_{im}[y^a X^c K^m_{ica} - \Gamma^m_{ki} A^k_a y^a - y^a \frac{\partial}{\partial x^i} A^m_a + \Gamma^k_{ai} A^m_k y^a$$
$$= c y^a (X^c K^m_{icaj} - g_{mj} \nabla_i A^m_a)$$

Thus we have

$$b[\pounds_v g_{ij} - 2\Omega g_{ij} - g_{mi}(\nabla_j X^m - A^m_j)] = c y^a (X^c K^m_{icaj} - g_{mj} \nabla_i A^m_a)$$

By means of Proposition 1 the left hand side of the above relation is a function of position alone. Applying $X_{\bar{k}} = \frac{\partial}{\partial y^k}$ to this relation gives

$$X^c K^m_{icaj} - g_{mj} \nabla_i A^m_a = 0.$$

Or,

$$X^c K^m_{icaj} = \nabla_i A_{ja}$$

From which

$$\nabla_i A_{ja} + \nabla_i A_{aj} = 0. \quad (6)$$

Now by replacing $X^{\bar{m}}$ in relation 4

$$2\Omega g_{ij} = g_{mj} X_{\bar{i}}\{y^h(\Gamma^m_{ha} X^a + \partial_h X^m)\} + g_{mi} X_{\bar{j}}(y^h(\Gamma^m_{ha} X^a + \partial_h X^m))$$
$$= g_{mi}(\Gamma^m_{ia} X^a + \partial_h X^m) + g_{mi}(\Gamma^m_{ja} X^a + \partial_j X^m)$$
$$= g_{mi} A^m_i + g_{mi} A^m_j.$$

Applying covariant derivative ∇_k to this relation gives

$$2 g_{ij} \nabla_k \Omega = \nabla_k A_{ji} + \nabla_k A_{ij}.$$

From relation 6, we get $\nabla_k \Omega = \frac{\partial}{\partial x_h} \Omega = 0.$

Since *M* is connected, the scalar function Ω is constant. Hence completes the proof.

Theorem 1. Let *M* be a connected *n*-dimensional Riemannian manifold and *TM* be its tangent bundle with metric $\tilde{g}$. Then every complete conformal vector field on

TM is homothetic, moreover, every horizontal or vertical conformal vector field on *TM* is a killing vector.

Proof. Let *M* be an *n* dimensional Riemannian manifold. *TM* its tangent bundle with the metric $\tilde{g}$ and *X* a complete (respectively horizontal or vertical) conformal vector field on *TM*. Then by means of Proposition 1 the scalar function $\Omega(x,y)$ in $£_x \tilde{g} = 2\Omega\,\tilde{g}$ is a function of position alone (respectively $\Omega = 0$), and by means of Proposition 2 it is constant. Thus, every complete conformal vector field on *TM* is homothetic and every horizontal or vertical conformal vector field on *TM* is a Killing vector. Hence completes the proof.

Theorem 2. Let *M* be a connected *n*-dimensional Riemannian manifold and *TM* be its tangent bundle with metric $\tilde{g}$. Then every inessential fiber preserving vector field on *TM* is homothetic.

Proof. Let *X* be an inessential fiber preserving conformal vector field on *TM* with components $(X^h, X^{\bar{h}})$, with respect to the adapted frame $(X_h, X_{\bar{h}})$. Using the same argument in proof of Proposition 1, it is obvious that we have relations 2, 3 and 4. From relation 4, we have

$$\Omega g_{ij} = g_{mi} X_{\bar{i}}(X^{\bar{m}}).$$

Since $\Omega(x,y)$ in $£_x \tilde{g} = 2\Omega\,\tilde{g}$ is supposed to be a function of position alone, by applying $X_{\bar{i}}$ to the above relation we have

$$X_{\bar{i}}(X_{\bar{i}}(X^{\bar{m}})) = 0.$$

Applying $X_{\bar{i}}$ to relation 4 again and using above relation gives

$$X_{\bar{i}}(X_{\bar{j}}(X^{\bar{m}})) = 0.$$

Thus we can write

$$X^{\bar{m}} = \alpha_a^m y^a + \beta^m, \qquad (7)$$

where α_a^m and β^m are certain functions of position alone. Replacing relation 7 in relation 3, we have

$$b(£_v g_{ij} - 2\Omega g_{ij}) = b g_{im}(\nabla_j X^m - \alpha_j^m) + c g_{jm}(y^b X^c K_{icb}^m - y^a \alpha_a^b \Gamma_{bi}^m - \beta^b \Gamma_{bi}^m - y^a \frac{\partial}{\partial x^i}\alpha_a^m - \frac{\partial}{\partial x^i}\beta^m + y^a \Gamma_{ai}^k \alpha_k^m)$$

$$= b g_{im}(\nabla_j X^m - \alpha_j^m) + c g_{jm}(y^b X^c K_{icb}^m - y^a \nabla_i \alpha_a^m) - c g_{jm} \nabla_i \beta^m$$

Therefore

$$b(£_v g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j X^m - \alpha_j^m)) + c g_{jm} \nabla_i \beta^m = c g_{jm} y^a (X^c X_{ica}^m - \nabla_i \alpha_a^m).$$

The left hand side of this relation is a function of position alone. From which by applying $X_{\bar{k}}$ we have

$$X^c K_{ica}^m = \nabla_i \alpha_a^m \qquad (8)$$

Replacing relation 7 in relation 4 we find

$$2\Omega g_{ij} = \alpha_{ji} + \alpha_{ij}.$$

The covariant derivative of this relation and using relation 8 gives

$$\nabla_k \Omega = \frac{\partial}{\partial x_h}\Omega = 0.$$

Since *M* is connected, then the scalar function Ω on *M* is constant. This completes the proof of Theorem 2.

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