Dhaka Univ. J. Sci. 58(1): 141-142 2010 (January)

Eigenvalue Analysis of 2D Helmholtz Equation on Quadrilateral Elements

Thowhida Akther[1], Md. Shafiqul Islam[2,*], Sanwar Uddin Ahmad[2]

[1]*Debidwar S. A. Govt. College, Comilla.*

[2]*Department of Mathematics, Dhaka University, Dhaka – 1000, Bangladesh*

Received on 10.08.2008. Accepted for Publication on 00.00.2009

I. Introduction

Eigenvalue analysis is one of the most important and fundamental tools for wave and vibration, and has been treated by the finite element method (FEM) [1 – 6], by the boundary element method (BEM) [7, 8], and also by the integral equation [9]. In this paper we study the Helmholtz equation which governs the time-harmonic wave problem $(\nabla^2 + k^2)u = 0$, since $k = 2\pi/\lambda$, where k is the wave number and λ is the wave length, and u is the field of interest which could be wave elevation, pressure, or an electro-magnetic potential, among many other possibilities [3]. Recently the same problem was considered by the method of the special integration scheme [3, 4] and the theory of integrations of the element matrices for rectangular, triangular and special quadrilateral finite elements for very short waves, and they used Maple and FORTRAN codes for the evaluation of integration abscissas and weights. Ortiz and Sanchez [5] also developed an analytical integration procedure for their triangular elements. They applied local co-ordinate rotation to the integrand so that the oscillatory component is transformed into the form depending only on one 'effective' direction with an 'equivalent' wave number. This simplifies the integration procedure and reduces the number operations. Hacker and Schreyer [6] studied exact analytical expressions for the eigenvalues only for four node rectangular elements. The quadrilateral element is more complicated to deal with than rectangular or triangular element due to the $\xi\eta$ term in the Jacobian. Thus the objective of this study is to evaluate the eigenvalue analysis for general quadrilateral elements and finally a comparison is made with the exact solutions.

II. FE Formulation of Helmholtz Equation

The partial differential equation (PDE) governing transient heat transfer on a 2-D region Ω

$$c\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a_{11}\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(a_{22}\frac{\partial u}{\partial y}) + a_0 u = f(x,y,t) \qquad (1a)$$

with the boundary conditions (b.c.),

$$u = \hat{u},\ q_n = a_{11}\frac{\partial u}{\partial x}n_x + a_{22}\frac{\partial u}{\partial x}n_y = \hat{q}_n \ \text{ on } \gamma \qquad (1b)$$

and the initial conditions (i.e. at $t = 0$) are of the form

$$u(x,y,0) = u_0(x,y) \ \text{ in } \Omega \qquad (1c)$$

Here t denotes the time and $c, a_{11}, a_{22}, a_0, u_0, f, \hat{u}, \hat{q}_n$ are given as functions of position and/or time.

The corresponding homogeneous boundary value problem is

$$c\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a_{11}\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(a_{22}\frac{\partial u}{\partial y}) + a_0 u = 0 \ \text{ in } \Omega \qquad (2a)$$

$$u = 0 \qquad q_n = 0 \ \text{ on } \gamma \qquad (2b)$$

Solution of equation (2) is known generally to behave according to

$$u(x,y,t) = \psi(x,y)\exp(-\beta t) \qquad (3)$$

which reduces equation (2) as

$$\frac{\partial}{\partial x}(a_{11}\frac{\partial \psi}{\partial x}) + \frac{\partial}{\partial y}(a_{22}\frac{\partial \psi}{\partial y}) + \lambda\psi = 0 \qquad (4a)$$

where $\lambda = c\beta + a_0$ and the b.c. (2b) reduces to

$$\psi = 0,\ \partial\psi/\partial\eta = 0 \qquad (4b)$$

The λ's and ψ's satisfying eqs. (4) are known as ***eigenvalues*** and ***eigenfunctions***, respectively. The differential equation (4), which represents the ***time-independent*** form of original equation, results from applying the technique of separation of variables to reduce the complexity of the analysis, is known as the ***Helmholtz equation***.

The weak form of (1) over an element $[e]$ is obtained by substituting a finite element approximation for the dependent variable u as

$$u(x,y,t) \approx \sum_{j=1}^{n} u_j^e(t)\,\psi_j^e(x,y) \qquad (5)$$

where u_j denotes the value of $u(x,y,t)$ at the spatial location (x_j, y_j) at time t.

The i-th element equation (in time) of the finite element model is

$$\sum_{j=1}^{n}(M_{ij}^e\frac{du_j^e}{dt} + K_{ij}^e u_j^e) = f_i^e + Q_i^e \qquad (6)$$

or in matrix form,

$$[M^e]\{\dot{u}^e\} + [K^e]\{u^e\} = \{f^e\}\{Q^e\} \qquad (7)$$

where a superposed dot on u denotes the time derivative ($\dot{u} = du/dt$) and

$$\begin{aligned} M_{ij}^e &= \iint_e c\,\psi_i\psi_j\,dx\,dy,\ f_i^e = \iint_e f\,\psi_i\,dx\,dy,\ Q_i^e = \int_{\tau^e} q_n\psi_i\,ds \\ K_{ij}^e &= \iint_e (a_{11}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + a_{22}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y} + a_0\psi_i\psi_j)\,dx\,dy \end{aligned} \qquad (8)$$

The matrices K_{ij}^e and M_{ij}^e are known as element matrix and mass matrix, respectively.

Substituting $u_j^e(t) = u_j\exp(-\lambda t)$ in (7) to obtain,

$$(-\lambda[M^e] + [K^e])\{U\} = \{0\} \qquad (9)$$

Assembling the element equations (9), we have

$$([K] - \lambda[M])\{U\} = \{0\} \qquad (10)$$

*Corresponding author (email: mdshafiqul@yahoo.com)

A non-trivial solution to equation (10) exists only if $\left|[K]-\lambda[M]\right|=0$, which when expanded results in an n-th degree polynomial in λ, where n is the number of nodes at which the solution is not known. The n roots, λ_j, j = 1, …, n of this polynomial gives the first n ***eigenvalues*** of the increased system.

III. Numerical Example

For the well-known transient heat conduction equation, the following PDE is considered [1],

$$\frac{\partial u}{\partial t}-\nabla^2 u=f \tag{11}$$

in a square region $D=\{(x,y):0<x<1,0<y<1\}$, subject to the boundary conditions for $t > 0$, shown in Fig. 1.

$$\frac{\partial u}{\partial t}(0,y,t)=0\,,\ \frac{\partial u}{\partial y}(x,0,t)=0\,,$$

$$u(x,1,t)=0\,,\ u(1,\ y,\ t)=0$$

and the initial condition

$$u(x,y,0)=0\,.$$

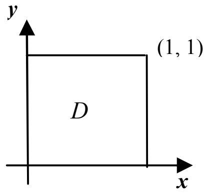

Fig 1. Domain *D*

The homogeneous form of (11) is,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{12a}$$

with boundary conditions,

$$\begin{aligned}&\frac{\partial u}{\partial t}(0,y,t)=0,\quad \frac{\partial u}{\partial y}(x,0,t)=0,\\ &u(x,1,t)=0,\ u(1,\ y,\ t)=0\end{aligned} \tag{12b}$$

The exact solution of (12) is

$$\begin{aligned}u(x,y,t)&=T_0\exp(-\lambda t)U(x,y)\\ &=T_0\exp(-\lambda t)(c_1\sin\sqrt{\mu}x+c_2\cos\sqrt{\mu}x)\times\\ &\quad(c_3\sin\sqrt{\lambda-\mu}y\ +c_4\cos\sqrt{\lambda-\mu}y)\end{aligned}$$

where λ is the required eigenvalue and the nontrivial $U(x,y)$ is the corresponding eigenfunction. The exact [1] eigenvalue λ is $\lambda_{m,n}=\pi^2(m^2+n^2)/4$, m, n = 1, 3, 5, … .

Now we evaluate the approximate eigenvalues using the mesh of quadrilateral elements, shown in Fig. 2.

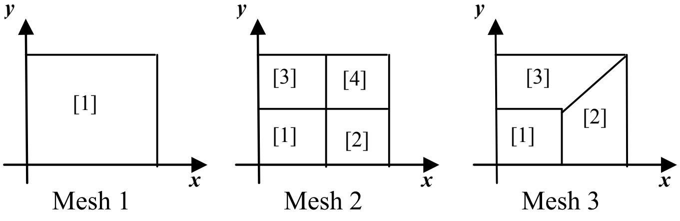

Fig. 2. Different Mesh of the domain of the problem.

The eigenvalues are calculated using the present formulation and compared to the exact solutions. The errors (%) are shown in Table 1.

Table. 1. Errors (%) analysis for the meshes in Fig.2.

Mesh1			Mesh 2			Mesh 3			Exact
Q4	Q8	Q9	Q4	Q8	Q9	Q4	Q8	Q9	
21.59	1.1	0.75	5.24	0.05	0.28	4.53	27.54	16.15	04.935
--	40.5	40.5	38.95	3.1	2.89	36.67	1.64	1.63	24.674
--	44.6	40.5	38.95	3.1	3.61	2.11	16.35	13.36	24.674
--	--	--	--	8.29	3.05	--	13.83	12.44	44.413
--	--	0.32	1.21	--	--	11.01	--	--	64.152
--	--	--	--	5.07	5.97	--	1.98	3.14	83.892
--	--	--	--	5.07	5.05	--	2.76	2.33	83.892
--	--	--	--	18.93	18.97	--	--	21.01	123.370
--	--	--	--	6.54	18.34	--	--	--	123.370
--	--	--	--	--	7.18	--	8.81	--	143.109
--	--	--	--	--		--	9.12	--	143.109
--	--	--	--	0.57	1.87	--	0.83	0.81	202.327
--	--	--	--	0.44	0.47	--	1.16	--	202.327
--	--	--	--	3.74	1.30	--	--	--	222.066
--	--	--	--	3.74	0.13	--	--	9.57	222.066
--	--	--	--	--	--	--	1.54	1.56	261.544
--	--	--	--	--	5.16	--	--	1.56	261.544
--	--	--	--	--	7.32	--	--	1.56	261.544
--	--	--	--	--	--	--	--	0.09	301.023
--	--	--	--	--	3.98	--	--	5.49	419.458

IV. Conclusions

In this paper the eigenvalue analysis of Helmholtz equation using FEM over quadrilateral element is studied. Two types of finite elements, such as regular and distorted quadrilateral, are discussed. All integrals of the element matrices and mass matrices are evaluated analytically using *MATHEMATICA*. We observe that regular Q8 elements give better results than those of distorted quadrilateral elements for small eigenvalues. On the other hand, for largest eigenvalues, Q9 elements may be recommended to get the results with high accuracy.

----

1. Reddy, J.N. (1993) - *An Introduction to the Finite Element Method*, Second edition, International Edn., McGraw-Hill London.
2. Bickford, W. B. (1990) - *A First Course in the Finite Element Method*, Second Edition, Irwin.
3. Bettess , P., Shirron, J., Laghrouche, O., Peseux, Bernard (2003) – A numerical integration scheme for special finite elements for the Helmholtz equation, *Int. Jn. Numer. Methods Engng*, **56**, 531 – 552.
4. Sugimoto, R., Bettess, P., Trevelyan, J. (2003)–A numerical integration scheme for special quadrilateral finite elements for the Helmholtz equation, *Commun. Numer. Methods Engng*, **19**, 233 – 245.
5. Ortiz , P., Sanchez , E. (2001) – An improved partition of unity finite element model for diffraction problems, *Int. Jn. Numer. Methods Engng*, **50**, 2727 – 2740.
6. Hacker, W.L., Schreyer, H.I. (1989) – Eigenvalue analysis of compatible and incompatible rectangular four node quadrilateral elements, *Int. Jn. Numer. Methods Engng*, Vol. 28, 687 – 703.
7. Kamiya , N., Andoh , E. (1993)– Helmholtz eigenvalue analysis by boundary element method, *Jn. Sound Vibration*, **160**, 279 – 287.
8. Kirkop, S.M., Amini, S. (1993) – Solution of the Helmholtz eigenvalue problem via the boundary element method, *Int. Jn. Numer. Methods Engg*, **36**, 321 – 330.
9. Mey, G. De (1976) – Calculation of eigenvalues of the Helmholtz equation by an integral equation, *Int. Jn. Numer. Methods Engg*, **10**, 59 – 66.